Order continuous operators on $CD_0(K, E)$ and $CD_w(K, E)$-spaces

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Abstract

In [2], Alpay and Ercan characterized order continuous duals of spaces $CD_0(K, E)$ and $CD_w(K, E)$ where $K$ is a compact Hausdorff space without isolated points and $E$ is a Banach lattice. In this note, we generalize their results to an arbitrary Dedekind complete Banach lattice $F$, that is to say, we characterize order continuous operators on these spaces taking values in an arbitrary Dedekind complete Banach lattice $F$.

Key Words: $CD_0(K)$-spaces, order continuous operators, isometric lattice isomorphism.

1. Introduction

Recall that a topological space is called basically disconnected if the closure of any $F_\sigma$-open set is open. A compact Hausdorff space that is basically disconnected will be referred to as a quasi-Stonean space. For a quasi-Stonean space $K$ without isolated points, the following function spaces were introduced by Abramovich and Wickstead [1]:

\[
\begin{align*}
\mathcal{l}_w^\infty(K) &= \{ f : f \text{ is real valued, bounded and the set} \\
&\quad \{ k : f(k) \neq 0 \} \text{ is countable} \};
\end{align*}
\]

\[
\begin{align*}
c_0(K) &= \{ f : f \text{ is real valued and the set} \\
&\quad \{ k : |f(k)| > \varepsilon \} \text{ is finite for each } \varepsilon > 0 \}.
\end{align*}
\]

These spaces were used to define $CD_0(K) = C(K) \oplus c_0(K)$ and $CD_w(K) = C(K) \oplus \mathcal{l}_w^\infty(K)$. Both spaces $CD_0(K)$ and $CD_w(K)$ are $AM$-spaces with strong order unit $1$ under the pointwise order and supremum norm. Properties such as Cantor property, Dedekind completeness, sequential order continuity of the norm in these spaces were studied in [1]. Further, Alpay and Ercan [2] relaxed the condition on the quasi-Stonean space $K$ and took it to be a compact Hausdorff space without isolated points and they defined the following vector-valued versions of $\mathcal{l}_w^\infty(K)$ and $c_0(K)$.

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Definition 1 For a set $K$ and a normed space $E$, let $C_0(K, E)$ be the space of all $E$-valued functions $f$ on $K$ such that for each $\varepsilon > 0$, the set $\{ s \in K : ||f(s)|| > \varepsilon \}$ is finite. Similarly, let $l_w(K, E)$ be the space of all bounded $E$-valued functions on $K$ with countable support.

The following vector-valued versions of the spaces $CD_0(K)$ and $CD_w(K)$ were given in [2].

Definition 2 Let $K$ be a compact Hausdorff space without isolated points and $E$ be a normed space. $CD_0(K, E)$ denotes the set of all $E$-valued functions on $K$ of the form $f + d$ such that $f \in C(K, E)$ and $d \in C_0(K, E)$. Similarly, $CD_w(K, E)$ denotes the set of all $E$-valued functions on $K$ of the form $f + d$ such that $f \in C(K, E)$ but $d \in l_w(K, E)$.

As order continuous operators as well as order continuous duals are very much in use here, it is useful to give their definitions. For more details on order continuous operators, see [3].

Definition 3 (1) A net $\{x_\alpha\}$ in a Riesz space is said to be decreasing to zero (in symbols $x_\alpha \downarrow 0$) whenever $\alpha \geq \beta$ implies $x_\alpha \leq x_\beta$ and $\inf\{x_\alpha\} = 0$ holds.

(2) A net $\{y_\alpha\}$ in a Riesz space is said to be order convergent to $x$, denoted by $x_\alpha \rightarrow^o x$ whenever there exists a net $\{y_\alpha\}$ with the same indexed set satisfying $|x_\alpha - x| \leq y_\alpha \downarrow 0$.

(3) A linear operator $T : E \rightarrow F$ between two Riesz spaces is said to be order continuous whenever $x_\alpha \rightarrow^o 0$ in $E$ implies $Tx_\alpha \rightarrow^o 0$ in $F$. The collection of all order continuous operators will be denoted by $L_n(E, F)$. It is useful to note that a positive operator $T : E \rightarrow F$ is order continuous if and only if $x_\alpha \downarrow 0$ in $E$ implies $Tx_\alpha \downarrow 0$ in $F$. The vector space $L_n(E, \mathbb{R})$ of all order continuous linear functionals is referred to as the order continuous dual of $E$ and denoted by $E_n^\sim$.

