Turk J Math
34 (2010) , 73 – 83.
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doi:10.3906/mat-0802-21

Korovkin type approximation theorem for functions of two variables in statistical sense

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Abstract

In this paper, using the concept of A-statistical convergence for double sequences, we investigate a Korovkin-type approximation theorem for sequences of positive linear operator on the space of all continuous real valued functions defined on any compact subset of the real two-dimensional space. Then we display an application which shows that our new result is stronger than its classical version. We also obtain a Voronovskaya-type theorem and some differential properties for sequences of positive linear operators constructed by means of the Bernstein polynomials of two variables.

Key Words: A-Statistical convergence of double sequence, Korovkin-type approximation theorem, Bernstein polynomials, Voronovskaya-type theorem.

1. Introduction

Let $\{L_n\}$ be a sequence of positive linear operators acting from $C(X)$ into $C(X)$, which is the space of all continuous real valued functions on a compact subset X of all the real numbers. In this case, Korovkin [9] first noticed necessary and sufficient conditions for the uniform convergence of $L_n(f)$ to a function f by using the test functions e_i defined by $e_i(x) = x^i$ $(i = 0, 1, 2)$. Later many researchers investigate these conditions for various operators defined on different spaces. Furthermore, in recent years, with the help of the concept of statistical convergence, various statistical approximation results have been proved ([1], [2], [3], [4], [5], [8]). Recall that every convergent sequence (in the usual sense) is statistically convergent but its converse is not always true. Also, statistical convergent sequences do need to be bounded. So, the usage of this method of convergence in the approximation theory provides us many advantages. Our primary interest in the present paper is to obtain a Korovkin-type approximation theorem for a sequences of positive linear operators defined on $C(D)$, which is the space of all continuous real valued functions on any compact subset of the real two-dimensional space. Also, we construct an example such that our approximation result works but its Pringsheim sense does not work. Finally, we obtain a Voronovskaya-type theorem and some differential properties in the statistical case for sequence of positive linear operators constructed by means of the Bernstein polynomials of two variables.

We first recall the concept of A-statistical convergence for double sequences.

2000 *AMS Mathematics Subject Classification:* Primary 41A25, 41A36, Secondary 40A05.

A double sequence $x = (x_{m,n})$ is said to be convergent in the Pringsheim's sense if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$, the set of all natural numbers, such that $|x_{m,n} - L| < \varepsilon$ whenever $m, n > N$. L is called the Pringsheim limit of x and denoted by $P - \lim x = L$ (see [12]). We shall denote such an x more briefly as "P-convergent". By a bounded double sequence we mean there exists a positive number K such that $|x_{m,n}| < K$ for all $(m,n) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$, the two-dimensional set of all positive integers. For bounded double sequences, we use the notation

$$||x||_{(\infty,2)} = \sup_{m,n} |x_{m,n}| < \infty.$$

Note that in contrast to the case for single sequences, a convergent double sequence is not necessarily bounded. Let $A = (a_{j,k,m,n})$ be a four-dimensional summability method. For a given double sequence $x = (x_{m,n})$, the A-transform of x, denoted by $Ax := ((Ax)_{j,k})$, is given by

$$(Ax)_{j,k} = \sum_{m,n=1,1}^{\infty,\infty} a_{j,k,m,n} x_{m,n}$$

provided the double series converges in the Pringsheim's sense for $(m,n) \in \mathbb{N}^2$.

A two dimensional matrix transformation is said to be regular if it maps every convergent sequence in to a convergent sequence with the same limit. The well-known characterization for two dimensional matrix transformations is known as Silverman-Toeplitz conditions ([7]). In 1926 Robinson [13] presented a four dimensional analog of regularity for double sequences in which he added an additional assumption of boundedness. This assumption was made because a double sequence, which is P-convergent, is not necessarily bounded. The definition and the characterization of regularity for four dimensional matrices is known as Robison-Hamilton conditions, or briefly, RH-regularity ([6], [13]).

