A fixed point theorem for a compact and connected set in Hilbert space

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Abstract

Let \((H, \langle, \rangle)\) be a real Hilbert space and let \(K\) be a compact and connected subset of \(H\). We show that every continuous mapping \(T : K \to K\) satisfying a mild condition has a fixed point.

Key Words: Fixed point, nonexpansive mapping, Hilbert space a fixed point theorem for a compact and connected set in Hilbert space

1. Introduction

Let \(K\) be a nonempty, close, convex and bounded subset of a real Hilbert space \(H\). Let \(T : K \to K\) be a continuous mapping.

If \(K\) is compact, then by The Schauder Fixed Point Theorem\([8]\) (a generalization of \([1]\)), \(T\) has a fixed point. If \(K = B\) is the closed unit ball of \(H\) and the dimension of \(H\) is finite, then by The Brouwer Fixed Point Theorem\([1]\), \(T\) has a fixed point. In the case where the dimension of \(H\) is infinite this is no longer the case\([4, p. 198\) and \(207]\). In this case, it is necessary to impose some extra conditions to assure the existence of a fixed point of \(T\). The conditions imposed are usually compactness or monotonicity or nonexpansiveness of \(T\)\([6]\). For instance, Browder\([2]\), Browder\([3]\)–Göhde\([5]\) and Kirk\([7]\) discovered in 1965 independently that the nonexpansiveness of \(T\) is a guarantee the existence of a fixed point of \(T\).

In the present paper, we impose an extra condition on \(T\) to obtain the same result. The extra condition imposed is this: For a certain number \(r > 0\), the inclusion \(T(\partial B_r) \subseteq B_r\) holds. Here, \(B_r\) denotes the closed ball, \(B_r = \{x \in H : \|x\| \leq r\}\) and \(\partial B_r\) is the boundary of the ball \(B_r\).

Moreover we impose some new conditions for The Schauder Fixed Point Theorem \([8]\), and for Theorem 1 in \([2]\). In addition, we obtain some results related to the imposed conditions. To explain these conditions let us define a subset \(A_T(x_0)\) of \(K\), for \(x_0 \in K\), as

\[A_T(x_0) = \{x \in K : \|x - x_0\| \leq \|T(x)\| - \|x_0\|\}\]

The main theorem of this paper obtains The Schauder Fixed Point Theorem, which states that every continuous mapping on a compact and convex subset of a Banach space has a fixed point, for compact and

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connected subset $K$ of $H$. To get this we replace the convexity of $K$ in this theorem with the condition $K = A^T(x_0)$ for some $x_0 \neq 0$. To obtain Theorem 1 in [2] for continuous mapping $T$, we replace the nonexpansiveness of $T$ with the following conditions:

(a) The existence of a point $a$ in $K$ satisfying $\|a\| = \|T(a)\|$; and

(b) $((f \circ P)(\partial B_r)) \cap \partial B_r \neq \emptyset,$

where $f$ is the mapping defined by, $f(x) = (x + T(x))/2$, $r = \sup\{||x|| : x \in K\}$ and $P : B_r \rightarrow K$ is the nearest point projection, that is, for $x \in B_r$,

$$||x - P(x)|| = \inf\{||x - u|| : u \in K\}.$$ 

We remark that the mapping $f$ also applies $K$ into itself.

**Proposition 1** Let $K$ be a nonempty and convex subset of $H$. Let $T : K \rightarrow K$ be a continuous mapping. Suppose that there exists a point $a \in K$ such that

$$||a|| = ||T(a)|| = ||f(a)||.$$ 

Then $a$ is a fixed point of $T$.

**Proof.** Let $||a|| = ||T(a)|| = ||f(a)||$. The equality $||a|| = ||f(a)||$ is equivalent to

$$<a, a> = \frac{1}{4} < a + T(a), a + T(a) > .$$

Developing this product, and taking into account the condition $||T(a)|| = ||a||$, we obtain the equality

$$||a||^2 = <a, T(a) > .$$

This equality in turn implies that $||a - T(a)||^2 = <a - T(a), a - T(a) >= 0$ so that $T(a) = a$. □

The following corollary is now obvious. Since the inequality $||a|| \leq ||f(a)||$ in turn implies that $||a|| \leq ||T(a)||$. 

