Hypersurfaces with constant mean curvature in a real space form

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Abstract

Let $M^n$ be an $n$-$\geq 3$)-dimensional complete connected and oriented hypersurface in $M^{n+1}(c)(c \geq 0)$ with constant mean curvature $H$ and with two distinct principal curvatures, one of which is simple. We show that (1) if $c = 1$ and the squared norm of the second fundamental form of $M^n$ satisfies a rigidity condition (1.3), then $M^n$ is isometric to the Riemannian product $S^{1}(\sqrt{1-a^2}) \times S^{n-1}(a)$; (2) if $c = 0$, $H \neq 0$ and the squared norm of the second fundamental form of $M^n$ satisfies $S \geq n^2H^2/(n-1)$, then $M^n$ is isometric to the Riemannian product $S^{n-1}(a) \times \mathbb{R}$ or $S^{1}(a) \times \mathbb{R}^{n-1}$.

Key Words: Hypersurface, scalar curvature, mean curvature, principal curvature.

1. Introduction

Let $M^{n+1}(c)$ be an $(n+1)$-dimensional connected Riemannian manifold with constant sectional curvature $c$. According to $c > 0$ or $c = 0$, it is called sphere space or Euclidean space, respectively, and it is denoted by $S^{n+1}(c)$ or $\mathbb{R}^{n+1}$. Let $M^n$ be an $n$-dimensional hypersurface in $S^{n+1}(1)$ or $\mathbb{R}^{n+1}$. As it is well known there are many rigidity results for hypersurfaces with constant mean curvature or constant scalar curvature $n(n-1)r$ in $S^{n+1}(1)$ or $\mathbb{R}^{n+1}$; for example, see [1], [2], [4], [5], [7] and the author of [3] and [6]. In [7], Wei proved the following theorem.

**Theorem 1.1 ([7])** Let $M^n$ be an $n$-$\geq 3$)-dimensional complete connected and oriented hypersurface in $S^{n+1}(1)$ with constant mean curvature $H$ and with two distinct principal curvatures, one of which is simple. If

$$
S \geq n + \frac{n^3 H^2}{2(n-1)} + \frac{n(n-2)}{2(n-1)} \sqrt{n^2H^4 + 4(n-1)H^2},
$$

then $M^n$ is isometric to the Riemannian product $S^{1}(a) \times S^{n-1}(\sqrt{1-a^2})$, where $a^2 = \frac{1}{2n(1+H^2)}[2 + nH^2 - \sqrt{n^2H^4 + 4(n-1)H^2}]$, and $S$ denotes the squared norm of the second fundamental form of $M^n$.

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Let $M^n$ be an $n$-dimensional complete connected and oriented hypersurface in $S^{n+1}(1)$ with constant mean curvature $H$ and with two distinct principal curvatures, one of which is simple. If
\[
S \leq n + \frac{n^3H^2}{2(n-1)} - \frac{n(n-2)}{2(n-1)} \sqrt{n^2H^4 + 4(n-1)H^2},
\]
then $M^n$ is isometric to the Riemannian product $S^1(a) \times S^{n-1}(\sqrt{1-a^2})$, where $a^2 = \frac{1}{2n(1+n^r)}[2 + nH^2 + \sqrt{n^2H^4 + 4(n-1)H^2}]$, and $S$ denotes the squared norm of the second fundamental form of $M^n$.

On the other hand, if $M^n$ is an $n$-dimensional complete connected and oriented hypersurface in $\mathbb{R}^{n+1}$ with constant scalar curvature $n(n-1)r$, Cheng [2] proved the following.

Let $M^n$ be an $n$-dimensional complete connected and oriented hypersurface in $\mathbb{R}^{n+1}$ with constant scalar curvature $n(n-1)r$ and with two distinct principal curvatures, one of which is simple. Then $M^n$ is isometric to the Riemannian product $S^{n-1}(a) \times \mathbb{R}$ or $S^1(a) \times \mathbb{R}^{n-1}$, if $S \geq \frac{n(n-1)r}{n-2}$.