Alpay and Ercan [2] proved that the spaces $CD_0(K, E)$ and $CD_w(K, E)$ are Banach lattices for a Banach lattice $E$. They investigated order properties of these spaces and characterized their order continuous duals.

The following definitions and theorems were given in [2].

Definition 4 Let $K$ be a compact Hausdorff space without isolated points and $E$ be a Banach lattice. Then $D_0(K, E_n^\sim)$ denotes the set of all mappings $\beta = \beta(k)$ from $K$ into $F_n^\sim$ satisfying

$$\sup_{||f|| \leq 1} \sum_k |\beta(k)||f(k)|| < \infty$$

for each $f \in CD_0(K, E)$ and $\sum_k |\beta(k)||f_\alpha(k)|| \downarrow 0$ whenever $f_\alpha \downarrow 0$.

As usual, $\sum_k |\beta(k)||f(k)||$ is the supremum of the sums

$$\sum_S |\beta(k)||f(k)||,$$

where $S \subset K$ and is finite. $D_0(K, E_n^\sim)$ is a normed Riesz space under pointwise operations and supremum norm.

Theorem 5 Let $K$ and $E$ be as above. Then $CD_0(K, E)_n^\sim$ and $D_0(K, E_n^\sim)$ are isometrically lattice isomorphic spaces.
Definition 6 Let $K$ be a compact Hausdorff space without isolated points and $E$ be a Banach lattice. Then $D_w(K, E_n^-)$ denotes the set of all mappings $\beta = \beta(k)$ from $K$ into $E_n^-$ satisfying
\[
\sup_{\|f\|\leq 1} \sum_k |\beta(k)||f(k)|| < \infty
\]
for each $f \in CD_w(K, E)$ and $\sum_k |\beta(k)||f_\alpha(k)|| \downarrow 0$ whenever $f_\alpha \downarrow 0$. As usual, $\sum_k |\beta(k)||f(k)||$ is the supremum of the sums
\[
\sum_S |\beta(k)||f(k)||,
\]
where $S \subset K$ and is finite. $D_w(K, E_n^-)$ is a normed Riesz space under pointwise operations and supremum norm.

Theorem 7 Let $K$ and $E$ be as above. Then $CD_w(K, E)_n^-$ and $D_w(K, E_n^-)$ are isometrically lattice isomorphic spaces.

2. Main results

Throughout this section, the symbol $\chi_k \otimes f$ denotes the vector-valued function which takes the value $f(k)$ at $k$ and 0 otherwise.

We start with the following definition which is not very commonly known.

Definition 8 Let $E$ and $F$ be two Banach lattices. The regular norm, denoted by $|| \cdot ||_r$ of a linear operator $T : E \rightarrow F$ with modulus $|T|$ is defined by
\[
||T||_r := ||T|| := \sup_{\|x\| \leq 1} ||T(x)||
\]
It is useful to note that $L_n(E, F)$ under the norm $|| \cdot ||_r$ is a Dedekind complete Banach lattice whenever $F$ is Dedekind complete.

In this section, we give a generalization of Theorem 5 and Theorem 7 in two directions. Firstly, we replace $CD_0(K, E)_n^-$ (or $CD_w(K, E)_n^-$) by $L_n(CD_0(K, E), F)$ (or $L_n(CD_w(K, E), F)$) where $E$ and $F$ are Banach lattices with $F$ Dedekind complete. We take $F$ as a Dedekind complete Banach lattice to ensure that $L_n(CD_0(K, E), F)$ (or $L_n(CD_w(K, E), F)$) is a Dedekind complete Banach lattice under the regular norm $|| \cdot ||_r$. Secondly, we replace $E_n^-$ by $L_n(E, F)$. We now give the following definition which is similar to Definition 4.

Definition 9 Let $K$ be a compact Hausdorff space without isolated points, $E$ and $F$ be two Banach lattices with $F$ Dedekind complete. We define $l^1(K, L_n(E, F))$ as the set of all mappings $\varphi = \varphi(k)$ from $K$ into $L_n(E, F)$ satisfying
\[
\sum_k |\varphi(k)||f(k)|| \in F
\]
for each $f \in CD_0(K, E)$ and $\sum_k |\varphi(k)||f_\alpha(k)|| \downarrow 0$ whenever $f_\alpha \downarrow 0$ in $CD_0(K, E)$.\]
As usual, \( \sum_k |\varphi(k)||f(k)| \) is the supremum of the sums
\[
\sum_S |\varphi(k)||f(k)|
\]
where \( S \subset K \) and is finite.
\( l^1(K, L_n(E, F)) \) is a Banach lattice under pointwise operations and supremum norm.