**Corollary 2** Let $K$ be a nonempty and convex subset of $H$ and let $T : K \rightarrow K$ be a continuous mapping. If there exists a point $a \in K$ such that $||T(a)|| \leq ||a|| \leq ||f(a)||$, then $a$ is a fixed point of $T$.

**Proposition 3** Let $K$ be a nonempty and convex subset of $H$ and let $T : K \rightarrow K$ be a continuous mapping. If there exists a point $x_0 \in K$ such that both sets $A^T(x_0)$ and $A^f(x_0)$ are at most countable then $x_0$ is a fixed point of the mapping $T$.

**Proof.** Remark that $A^f(x_0) = \{x \in K : ||x - x_0|| \leq ||f(x)|| - \|x_0\|\}$. Let both $A^T(x_0)$ and $A^f(x_0)$ be at most countable. We shall show that $\|x_0\| = \|T(x_0)\|$ and $\|x_0\| = \|f(x_0)\|$. Let show that $\|x_0\| = \|T(x_0)\|$. On the contrary, assume that $\|x_0\| \neq \|T(x_0)\|$. Let $\varepsilon = (\|x_0\| - \|T(x_0)\|)/2$. Since $T$ is continuous, there exists $\delta > 0$ such that if $x \in K$ and $\|x - x_0\| < \delta$ then $\|T(x)\| - \|T(x_0)\| < \varepsilon$. Let $\delta_0 = \min\{\varepsilon, \delta\}$. We claim that $B_{\delta_0}(x_0) \cap K \subset A^T(x_0)$. Indeed, let $x \in K$ and $\|x - x_0\| < \delta_0$. Then,
\[ \| x - x_0 \| < 2\varepsilon - \varepsilon \leq \| x_0 \| - \| T(x_0) \| - \| T(x) \| - \| T(x_0) \|. \]

From here we conclude that
\[ \| x - x_0 \| \leq \| T(x) \| - \| x_0 \|. \]

That means \( x \in A^T(x_0) \). This implies that \( B_{\delta}(x_0) \cap K \subset A^T(x_0) \). But this is impossible, since \( K \) is convex and \( A^T(x_0) \) is at most countable. To prove that \( \| x_0 \| = \| f(x_0) \| \), it is enough to replace \( T \) with \( f \) and repeat the proof. By Proposition 1, \( x_0 \) is a fixed point of \( T \).

Next is the main theorem of this paper.

**Theorem 4** Let \( K \) be a nonempty, compact and connected subset of \( H \) and let \( T : K \to K \) be a continuous mapping. Assume that there exists a point \( x_0 \in K \), \( x_0 \neq 0 \), such that \( A^T(x_0) = K \). Then \( T \) has a fixed point.

**Proof.** Let \( a \in K \) be fixed. Now we define the sequences \( \alpha_n =\| T^n(a) - x_0 \| \) and \( \beta_n =\| T^n(a) \| - \| x_0 \| \).

Since \( A^T(x_0) = K \) for some \( x_0 \in K \), \( x_0 \neq 0 \), the sequence \( (T^n(a))_{n \in \mathbb{N}} \) is in the set \( A^T(x_0) \). Hence we have
\[ \| T(a) - x_0 \| \leq \| T^2(a) \| - \| x_0 \| \leq \| T^2(a) - x_0 \| , \]
and so on. In this way we get,
\[ \| T^n(a) - x_0 \| \leq \| T^{n+1}(a) \| - \| x_0 \| \leq \| T^{n+1}(a) - x_0 \| , \]
for all \( n = 0, 1, 2, \ldots \) (Here \( T^n = T \circ T \circ T \circ \ldots \circ T \) \( (n - \text{times}) \) and \( a = T^0(a) \)).