In this paper, we shall also investigate $n$-dimensional hypersurfaces with constant mean curvature $H$ in $S^{n+1}(c)$ or $\mathbb{R}^{n+1}$ and obtain the following results:

Let $M^n$ be an $n$-dimensional complete connected and oriented hypersurface in $S^{n+1}(1)$ with constant mean curvature $H$ and with two distinct principal curvatures, one of which is simple. If
\[
S \leq n + \frac{n^3H^2}{2(n-1)} - \frac{n(n-2)}{2(n-1)} \sqrt{n^2H^4 + 4(n-1)H^2},
\]
then $M^n$ is isometric to the Riemannian product $S^1(a) \times S^{n-1}(\sqrt{1-a^2})$, where $a^2 = \frac{1}{2n(1+n^r)}[2 + nH^2 + \sqrt{n^2H^4 + 4(n-1)H^2}]$.

Let $M^n$ be an $n$-dimensional complete connected and oriented hypersurface in $\mathbb{R}^{n+1}$ with non-zero constant mean curvature $H$ and with two distinct principal curvatures, one of which is simple. If
\[
S \geq \frac{n^2H^2}{n-1},
\]
then $M^n$ is isometric to the Riemannian product $S^{n-1}(a) \times \mathbb{R}$ or $S^1(a) \times \mathbb{R}^{n-1}$.

2. Preliminaries

Let $M^{n+1}(c)$ be an $(n+1)$-dimensional connected Riemannian manifold with constant sectional curvature $c(\geq 0)$. Let $M^n$ be an $n$-dimensional complete connected and oriented hypersurface in $M^{n+1}(c)$. We choose a local orthonormal frame $e_1, \ldots, e_{n+1}$ in $M^{n+1}(c)$ such that $e_1, \ldots, e_n$ are tangent to $M^n$. Let $\omega_1, \ldots, \omega_{n+1}$ be the dual coframe. We use the following convention on the range of indices:

$$1 \leq A, B, C, \cdots \leq n + 1; \quad 1 \leq i, j, k, \cdots \leq n.$$
The structure equations of $M^{n+1}(c)$ are given by

\[ d\omega_A = \sum_B \omega_{AB} \land \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \quad (2.1) \]

\[ d\omega_{AB} = \sum_C \omega_{AC} \land \omega_{CB} + \Omega_{AB}, \quad (2.2) \]

where

\[ \Omega_{AB} = -\frac{1}{2} \sum_{C,D} K_{ABCD} \omega_C \land \omega_D, \quad (2.3) \]

\[ K_{ABCD} = c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \quad (2.4) \]

Restricting to $M^n$ such that

\[ \omega_{n+1} = 0, \quad (2.5) \]

\[ \omega_{n+1} = \sum_j h_{ij} \omega_j, \quad h_{ij} = h_{ji}, \quad (2.6) \]

the structure equations of $M^n$ are

\[ d\omega_i = \sum_j \omega_{ij} \land \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \quad (2.7) \]

\[ d\omega_{ij} = \sum_k \omega_{ik} \land \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \land \omega_l, \quad (2.8) \]

\[ R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}), \quad (2.9) \]

\[ R_{ij} = (n-1)c\delta_{ij} + nHh_{ij} - \sum_k h_{ik}h_{kj}, \quad (2.10) \]

\[ n(n-1)(r-c) = n^2H^2 - S, \quad (2.11) \]

where $n(n-1)r$ is the scalar curvature, $H$ is the mean curvature and $S$ is the squared norm of the second fundamental form of $M^n$.