Recall that a four dimensional matrix $A = (a_{j,k,m,n})$ is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit. The Robinson- Hamilton conditions state that a four dimensional matrix $A = (a_{j,k,m,n})$ is RH-regular if and only if

(i) $P - \lim_{j,k} a_{j,k,m,n} = 0$ for each k and l;

(ii) $P - \lim_{j,k} \sum_{m,n=1,1}^{\infty,\infty} a_{j,k,m,n} = 1$;

(iii) $P - \lim_{j,k} \sum_{m=1}^{\infty} |a_{j,k,m,n}| = 0$ for each $n \in \mathbb{N}$;

(iv) $P - \lim_{j,k} \sum_{n=1}^{\infty} |a_{j,k,m,n}| = 0$ for each $m \in \mathbb{N}$;

(v) $\sum_{m,n=1,1}^{\infty,\infty} |a_{j,k,m,n}|$ is P-convergent; and

(vi) There exits finite positive integers A and B such that $\sum_{m,n>B} |a_{j,k,m,n}| < A$ holds for every $(j,k) \in \mathbb{N}^2$.

Now let $A = (a_{j,k,m,n})$ be a nonnegative RH-regular summability matrix, and let $K \subset \mathbb{N}^2$. Then

A-density of K is given by

$$\delta^2_{(A)}(K) := P - \lim_{j,k} \sum_{(m,n)\in K} a_{j,k,m,n},$$

provided that the limit on the right-hand side exists in the Pringsheim sense. A real double sequence $x = (x_{m,n})$ is said to be A-statistically convergent to L if, for every $\varepsilon > 0$,

$$\delta^2_{(A)}(\{(m,n) \in \mathbb{N}^2 : |x_{m,n} - L| \geq \varepsilon\}) = 0.$$

In this case, we write $st^2_{(A)} - \lim x = L$. Clearly, a P-convergent double sequence is A-statistically convergent to the same value but its converse it is not always true. Also, note that an A-statistically convergent double sequence need not be bounded. For example, consider the double sequence $x = (x_{m,n})$ given by

$$x_{m,n} = \begin{cases} mn, & \text{if } m \text{ and } n \text{ are squares,} \\ 1, & \text{otherwise.} \end{cases}$$

We should note that if we take $A = C(1,1)$,which is double Cesáro matrix, then $C(1,1)$-statistical convergence coincides with the notion of statistical convergence for double sequence, which was introduced in ([10], [11]).

2. A Korovkin-type approximation theorem

By $C(D)$, we denote the space of all continuous real valued functions on any compact subset of the real two-dimensional space. This space is equipped with the supremum norm

$$\|f\|_{C(D)} = \sup_{(x,y)\in D} |f(x,y)|, \ (f \in C(D)).$$

Let L be a linear operator from $C(D)$ into $C(D)$. Then, as usual, we say that L is positive linear operator provided that $f \geq 0$ implies $Lf \geq 0$. Also, we denote the value of Lf a point $(x,y) \in D$ by $L(f;x,y)$.

Now we have the following main result.

Theorem 2.1 *Let $A = (a_{j,k,m,n})$ be a nonnegative RH-regular summability matrix method. Let $\{L_{m,n}\}$ be a double sequence of positive linear operators acting from $C(D)$ into itself. Then, for all $f \in C(D)$,*

$$st^2_{(A)} - \lim_{m,n} \|L_{m,n}f - f\|_{C(D)} = 0 \tag{1}$$

if and only if

$$st^2_{(A)} - \lim_{m,n} \|L_{m,n}f_i - f_i\|_{C(D)} = 0, \ (i = 0,1,2,3), \tag{2}$$

where $f_0(x,y) = 1$, $f_1(x,y) = x$, $f_2(x,y) = y$ and $f_3(x,y) = x^2 + y^2$.