Hence, for all \( n \), we get
\[ \alpha_n \leq \beta_{n+1} \leq \alpha_{n+1} . \]

The last inequalities show that \( (\alpha_n)_{n \in \mathbb{N}} \) and \( (\beta_n)_{n \in \mathbb{N}} \) are increasing sequences. Moreover, since \( K \) is bounded, they converge and approach the same limit. On the other hand, since \( K \) is compact, the sequence \( (T^n(a))_{n \in \mathbb{N}} \) has a convergent subsequence. We show this subsequence \( (T^{n_k}(a))_{k \in \mathbb{N}} \). Let \( \lim_{n \to \infty} T^{n_k}(a) = b \in K \). Since \( T \) is continuous, we get
\[ \lim_{n \to \infty} \| T^{n_k+p}(a) \| = \| T^p(b) \| , \]
for all \( p = 0, 1, 2, \ldots \) Now \( \alpha_{n_k+p} =\| T^{n_k+p}(a) - x_0 \| \) and \( \beta_{n_k+p} =\| T^{n_k+p}(a) \| - \| x_0 \| \) are all subsequences of \( (\alpha_n)_{n \in \mathbb{N}} \) and \( (\beta_n)_{n \in \mathbb{N}} \). Since they have the same limit, we get
\[ \| b - x_0 \| = \| b \| - \| x_0 \| = \| T(b) - x_0 \| = \| T(b) \| - \| x_0 \| = \cdots . \]

That is,
\[ \| T^n(b) - x_0 \| = \| T^n(b) \| - \| x_0 \| = \| T^{n+1}(b) - x_0 \|. \]

From here we conclude that, for all \( n \in \mathbb{N} \),
Hence relations (4) from here, we get for all 

\[ x_0 = t_0b = t_1T(b) = t_2T^2(b) = \cdots = t_nT^n(b) = \cdots \]

(2) hold. Now there exist two cases:

**Case 1.** Assume that the equality \( \parallel T^n(b) = T^{n+1}(b) \parallel \) holds for some \( n \). Then the point \( T^n(b) \) is a fixed point of \( T \). Indeed by (2),

\[ \parallel x_0 \parallel = t_n \parallel T^n(b) \parallel = t_{n+1} \parallel T^{n+1}(b) \parallel . \]

Hence \( t_n = t_{n+1} \). Again by (2), we get \( T^n(b) = T^{n+1}(b) = T(T^n(b)) \).

**Case 2.** Suppose that \( \parallel T^n(b) \parallel \neq \parallel T^{n+1}(b) \parallel \) for all \( n \). In this case \( x_0 \) is a fixed point of \( T \). Indeed, by taking square of the equalities in (1), we obtain,

\[ \parallel x_0 \parallel = (\parallel b \parallel + \parallel T(b) \parallel )/2 = (\parallel T(b) \parallel + \parallel T^2(b) \parallel )/2 = \cdots . \]

(3) From here, we get for all \( n \)

\[ \parallel b \parallel = \parallel T^2(b) \parallel = \parallel T^{2n}(b) \parallel \quad \text{and} \quad \parallel T(b) \parallel = \parallel T^3(b) \parallel = \parallel T^{2n+1}(b) \parallel . \]

(4) Together with (2), the equalities in (4) imply that both

\[ \parallel x_0 \parallel = t_0 \parallel b \parallel = t_2 \parallel T^2(b) \parallel = t_{2n} \parallel T^{2n}(b) \parallel \]

(5) and

\[ \parallel x_0 \parallel = t_1 \parallel T(b) \parallel = t_3 \parallel T^3(b) \parallel = t_{2n+1} \parallel T^{2n+1}(b) \parallel . \]

(6) Hence relations (4), (5) and (6) give us for all \( n \)

\[ t_0 = t_2 = t_{2n} \quad \text{and} \quad t_1 = t_3 = t_{2n+1} . \]