Let $M^n$ be an $n (n \geq 3)$-dimensional complete connected and oriented hypersurface in $M^{n+1}(c)$ with constant mean curvature and with two distinct principal curvatures, one of which is simple. Without loss of generality, we may assume

\[ \lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = \lambda, \quad \lambda_n = \mu, \quad (2.12) \]

where $\lambda_i$ for $i = 1, 2, \cdots, n$ are the principal curvatures of $M^n$. We have

\[ (n-1)\lambda + \mu = nH, \quad S = (n-1)\lambda^2 + \mu^2. \quad (2.13) \]

From (2.13) and (2.11), we have, for $c = 1$, that

\[ \lambda\mu = (n-1)(r-1) - (n-2)H^2 + (n-2)H \sqrt{H^2 - (r-1)}, \quad (2.14) \]

303
Example 2.1 Let \(M\) be an \((n - 1)\)-dimensional connected hypersurface with constant mean curvature \(H\) and with two distinct principal curvatures \(\lambda\) and \(\mu\) with multiplicities \((n - 1)\) and 1, respectively. Then \(M\) is a locus of moving \((n - 1)\)-dimensional submanifold \(M_{1,n-1}(s)\) along which the principal curvature \(\lambda\) of multiplicity \(n - 1\) is constant and which is locally isometric to an \((n - 1)\)-dimensional sphere \(S^{n-1}(a)\) of constant curvature \(a\). Thus \(M\) is equivalent to its first order integral

\[
\lambda \mu = (n - 1)(r - 1) - (n - 2)H^2 - (n - 2)H\sqrt{H^2 - (r - 1)},
\]

on \(M\).

On the other hand, from (2.13) and (2.11), we have, for \(c = 0\), that

\[
\lambda \mu = (n - 1)r - (n - 2)H^2 + (n - 2)H\sqrt{H^2 - r},
\]

on \(M\), or

\[
\lambda \mu = (n - 1)r - (n - 2)H^2 - (n - 2)H\sqrt{H^2 - r},
\]

on \(M\).

Example 2.2 Let \(M_{k,n-k} := S^{n-k}(a) \times \mathbb{R}^k\). Then \(M_{k,n-k}\) has two distinct constant principal curvatures 0 and \(\sqrt{a}\) with multiplicities \(k\) and \(n-k\), respectively. It is easily seen that \(a^2 = \frac{1}{2n(1+H^2)}[2 + nH^2 \pm \sqrt{n^2H^4 + 4(n-1)H^2}]\) and \(S = n + \frac{n^2H^2}{2(n-1)} + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}\).

Example 2.3 Let \(M_{1,n-1} := S^1(a) \times S^{n-1}(\sqrt{1-a^2})\). Then \(M_{1,n-1}\) has two distinct constant principal curvatures \(-\frac{a}{\sqrt{1-a^2}}\) and \(\frac{\sqrt{1-a^2}}{a}\) with multiplicities \(n - 1\) and 1, respectively. It is easily seen that \(\frac{1}{2n(1+H^2)}[2 + nH^2] \pm \sqrt{n^2H^4 + 4(n-1)H^2}\) and \(S = n + \frac{n^2H^2}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}\).

3. Proof of theorem

In order to prove Theorem 1.4, we need the following propositions due to [7].

Proposition 3.1 ([7]) Let \(M^n\) be an \(n\) \((n \geq 3)\)-dimensional connected hypersurface with constant mean curvature \(H\) and with two distinct principal curvatures \(\lambda\) and \(\mu\) with multiplicities \((n - 1)\) and 1, respectively. Then \(M^n\) is a locus of moving \((n - 1)\)-dimensional submanifold \(M_{1,n-1}^{n-1}(s)\) along which the principal curvature \(\lambda\) of multiplicity \(n - 1\) is constant and which is locally isometric to an \((n - 1)\)-dimensional sphere \(S^{n-1}(a(s)) = E^n(s) \cap S^{n+1}(1)\) of constant curvature \(\varpi = |\lambda - H|^{-\frac{1}{2}}\) satisfies the ordinary differential equation of order 2

\[
\frac{d^2\varpi}{ds^2} + \varpi[1 + H^2 + (2 - n)H\varpi^{-n} + (1 - n)\varpi^{-2n}] = 0,
\]

for \(\lambda - H > 0\) or

\[
\frac{d^2\varpi}{ds^2} + \varpi[1 + H^2 + (n - 2)H\varpi^{-n} + (1 - n)\varpi^{-2n}] = 0,
\]

for \(\lambda - H < 0\), where \(E^n(s)\) is an \(n\)-dimensional linear subspace in the Euclidean space \(R^{n+2}\) which is parallel to a fixed \(E^n(s_0)\).