We now give the following theorem which is the main result of this note.

**Theorem 10** Let \( K \) be a compact Hausdorff space without isolated points, \( E \) and \( F \) be two Banach lattices with \( F \) Dedekind complete. Then \( L_n(CD_0(K, E), F) \) is isometrically lattice isomorphic to \( l^1(K, L_n(E, F)) \).

**Proof.** Let us define a map
\[
\phi : L_n(CD_0(K, E), F) \to l^1(K, L_n(E, F))
\]
at \( e \in E \) by the formula
\[
\phi(G)(k)(e) = G(\chi_k \otimes e)
\]
for each \( G \in L_n(CD_0(K, E), F) \) and \( k \in K \). It is clear that \( \phi \) is a linear map. Using the linearity of \( \phi \) and the fact that \( \phi(G^+)(k) \) and \( \phi(G^-)(k) \) are order bounded \( F \)-valued operators for each \( G \) on \( CD_0(K, E) \), \( \phi(G)(k) \) is order bounded.

Moreover, if \( e_\alpha \downarrow 0 \) in \( E \), then \( \chi_k \otimes e_\alpha \downarrow 0 \) in \( CD_0(K, E) \) for each \( k \in K \). Using the order continuity of \( G \), we have that \( G(\chi_k \otimes e) \) is order convergent to 0 so that \( \phi(G)(k) \in L_n(E, F) \) for each \( G \in L_n(CD_0(K, E), F) \).

We thus have a map \( \phi(G) \) from \( K \) into \( L_n(E, F) \).

Now we will show that
\[
\sum_{k \in K} |\phi(G)(k)||f(k)| \in F, \ (f \in CD_0(K, E)).
\]
Let \( S \) be a finite subset of \( K \) and \( G \in L_n(CD_0(K, E), F) \). Then
\[
\sum_{k \in S} |\phi(G)(k)||f(k)| = \sum_{k \in S} |\phi(G^+ - G^-)(k)||f(k)|
\]
\[
\leq \sum_{k \in S} \phi(G^+)(k)||f(k)|| + \sum_{k \in S} \phi(G^-)(k)||f(k)||
\]
\[
= \sum_{k \in S} G^+(\chi_k \otimes |f|) + \sum_{k \in S} G^-(\chi_k \otimes |f|)
\]
\[
= G^+(\sum_{k \in S} \chi_k \otimes |f|) + G^-(\sum_{k \in S} \chi_k \otimes |f|)
\]
for each \( f \in CD_0(K, E) \). As \( \sum_{k \in S} \chi_k \otimes |f| \uparrow |f| \), \( G^+ \) and \( G^- \) are order continuous, we obtain
\[
\sum_{k \in S} |\phi(G)(k)||f(k)| \leq G^+(|f|) + G^-(|f|) = |G||f|.
\]
Hence
\[
\sum_{k \in K} |\phi(G)(k)||f(k)|| \in F,
\]
since \(F\) is Dedekind complete. We also have to show that
\[
\sum_{k} |\phi(G)(k)||f_{\alpha}(k)|| \downarrow 0 \text{ in } F
\]
for each \(f_{\alpha} \in CD_{0}(K, E)\) such that \(f_{\alpha} \downarrow 0\). It is enough to show this for positive elements in \(L_{n}(CD_{0}(K, E), F)\). Let us take \(0 \leq G \in L_{n}(CD_{0}(K, E), F)\) and \(f_{\alpha} \downarrow 0\) in \(CD_{0}(K, E)\). For a fixed \(\alpha\), we have \(\sum_{k \in S} \chi_{k} \otimes f_{\alpha} \uparrow 0\) \(f_{\alpha}\). As \(G\) is order continuous and positive,
\[
G \left(\sum_{k \in S} \chi_{k} \otimes f_{\alpha}\right) = \sum_{k \in S} G(\chi_{k} \otimes f_{\alpha}) \uparrow G(f_{\alpha}),
\]
so that
\[
\sum_{k \in K} |\phi(G)(k)||f_{\alpha}(k)|| = \sum_{k \in K} \phi(G)(k)(f_{\alpha}(k)) = \sum_{k \in K} G(\chi_{k} \otimes f_{\alpha}) = G(f_{\alpha}) \downarrow 0.
\]

Hence the map \(\phi(G)\) is an element of \(l^{1}(K, L_{n}(E, F))\).