By (2), \( b = T^2(b) = T^{2n}(b) \). Let \( g : K \to R \) be a function defined by

\[ g(x) = \parallel b \parallel - \parallel T^2(x) \parallel - \parallel T(b) \parallel - \parallel T^2(x) \parallel . \]

Clearly \( g \) is continuous and \( g(b) < 0 \) and \( 0 < g(T(b)) \) hold. Since \( K \) is connected, by The Intermediate Value Theorem, there exists a point \( c \) in \( K \) such that

\[ g(c) = \parallel b \parallel - \parallel T^2(c) \parallel - \parallel T(b) \parallel - \parallel T^2(c) \parallel = 0. \]

That is, \( \parallel T^2(c) \parallel = (\parallel b \parallel + \parallel T(b) \parallel ) / 2 \). By (3), \( \parallel x_0 \parallel = \parallel T^2(c) \parallel \). On the other hand since \( T(c) \in A^T(x_0) = K \), we get \( \parallel x_0 - T(c) \parallel \leq \parallel x_0 - T^2(c) \parallel = 0 \). Hence \( x = T(c) \). Similarly since \( c \in A^T(x_0) \), we get \( \parallel x_0 - c \parallel \leq \parallel x_0 - T(c) \parallel = 0 \). Hence \( x_0 = c \). From here we conclude that \( x_0 \) is a fixed point of \( T \).

We use the symbol \( \overline{A} \) to denote the closure of a set \( A \).
Corollary 5 Let $K$ and $T$ be as in Theorem 4. If $T$ has no fixed point then

$$A^T(x) \cap \overline{K/A^T(x)} \neq \emptyset$$

for all $x \in K$, $x \neq 0$. That is, for all $x \in K$, there exists a point $y$ in $K$ such that

$$||x - y|| = ||x|| - ||T(y)||.$$

Proof. On the contrary, suppose that $A^T(x) \cap \overline{K/A^T(x)} = \emptyset$ for some $x \in K$. Then the set $A^T(x)$ is both open and closed in $K$. Since $K$ is connected and $x \in A^T(x) \neq \emptyset$ we must have $A^T(x) = K$. By Theorem 4, $T$ has a fixed point, which is not the case. □

Lemma 6 Let $K$ be a nonempty, closed, convex and bounded subset of $H$ and $0 \in K$. Let $\tilde{T} : K \to K$ be a continuous mapping. Assume that there exists at least one point $a \in K$ such that the equality $||T(a)|| = ||a||$ holds. Then,

(a) The set $F = \{x \in K : ||f(x)|| = ||f(x)|| \}$ is nonempty.

(b) The quantity $\delta(f) = \inf\{|||x|| : x \in F\}$ is zero iff $T(0) = 0$.

Proof. (a) If $f(0) = 0$, then there is nothing to prove. If

$$a \in A^f(0) = F \cup \{x \in K : ||x|| < ||f(x)||\}$$

then, $||a|| \leq ||f(a)||$. By Corollary 2, $a$ is a fixed point of both $T$ and $f$. Hence $a \in F \neq \emptyset$. Hence we suppose that $0 < ||f(0)||$ and $a \notin A^f(0)$. Let $g : K \to R$ be a continuous function defined by, $g(x) = ||x|| - ||f(x)||$. Since $g(0) < 0$ and $g(a) > 0$ and since $K$ is connected, by the intermediate value theorem, there is a point $b \in K$ such that $g(b) = 0$. Hence $b \in F \neq \emptyset$.

(b) Since $F \neq \emptyset$, the quantity $\delta(f) = \inf\{|||x|| : x \in F\}$ exists. This quantity is zero iff $T(0) = 0$. Indeed, if $T(0) = 0$ then $0 \in F$ so that $\delta(f) = 0$. Conversely, if $\delta(f) = 0$ then there is a sequence $(x_n)_{n \in N}$ in $F$ such that $||x_n|| \to 0$, as $n \to \infty$. Since $f$ is continuous on $K$ and since $||x_n|| = ||f(x_n)||$, we see that $f(0) = 0$. This implies that $T(0) = 0$, too □

Let $K$ be a nonempty, closed, convex and bounded subset of $H$. In this case the quantity $\sup\{|||x|| : x \in K\} = r$ exists and $K \subset B_r$. Let $P : B_r \to K$ be the nearest point projection, that is, for $x \in B_r$,

$$||x - P(x)|| = \inf\{|||x - u|| : u \in K\}.$$ 

Now we give the next corollary below.