Lemma 3.1 ([7]) Equation (3.1) or (3.2) is equivalent to its first order integral

\[
\left(\frac{d\varpi}{ds}\right)^2 + (1 + H^2)\varpi^2 + 2H\varpi^{-n} + \varpi^{-2n} = C,
\]

(3.3)
for \( \lambda - H > 0 \) or
\[
\left( \frac{d\varpi}{ds} \right)^2 + (1 + H^2)\varpi^2 - 2H\varpi^{2-n} + \varpi^{2-2n} = C,
\]
(3.4)
for \( \lambda - H < 0 \), where \( C \) is a constant. Moreover, the constant solution of (3.1) or (3.2) corresponds to the Riemannian product \( S^1(a) \times S^{n-1}(\sqrt{1-a^2}) \).

By the same method in [7], we can prove the following proposition.

**Proposition 3.2** Let \( M^n \) be an \( n \) \((n \geq 3)\)-dimensional complete connected hypersurface in \( S^{n+1}(1) \) with constant mean curvature \( H \) and with two distinct principal curvatures \( \lambda \) and \( \mu \) with multiplicities \((n-1)\) and \( 1 \), respectively. If \( \lambda \mu + 1 \geq 0 \), then \( M^n \) is isometric to the Riemannian product \( S^1(a) \times S^{n-1}(\sqrt{1-a^2}) \).

**Proof.** Let \( \lambda \) and \( \mu \) be the two distinct principal curvatures of \( M^n \) with multiplicities \((n-1)\) and \( 1 \), respectively. Then, from \( nH = (n-1)\lambda + \mu \), we have \( \lambda \mu = nH\lambda - (n-1)\lambda^2 \). Let \( \varpi = |\lambda - H|^{-\frac{1}{2}} \). Then we have \( \lambda = H + \varpi^{-n} \) for \( \lambda - H > 0 \) and \( \lambda = H - \varpi^{-n} \) for \( \lambda - H < 0 \). If \( \lambda - H > 0 \), we have
\[
\lambda \mu + 1 = 1 + H^2 + (2 - n)H\varpi^{-n} + (1 - n)\varpi^{-2n},
\]
and if \( \lambda - H < 0 \), we have
\[
\lambda \mu + 1 = 1 + H^2 + (n - 2)H\varpi^{-n} + (1 - n)\varpi^{-2n}.
\]

Therefore, if \( \lambda \mu + 1 \geq 0 \), we obtain
\[
1 + H^2 + (2 - n)H\varpi^{-n} + (1 - n)\varpi^{-2n} \geq 0,
\]
for \( \lambda - H > 0 \) and
\[
1 + H^2 + (n - 2)H\varpi^{-n} + (1 - n)\varpi^{-2n} \geq 0,
\]
for \( \lambda - H < 0 \). From (3.1) and (3.2), we have \( \frac{d^2 \varpi}{ds^2} \leq 0 \). Thus \( \frac{d\varpi}{ds} \) is a monotonic function of \( s \in (-\infty, +\infty) \). Therefore, \( \varpi(s) \) must be monotonic when \( s \) tends to infinity. From (3.3) and (3.4), we know that the positive function \( \varpi(s) \) is bounded from above. Since \( \varpi(s) \) is bounded and is monotonic when \( s \) tends infinity, we find that both \( \lim_{s \to -\infty} \varpi(s) \) and \( \lim_{s \to +\infty} \varpi(s) \) exist and then we have
\[
\lim_{s \to -\infty} \frac{d\varpi(s)}{ds} = \lim_{s \to +\infty} \frac{d\varpi(s)}{ds} = 0.
\]
(3.5)
By the monotonicity of \( \frac{d\varpi}{ds} \), we see that \( \frac{d\varpi}{ds} = 0 \) and \( \varpi(s) \) is a constant. Then, by Lemma 3.1, it is easily see that \( M^n \) is isometric to the Riemannian product \( S^1(a) \times S^{n-1}(\sqrt{1-a^2}) \). This completes the proof of Proposition 3.2. \( \square \)