We now show that \(\phi\) is bipositive. It is easy to show that \(\phi(G) \geq 0\) whenever \(G \geq 0\). Conversely, assume that \(\phi(G) \geq 0\) for some \(G \in L_{n}(CD_{0}(K, E), F)\) and take \(0 \leq f \in CD_{0}(K, E)\). We have \(\sum_{k \in S} G(\chi_{k} \otimes f) \rightarrow G(f)\), since \(\sum_{k \in S} \chi_{k} \otimes f \uparrow 0\) \(f\) in \(CD_{0}(K, E)\). As \(G(\chi_{k} \otimes f) = \phi(G)(k)(f) \geq 0\) and thus \(G(f) \geq 0\) for each \(0 \leq f \in CD_{0}(K, E)\), i.e., \(G \geq 0\).

To show that \(\phi\) is one-to-one, let \(\phi(G) = 0\) for some \(G \in L_{n}(CD_{0}(K, E), F)\). Then \(G(\chi_{k} \otimes f) = 0\) for each \(k \in K\) and \(0 \leq f \in CD_{0}(K, E)\). As \(G\) is order continuous and \(\sum_{k \in S} \chi_{k} \otimes f \uparrow 0\), this gives that \(0 = \sum_{k \in S} G(\chi_{k} \otimes f) \rightarrow G(f)\) or \(G(f) = 0\). As \(CD_{0}(K, E)\) is a vector lattice, we get \(G = 0\).

To show that \(\phi\) is surjective, let us take an arbitrary \(0 \leq \alpha \in l^{1}(K, L_{n}(E, F))\) and define \(G : CD_{0}(K, E)_{+} \rightarrow F_{+}\) by \(G(f) = \sum_{k \in K} \alpha(k)(f(k))\). As \(G\) is additive on \(CD_{0}(K, E)\) and so \(G(f) = G(f^{+}) - G(f^{-})\) extends \(G\) to \(CD_{0}(K, E)\). We now verify that \(\phi(G) = \alpha\). If \(0 \leq e \in E\), then
\[
\phi(G)(k_{0})(e) = G(\chi_{k_{0}} \otimes e) = \sum_{k \in K} \alpha(k)(\chi_{k_{0}} \otimes e)(k) = \alpha(k_{0})e.
\]
Since \(e \in E\) is arbitrary, we conclude that \(\phi(G)(k_{0}) = \alpha(k_{0})\) and \(k_{0}\) is arbitrary, we have \(\phi(G) = \alpha\).
Finally we show that $\phi$ is an isometry. Assume that $G \in L_n(CD_0(K, E), F)$ and $f \in CD_0(K, E)$. Then

$$||G||_r = \sup_{||f|| \leq 1} ||G|f|| = \sup_{||f|| \leq 1} ||G(|f|)||$$

$$= \sup_{||f|| \leq 1} ||G\left(\sum_{k \in K} \chi_k \otimes |f|\right)||$$

$$= \sup_{||f|| \leq 1} \sum_{k \in K} ||G(\chi_k \otimes |f|)||$$

$$= ||\phi(|G|)|| = ||\phi(G)||_r.$$

This completes the proof. \qed

**Definition 11** Let $K$ be a compact Hausdorff space without isolated points, $E$ and $F$ be two Banach lattices with $F$ Dedekind complete. Then we define $l^1_w(K, L_n(E, F))$ as the set of all maps $\varphi = \varphi(k)$ from $K$ into $L_n(E, F)$ satisfying

$$\sum_k |\varphi(k)||(f(k))| \in F$$

for each $f \in CD_w(K, E)$ and $\sum_k |\varphi(k)||(f_\alpha(k))| \downarrow 0$ whenever $f_\alpha \downarrow 0$ in $CD_w(K, E)$.

$l^1_w(K, L_n(E, F))$ is a Banach lattice under pointwise operations and supremum norm. The following theorem is similar to Theorem 10 so we omit its proof.

**Theorem 12** Let $K$ be a compact Hausdorff space without isolated points, $E$ and $F$ be two Banach lattices with $F$ Dedekind complete. Then $L_n(CD_w(K, E), F)$ is isometrically lattice isomorphic to $l^1_w(K, L_n(E, F))$.

**References**


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