Proof. Since each $f_i \in C(D)$, $(i = 0,1,2,3)$, the implication (1)$\Rightarrow$(2) is obvious. Suppose now that (2) holds. By the continuity of f on compact set D, we can write

$$|f(x,y)| \leq M,$$

where $M := \|f\|_{C(D)}$. Also, since f is continuous on D, we write that for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|f(u,v) - f(x,y)| < \varepsilon$ for all $(u,v) \in D$ satisfying $|u - x| < \delta$ and $|v - y| < \delta$. Hence, we get

$$|f(u,v) - f(x,y)| < \varepsilon + \frac{2M}{\delta^2}\left\{(u-x)^2 + (v-y)^2\right\}. \tag{3}$$

Since $L_{m,n}$ is linear and positive, we obtain

$$\begin{aligned}
|L_{m,n}(f;x,y) - f(x,y)| &= |L_{m,n}(f(u,v) - f(x,y);x,y) \\
&\quad - f(x,y)(L_{m,n}(f_0;x,y) - f_0(x,y))| \\
&\leq L_{m,n}(|f(u,v) - f(x,y)|;x,y) \\
&\quad + M|L_{m,n}(f_0;x,y) - f_0(x,y)| \\
&\leq \left|L_{m,n}\left(\varepsilon + \frac{2M}{\delta^2}\left\{(u-x)^2 + (v-y)^2\right\};x,y\right)\right| \\
&\quad + M|L_{m,n}(f_0;x,y) - f_0(x,y)| \\
&\leq \left(\varepsilon + M + \frac{2M}{\delta^2}\left(E^2 + F^2\right)\right)|L_{m,n}(f_0;x,y) - f_0(x,y)| \\
&\quad + \frac{4M}{\delta^2}E\,|L_{m,n}(f_1;x,y) - f_1(x,y)| \\
&\quad + \frac{4M}{\delta^2}F\,|L_{m,n}(f_2;x,y) - f_2(x,y)| \\
&\quad + \frac{2M}{\delta^2}|L_{m,n}(f_3;x,y) - f_3(x,y)| + \varepsilon,
\end{aligned}$$

where $E := \max|x|$, $F := \max|y|$. Taking supremum over $(x,y) \in D$ we get

$$\begin{aligned}
\|L_{m,n}f - f\|_{C(D)} \leq\ & K\Big\{\|L_{m,n}f_0 - f_0\|_{C(D)} + \|L_{m,n}f_1 - f_1\|_{C(D)} \\
& + \|L_{m,n}f_2 - f_2\|_{C(D)} + \|L_{m,n}f_3 - f_3\|_{C(D)}\Big\} + \varepsilon
\end{aligned} \tag{4}$$

where $K = \max\left\{\varepsilon + M + \frac{2M}{\delta^2}\left(E^2 + F^2\right), \frac{4M}{\delta^2}E, \frac{4M}{\delta^2}F, \frac{2M}{\delta^2}\right\}$.

Now, for a given $\varepsilon' > 0$, choose $\varepsilon > 0$ such that $\varepsilon < \varepsilon'$. Then, setting

$$\begin{aligned}
D &:= \left\{(m,n) \in \mathbb{N}^2 : \|L_{m,n}f - f\|_{C(D)} \geq \varepsilon'\right\}, \\
D_i &:= \left\{(m,n) \in \mathbb{N}^2 : \|L_{m,n}f_i - f_i\|_{C(D)} \geq \frac{\varepsilon' - \varepsilon}{4K}\right\}, \ i = 0,1,2,3,
\end{aligned}$$

it is easy to see that

$$D \subseteq \bigcup_{i=0}^{3} D_i$$

which gives, for all $(j,k) \in \mathbb{N}^2$,

$$\sum_{(m,n)\in D} a_{j,k,m,n} \le \sum_{i=0}^{3} \sum_{(m,n)\in D_i} a_{j,k,m,n}.$$

Letting $j,k \to \infty$ (in any manner) and using (2), we obtain (1). The proof is complete. $\square$

Remark 1 If we replace the matrix A in Theorem 2.1 by identity double matrix, then we immediately get the following classical result which was first introduced by Volkov [15].

Corollary 2.2 *[15]Let $\{L_{m,n}\}$ be a sequence of positive linear operators acting from $C(D)$ into itself. Then, for all $f \in C(D)$,*

$$P - \lim_{m,n} \|L_{m,n}f - f\|_{C(D)} = 0$$

if and only if

$$P - \lim_{m,n} \|L_{m,n}f_i - f_i\|_{C(D)} = 0, \ (i = 0,1,2,3),$$

where $f_0(x,y) = 1$, $f_1(x,y) = x$, $f_2(x,y) = y$ and $f_3(x,y) = x^2 + y^2$.