On the other hand, if \( \lambda \mu + 1 \leq 0 \), from above, we can obtain \( \frac{d^2 \varpi}{ds^2} \geq 0 \). Combining \( \frac{d^2 \varpi}{ds^2} \geq 0 \) with the boundedness of \( \varpi(s) \), similar to the proof of Proposition 3.2, we know that \( \varpi(s) \) is constant. Then, by Lemma 3.1, it is easily see that \( M^n \) is isometric to the Riemannian product \( S^1(a) \times S^{n-1}(\sqrt{1-a^2}) \). Therefore, we have the following proposition.
Proposition 3.3 Let $M^n$ be an $n$ ($n \geq 3$)-dimensional complete connected hypersurface in $S^{n+1}(1)$ with constant mean curvature $H$ and with two distinct principal curvatures $\lambda$ and $\mu$ with multiplicities $(n-1)$ and 1, respectively. If $\lambda \mu + 1 \leq 0$, then $M^n$ is isometric to the Riemannian product $S^1(a) \times S^{n-1}(\sqrt{1-a^2})$.

Proof of theorem 1.4 Since $M^n$ has two distinct principal curvatures $\lambda$ and $\mu$, if $H = 0$ on $M^n$, from (1.3) we have $S = n$, then $M^n$ is isometric to a Clifford torus $S^1(\sqrt{\frac{1}{a}}) \times S^{n-1}(\sqrt{\frac{n-1}{a}})$. Therefore, we next only consider $H \neq 0$ on $M^n$. Since $M^n$ is oriented and the mean curvature $H$ is constant, we can choose an orientation for $M^n$ such that $H > 0$. From (2.11), we know that (1.3) is equivalent to

$$\frac{n(n-2)}{2(n-1)}[nH^2 - \sqrt{n^2H^4 + 4(n-1)H^2} + 2(n-1)]$$

$$\leq n(n-1)r \leq \frac{n(n-2)}{2(n-1)}[nH^2 + \sqrt{n^2H^4 + 4(n-1)H^2} + 2(n-1)],$$

that is

$$\frac{1}{2(n-1)^2}[n^2H^2 - n\sqrt{n^2H^4 + 4(n-1)H^2} + 2(n-1)]$$

$$\leq \frac{n(r-1) + 2}{n-2} \leq \frac{1}{2(n-1)^2}[n^2H^2 + n\sqrt{n^2H^4 + 4(n-1)H^2} + 2(n-1)],$$

where $n(n-1)r$ is the scalar curvature of $M^n$.

We define the function

$$f(x) = (n-1)^2x^2 - [n^2H^2 + 2(n-1)]x + 1.$$  \hspace{1cm} (3.7)

Since $f(0) = 1$, we know that function (3.7) has two positive real roots

$$x_{1,2} = \frac{1}{2(n-1)^2}[n^2H^2 \pm n\sqrt{n^2H^4 + 4(n-1)H^2} + 2(n-1)].$$  \hspace{1cm} (3.8)

It can be easily checked that $x_1 \leq x_2$ and if $x_1 \leq x \leq x_2$, then $f(x) \leq 0$.