Corollary 7 Let $K$, $T$ and $a$ be as in lemma 6. Suppose that

$$((f \circ P)(\partial B_r)) \cap \partial B_r \neq \emptyset.$$

Then $T$ has a fixed point.

Proof. Let $((f \circ P)(\partial B_r)) \cap \partial B_r \neq \emptyset$. In this case, there is a $x \in \partial B_r$ such that $||(f \circ P)(x)|| = ||x|| = r$. The equality $||(f \circ P)(x)|| = r$ implies that both
Theorem 11 \[ \|P(x)\| \leq ||(f \circ P)(x)|| \text{ and } ||T(P(x))|| \leq ||(f \circ P)(x)||. \]

As \((f \circ P)(x) = (P(x) + T(P(x))) / 2\), we have \(\|P(x)\| = \|T(P(x))\| = ||(f \circ P)(x)||\). By Proposition 1, \(P(x)\) is a fixed point of \(\hat{T}\).

In the previous corollary the nearest point projection can be replaced with the radial retraction, which uniquely defined for a ball in any strictly convex normed space.

Theorem 8 Let \(K = B\) be the closed unit ball of \(H\) and let \(T : K \to K\) be a continuous mapping. Suppose that for each \(r \geq \delta(f)\), the inclusion \(T(\partial B_r) \subseteq B_r\) holds. Then \(T\) has a fixed point in \(B\).

Proof. If we take \(a \in K\) with \(\|a\| = 1\) and repeat the proof of Lemma 6(a), we see that the set \(F \neq \emptyset\). By Lemma 6(b), If \(\delta(f) = 0\) then zero is a fixed point of \(T\) so that there is nothing to prove in this case. Hence we suppose that \(\delta(f) > 0\).

Case 1. \(f(\partial B) \cap \partial B \neq \emptyset\). In this case, there is a point \(x \in \partial B\) such that \(\|f(x)\| = 1 = \|x\|\). As \(f(x) = (x + T(x)) / 2\), the equality \(\|f(x)\| = \|x\|\) implies that \(\|T(x)\| = \|x\|\). Since \(\|x\| = 1\), this is possible only if \(\|T(x)\| = \|x\|\). By Proposition 1, \(T(x) = x\).

Case 2. \(f(\partial B) \cap \partial B = \emptyset\). In this case, for all \(x \in \partial B\), \(\|f(x)\| < 1\) so that \(\|f(x)\| < \|x\|\). We fix a \(y \in \partial B\). Since \(\|f(y)\| < \|y\|\) and \(\|f(0)\| > 0\), as in the proof of Lemma 6(a), the function \(g(x) = \|x\| - \|f(x)\|\) vanishes at some point \(a \in B\). That is, \(\|a\| = \|f(a)\|\). This point \(a\) belongs to \(F\) so that \(\|a\| \geq \delta(f)\). As \(a \in \partial B_r\), where \(r = \|a\|\) and since \(T(\partial B_r) \subseteq B_r\), we have \(\|T(a)\| \leq \|a\|\). On the other hand, since \(\|a\| = \|f(a)\|\), we also have \(\|T(a)\| \geq \|a\|\). Hence \(\|T(a)\| = \|a\|\). By Proposition 1, \(T(a) = a\). Hence \(T\) has a fixed point in \(B\).

The next corollaries are now obvious.