Remark 2 We now exhibit an example of a sequence of positive linear operators of two variables satisfying the conditions of Theorem 2.1 but that does not satisfy the conditions of the Korovkin theorem. Now we consider the following Bernstein operators (see [14]) given by

$$B_{m,n}(f;x,y) = \sum_{k=0}^{m}\sum_{j=0}^{n} f\left(\frac{k}{m},\frac{j}{n}\right) C_m^k x^k (1-x)^{m-k} C_n^j y^j (1-y)^{n-j}, \tag{5}$$

where $(x,y) \in D = [0,1] \times [0,1]$; $f \in C(D)$. Also, observe that

$$\begin{aligned}
B_{m,n}(f_0;x,y) &= f_0(x,y), \\
B_{m,n}(f_1;x,y) &= f_1(x,y), \\
B_{m,n}(f_2;x,y) &= f_2(x,y), \\
B_{m,n}(f_3;x,y) &= f_3(x,y) + \frac{x-x^2}{m} + \frac{y-y^2}{n},
\end{aligned}$$

where $f_0(x,y) = 1$, $f_1(x,y) = x$, $f_2(x,y) = y$ and $f_3(x,y) = x^2 + y^2$. Then, by Corollary 2.2, we know that, for any $f \in C(D)$,

$$P - \lim_{m,n} \|B_{m,n}f - f\|_{C(D)} = 0. \tag{6}$$

Now we take $A = C(1,1)$ and define a double sequence $(\gamma_{m,n})$ by

$$\gamma_{m,n} = \begin{cases} 1, & \text{if } m \text{ and } n \text{ are squares,} \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

It is clear that

$$st^2_{C(1,1)} - \lim \gamma_{m,n} = 0. \tag{8}$$

Now using (5)and (7), we define the following positive linear operators on $C(D)$ as follows:

$$L_{m,n}(f;x,y) = (1+\gamma_{m,n})\, B_{m,n}(f;x,y). \tag{9}$$

So, by the Theorem 2.1 and (8), we see that

$$st^2_{C(1,1)} - \lim_{m,n} \|L_{m,n}f - f\|_{C(D)} = 0.$$

Also, since $(\gamma_{m,n})$ is not P-convergent, we can say that the Korovkin theorem for positive linear operators of two variables in the Pringsheim's sense does not work for our operators defined by (9).

3. A Voronovskaya-type theorem

In this section, we obtain a Voronovskaya-type theorem and some differential properties in the $C(1,1)$-statistical case for the positive linear operators $\{L_{n,n}\}$ given by (9) for $n = m$.

Lemma 3.1 *Let $x, y \in [0,1]$. Then, we get*

$$st^2_{C(1,1)} - \lim_n n^2 L_{n,n}\left((u-x)^4; x, y\right) = 3x^2(1+x)^2, \tag{10}$$

and

$$st^2_{C(1,1)} - \lim_n n^2 L_{n,n}\left((v-y)^4; x, y\right) = 3y^2(1+y)^2. \tag{11}$$

Proof. We shall prove only (10) because the proof of (11) is similar. After some simple calculations, we can write form (10) that

$$n^2 L_{n,n}\left((u-x)^4; x, y\right) = (1+\gamma_{nn})\left[3x^4 - 6x^3 + 3x^2 + \frac{-6x^4+12x^3-7x^2+x}{n}\right].$$

Hence, we obtain

$$\left|n^2 L_{n,n}\left((u-x)^4; x, y\right) - 3f_2(x,y)\left[f_2(x,y) - 2f_1(x,y) + f_0(x,y)\right]\right| \le 12\gamma_{nn} + (1+\gamma_{nn})\frac{26}{n} \tag{12}$$

for every $x \in [0,1]$. Since $st^2_{C(1,1)} - \lim_n \gamma_{nn} = 0$, it is easy see that

$$st^2_{C(1,1)} - \lim_n \left[12\gamma_{nn} + (1+\gamma_{nn})\frac{26}{n}\right] = 0. \tag{13}$$