Now we set $x = \frac{n(r-1) + 2}{n-2}$, from (3.6), we have

$$f(\frac{n(r-1) + 2}{n-2}) \leq 0.$$  \hspace{1cm} (3.9)

If there exists a point $p$ on $M^n$ such that (2.14) and (2.15) hold at $p$, that is, we have $H = 0$ or $H^2 = r - 1$ at $p$. If $H = 0$ at $p$, we have a contradiction to $H \neq 0$ on $M^n$. If $H^2 = r - 1$ at $p$, from (2.11) we have $S = nH^2$ at $p$, that is, $p$ is a umbilical point on $M^n$, this is a contradiction to $M^n$ has no umbilical points. Therefore, we only consider two cases:

**Case (1)** If (2.14) holds on $M^n$, next we shall prove that $\lambda \mu + 1 \geq 0$ on $M^n$. We consider three subcases:

(i) If $1 + (n-1)(r-1) - (n-2)H^2 \geq 0$ on $M^n$, then from (2.14), it is obvious that $\lambda \mu + 1 \geq 0$ on $M^n$.  

306
Therefore, we have
\[(n-2)H\sqrt{H^2-(r-1)} < -[1 + (n-1)(r-1) - (n-2)H^2] ,\]
Therefore, we have
\[(n-2)^2H^2[H^2-(r-1)] < [1 + (n-1)(r-1) - (n-2)H^2]^2 ,\]
that is, \( f\left(\frac{n(r-1)+2}{n-2}\right) > 0 . \) This is a contradiction to (3.9); we deduce that \( \lambda \mu + 1 \geq 0 \) on \( M^n . \)

(iii) If \( 1 + (n-1)(r-1) - (n-2)H^2 \geq 0 \) at a point \( p \) of \( M^n \) and \( 1 + (n-1)(r-1) - (n-2)H^2 < 0 \) at other points of \( M^n \), in this case, from (i) and (ii), we have at point \( p \), \( \lambda \mu + 1 \geq 0 \) and at other points of \( M^n \), also \( \lambda \mu + 1 \geq 0 \). Therefore, we obtain \( \lambda \mu + 1 \geq 0 \) on \( M^n \).

Therefore, we know that if (2.15) holds on \( M^n \), then \( \lambda \mu + 1 \geq 0 \) on \( M^n \). By Proposition 3.2, we obtain that \( M \) is isometric to the Riemannian product \( S^1(a) \times S^{n-1}(\sqrt{1-a^2}) \). From Example 2.1, we have
\[ a^2 = \frac{2nH^2+\sqrt{n^2H^4+4(n-1)H^2}}{2nH^2}. \]

**Case (2)** If (2.15) holds on \( M^n \), we consider three subcases:
(i) If \( 1 + (n-1)(r-1) - (n-2)H^2 \leq 0 \) on \( M^n \), then from (2.15), it is obvious that \( \lambda \mu + 1 \leq 0 \) on \( M^n \).
(ii) If \( 1 + (n-1)(r-1) - (n-2)H^2 > 0 \) on \( M^n \), suppose \( \lambda \mu + 1 > 0 \) on \( M^n \), from (2.15), we have
\[ 1 + (n-1)(r-1) - (n-2)H^2 > (n-2)H\sqrt{H^2-(r-1)}. \]
Therefore, we have
\[ (1 + (n-1)(r-1) - (n-2)H^2)^2 > (n-2)^2H^2[H^2-(r-1)], \]
that is, \( f\left(\frac{n(r-1)+2}{n-2}\right) > 0 . \) This is a contradiction to (3.9); we deduce that \( \lambda \mu + 1 \leq 0 \) on \( M^n \).

(iii) If \( 1 + (n-1)(r-1) - (n-2)H^2 \leq 0 \) at a point \( p \) of \( M^n \) and \( 1 + (n-1)(r-1) - (n-2)H^2 > 0 \) at other points of \( M^n \), in this case, from (i) and (ii), we have at point \( p \), \( \lambda \mu + 1 \leq 0 \) and at other points of \( M^n \), also \( \lambda \mu + 1 \leq 0 \). Therefore, we obtain \( \lambda \mu + 1 \leq 0 \) on \( M^n \).

Therefore, we know that if (2.15) holds on \( M^n \), then \( \lambda \mu + 1 \leq 0 \) on \( M^n \). By Proposition 3.3, we obtain that \( M \) is isometric to the Riemannian product \( S^1(a) \times S^{n-1}(\sqrt{1-a^2}) \). From Example 2.1, we have
\[ a^2 = \frac{2nH^2+\sqrt{n^2H^4+4(n-1)H^2}}{2nH^2}. \] This completes the proof of Theorem 1.4.