Corollary 9 Suppose that for each \(x \in B\) with \(\|x\| \geq \delta(f)\), we have \(\|T(x)\| \leq \|x\|\). Then \(T\) has a fixed point in \(B\). \(\Box\)

Corollary 10 Let \(K\) be a closed, convex and bounded subset of \(H\) with \(0 \in K\), and let \(T : K \to K\) be a continuous mapping. Suppose that the inclusion \(T(\partial C) \subseteq C\) holds for all convex and closed subsets of \(K\). Then \(T\) has a fixed point in \(K\).

Theorem 11 Let \(H\) be infinite dimensional and let \(K = B\) be the closed unit ball of \(H\). Let \(T : K \to K\) be a continuous mapping. Set

\[ M_i = \{x \in B : x = x_i e_i\}, \]

where \(e_i = (0, 0, ..., y_i, 0, ..), y_i = 1\). Then \(F \cap M_i \neq \emptyset\), for all \(i\).

Proof. Let \(H = \ell_2\). We give the proofs without loss of generality for \(M = M_1\).

\(a\) If \(f(0) = 0\) then \(0 \in F \cap M\). If \(a = (x_1, y_1, y_2, ...) \in F\) where \(|x_1| = 1\) then \(a \in F \cap M\). So we suppose that \(\{0, e_1, -e_1\} \cap F = \emptyset\). Now define the function \(g : M \to R, g(x) = \|x\| - \|f(x)\|\). Then since \(g(0) < 0\) and \(g(e_1) > 0\) and since \(M\) is convex, by the intermediate value theorem, there is a point \(b \in M\) such that
\[ g(b) = 0. \] That is \( b \in F \cap M \neq \emptyset. \]

For the next corollary we put \( \sup \{ \| x \| : x \in F \cap M_i \} = \| x_i \| = r_i \) for some \( x_i \in F \cap M_i \). If \( r_i = 1 \) for some \( i \), then \( \| x_i \| = \| f(x_i) \| = 1 \geq \| T(x_i) \| \). By Proposition 1, \( x_i \) is a fixed point of \( T \).

**Corollary 12** Let \( B \) be the closed unit ball of \( H \) and \( T : B \to B \) be a continuous mapping. If the inclusion \( T(\partial B_{r_i}) \subseteq B_{r_i} \), for some \( i \), then \( T \) has a fixed point.

**Proof.** We remark that \( r_i = \| x_i \| \) for some \( x_i \in F \cap M_i \), for all \( i \). If \( r_i = 1 \) for some \( i \), then

\[ \| x_i \| = \| f(x_i) \| = 1 \geq \| T(x_i) \| . \]

By Proposition 1, \( x_i \) is a fixed point of \( T \). Let \( 0 < r_i < 1 \) for all \( i \). Then since \( T(\partial B_{r_i}) \subseteq B_{r_i} \) and \( x_i \in \partial B_{r_i} \),

\[ \| T(x_i) \| \leq r_i = \| x_i \| = \| f(x_i) \| . \]

By Proposition 1, \( x_i \) is a fixed point of \( T \). \( \square \)

**Example 13** Let \( H = \ell_2 \) and \( B \) its closed unit ball.

1- For \( x = (x_1, x_2, ...) \in B \), let \( T(x) = (1 - \| x \|, \| x \|, x_3, x_4, ...) \). Then clearly \( T \) takes \( B \) into itself.

By a simple calculation, we have \( r_1 = 1 \sqrt{3}, r_2 = 1 \) and \( r_i = \sqrt{2} - 1 \), for all \( i = 3, 4, ... \). It is clear that

\[ T(\partial B_{r_2}) \subseteq B_{r_2} \]. By Corollary 12, \( T \) has a fixed point. Moreover, \( T \) is not a nonexpansive mapping.

2- For \( x = (x_1, x_2, ...) \in B \), let \( T(x) = (1 - \| x \|, x_2, x_3, x_4, ...) \). Then we have, \( r_1 = 1 / 2 \) and \( r_i = 1 \) for all \( i \geq 2 \). It is clear that

\[ T(\partial B_{r_1}) \subseteq B_{r_1} \]. By Corollary 12, \( T \) has a fixed point.

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