In order to prove Theorem 1.5, we need the following Proposition 3.4, which can be proved by the same method due to Otsuki [5], also see Cheng [2].

**Proposition 3.4** Let \( M^n \) be an \( n \) \((n \geq 3)\)-dimensional complete oriented hypersurface in \( \mathbb{R}^{n+1} \) with constant mean curvature \( H \) and with two distinct principal curvatures, one of which is simple. Then \( M^n \) is isometric to one of the following hypersurfaces:
(1) \( S^1(a) \times \mathbb{R}^{n-1} \),
(2) a complete non-compact hypersurface of revolution \( S^{n-1}(a(s)) \times M^1 \), where \( S^{n-1}(a(s)) \) is of constant curvature \( \left(\frac{d\log|\lambda-H|}{ds}\right)^2+\lambda^2 \) and \( M^1 \) is a plane curve and \( \omega = |\lambda-H|^\frac{1}{\lambda} \) satisfies the following ordinary
differential equation of order 2

\[
\frac{d^2 \varpi}{ds^2} + \varpi [H^2 + (2 - n)H\varpi^{-n} + (1 - n)\varpi^{-2n}] = 0, \tag{3.10}
\]

for \( \lambda - H > 0 \) or
\[
\frac{d^2 \varpi}{ds^2} + \varpi [H^2 + (n - 2)H\varpi^{-n} + (1 - n)\varpi^{-2n}] = 0, \tag{3.11}
\]

for \( \lambda - H < 0 \).

By a similar method in \([7]\), we can prove the following lemma.

**Lemma 3.2** Equation (3.10) or (3.11) is equivalent to its first order integral

\[
(d\varpi/ ds)^2 + H^2 \varpi^2 + 2H \varpi^{2-n} + \varpi^{2-2n} = C, \tag{3.12}
\]

for \( \lambda - H > 0 \) or
\[
(d\varpi/ ds)^2 + H^2 \varpi^2 - 2H \varpi^{2-n} + \varpi^{2-2n} = C, \tag{3.13}
\]

for \( \lambda - H < 0 \), where \( C \) is a constant. Moreover, the constant solution of (3.10) or (3.11) corresponds to the Riemannian product \( S^{n-1}(a) \times \mathbb{R} \) or \( S^1(a) \times \mathbb{R}^{n-1} \).

By the similar method in the proof of Proposition 3.2 and Proposition 3.3, we can also prove the following:

**Proposition 3.5** Let \( M^n \) be an \( (n \geq 3) \)-dimensional complete connected and oriented hypersurface in \( \mathbb{R}^{n+1} \) with constant mean curvature \( H \) and with two distinct principal curvatures, one of which is simple. If \( \lambda \mu \geq 0 \), then \( M^n \) is isometric to the Riemannian product \( S^{n-1}(a) \times \mathbb{R} \) or \( S^1(a) \times \mathbb{R}^{n-1} \).

**Proposition 3.6** Let \( M^n \) be an \( (n \geq 3) \)-dimensional complete connected and oriented hypersurface in \( \mathbb{R}^{n+1} \) with constant mean curvature \( H \) and with two distinct principal curvatures, one of which is simple. If \( \lambda \mu \leq 0 \), then \( M^n \) is isometric to the Riemannian product \( S^{n-1}(a) \times \mathbb{R} \) or \( S^1(a) \times \mathbb{R}^{n-1} \).

**Proof of theorem 1.5** From (2.11), we know that

\[
S \geq \frac{n^2 H^2}{n - 1} \text{ is equivalent to } n^2 H^2 \geq \frac{n(n - 1)^2 r}{n - 2}. \tag{3.14}
\]

If there exists a point \( p \) on \( M^n \) such that (2.16) and (2.17) hold at \( p \), that is, we have \( H = 0 \) or \( H^2 = r \) at \( p \). If \( H = 0 \) at \( p \), this is a contradiction because of the assumption \( H \neq 0 \). If \( H^2 = r \) at \( p \), from (2.11) we have \( S = nH^2 \) at \( p \), that is, \( p \) is a umbilical point on \( M^n \), this is a contradiction to \( M^n \) has no umbilical points. Therefore, we only consider two cases.

**Case (1)** If (2.16) holds on \( M^n \), next we shall prove that \( \lambda \mu \geq 0 \) on \( M^n \). We consider three subcases:

(i) If \( (n - 1)r - (n - 2)H^2 \geq 0 \) on \( M^n \), then from (2.16), it is obvious that \( \lambda \mu \geq 0 \) on \( M^n \).

(ii) If \( (n - 1)r - (n - 2)H^2 < 0 \) on \( M^n \), suppose \( \lambda \mu < 0 \) on \( M^n \), from (2.16), we have

\[
(n - 2)H\sqrt{H^2 - r} < -[(n - 1)r - (n - 2)H^2].
\]
Therefore, we have
\[(n - 2)^2H^2(H^2 - r) < [(n - 1)r - (n - 2)H^2]^2,\]
that is, \[n^2H^2 < \frac{n(n-1)^2r}{n-2}.\] This is a contradiction to (3.14), we deduce that \(\lambda \mu \geq 0\) on \(M^n\).

(iii) If \((n - 1)r - (n - 2)H^2 \geq 0\) at a point \(p\) of \(M^n\) and \((n - 1)r - (n - 2)H^2 < 0\) at other points of \(M^n\), in this case, from (i) and (ii), we have at point \(p\), \(\lambda \mu \geq 0\) and at other points of \(M^n\), also \(\lambda \mu \geq 0\). Therefore, we obtain \(\lambda \mu \geq 0\) on \(M^n\).

Therefore, we know that if (2.16) holds on \(M^n\), then \(\lambda \mu \geq 0\) on \(M^n\). By Proposition 3.5, we obtain that \(M^n\) is isometric to the Riemannian product \(S^{n-1}(a) \times \mathbb{R}\) or \(S^1(a) \times \mathbb{R}^{n-1}\).

**Case (2)** If (2.17) holds on \(M^n\), we consider three subcases:

(i) If \((n - 1)r - (n - 2)H^2 \leq 0\) on \(M^n\), then from (2.17), it is obvious that \(\lambda \mu \leq 0\) on \(M^n\).

(ii) If \((n - 1)r - (n - 2)H^2 > 0\) on \(M^n\), suppose \(\lambda \mu > 0\) on \(M^n\), from (2.17), we have
\[
(n - 1)r - (n - 2)H^2 > (n - 2)H\sqrt{H^2 - r}.
\]

Therefore, we have
\[
[(n - 1)r - (n - 2)H^2]^2 > (n - 2)^2H^2(H^2 - r),
\]
that is \[n^2H^2 < \frac{n(n-1)^2r}{n-2}.\] This is a contradiction to (3.14), we deduce that \(\lambda \mu \leq 0\) on \(M^n\).

(iii) If \((n - 1)r - (n - 2)H^2 \leq 0\) at a point \(p\) of \(M^n\) and \((n - 1)r - (n - 2)H^2 > 0\) at other points of \(M^n\), in this case, from (i) and (ii), we have at point \(p\), \(\lambda \mu \leq 0\) and at other points of \(M^n\), also \(\lambda \mu \leq 0\). Therefore, we obtain \(\lambda \mu \leq 0\) on \(M^n\).

Therefore, we know that if (2.17) holds on \(M^n\), then \(\lambda \mu \leq 0\) on \(M^n\). By Proposition 3.6, we obtain that \(M^n\) is isometric to the Riemannian product \(S^{n-1}(a) \times \mathbb{R}\) or \(S^1(a) \times \mathbb{R}^{n-1}\). This completes the proof of Theorem 1.5.

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