Combining (12) and (13), the proof is complete. □

Theorem 3.2 *For every* $f \in C(D)$ *such that* $f_x, f_y, f_{xx}, f_{xy}, f_{yy} \in C(D)$, *we have*

$$st^2_{C(1,1)} - \lim_n n\{L_{n,n}(f;x,y) - f(x,y)\} = \frac{1}{2}\left\{\left(x - x^2\right) f_{xx}(x,y) + \left(y - y^2\right) f_{yy}(x,y)\right\}.$$

Proof. Let $(x,y) \in D$ and $f_x, f_y, f_{xx}, f_{xy}, f_{yy} \in C(D)$. Define the function ϕ by

$$\phi_{(x,y)}(u,v) = \begin{cases} \dfrac{f(u,v)-f(x,y)-f_x(u-x)-f_y(v-y)-\frac{1}{2}\left\{f_{xx}(u-x)^2+2f_{xy}(u-x)(v-y)+f_{yy}(v-y)^2\right\}}{\sqrt{(u-x)^4+(v-y)^4}} & , \quad (u,v) \neq (x,y), \\ 0 & , \quad (u,v) = (x,y). \end{cases}$$

Then by assumption we get $\phi_{(x,y)}(x,y) = 0$ and $\phi_{(x,y)}(.,.) \in C(D)$. By the Taylor formula for $f \in C(D)$, we have

$$\begin{aligned} f(u,v) &= f(x,y) + f_x(u-x) + f_y(v-y) + \frac{1}{2}\Big\{f_{xx}(u-x)^2 \\ &\quad + 2f_{xy}(u-x)(v-y) + f_{yy}(v-y)^2\Big\} \\ &\quad + \phi_{(x,y)}(u,v)\sqrt{(u-x)^4 + (v-y)^4}. \end{aligned}$$

Since the operator $L_{n,n}$ is linear, we obtain

$$\begin{aligned} L_{n,n}(f;x,y) &= f(x,y)(1+\gamma_{nn}) + f_x L_{n,n}(u-x;x,y) \\ &\quad + f_y L_{n,n}(v-y;x,y) + \frac{1}{2}\Big\{f_{xx} L_{n,n}\left((u-x)^2;x,y\right) \\ &\quad + 2f_{xy} L_{n,n}((u-x)(v-y);x,y) + f_{yy} L_{n,n}\left((v-y)^2;x,y\right)\Big\} \\ &\quad + L_{n,n}\left(\phi_{(x,y)}(u,v)\sqrt{(u-x)^4+(v-y)^4};x,y\right). \end{aligned}$$

Note that, if $g \in C(D)$ and if $g(u,v) = g_1(u) g_2(v)$ for all $(u,v) \in D$, then

$$L_{n,n}(g;x,y) = (1+\gamma_{m,n}) B_n(g_1;x) B_n(g_2;y)$$

for every $(u,v) \in D$ and $n \in \mathbb{N}$. Now using this property, we obtain

$$\begin{aligned} L_{n,n}(f;x,y) &= f(x,y)(1+\gamma_{nn}) + \frac{1+\gamma_{nn}}{2n}\left\{\left(x-x^2\right) f_{xx}(x,y)\right. \\ &\quad \left. + \left(y-y^2\right) f_{yy}(x,y)\right\} \\ &\quad + L_{n,n}\left(\phi_{(x,y)}(u,v)\sqrt{(u-x)^4+(v-y)^4};x,y\right) \end{aligned}$$

which yields

$$\begin{aligned} n\{L_{n,n}(f;x,y) - f(x,y)\} &= n\gamma_{nn} f(x,y) + \frac{1+\gamma_{nn}}{2}\left\{\left(x-x^2\right) f_{xx}(x,y)\right. \\ &\quad \left. + \left(y-y^2\right) f_{yy}(x,y)\right\} \\ &\quad + nL_{n,n}\left(\phi_{(x,y)}(u,v)\sqrt{(u-x)^4+(v-y)^4};x,y\right). \end{aligned} \tag{14}$$

Applying the Cauchy-Schwarz inequality for the third term on the right-hand side of (14), we get

$$\begin{aligned}
& n\left|L_{n,n}\left(\phi_{(x,y)}(u,v)\sqrt{(u-x)^4+(v-y)^4};x,y\right)\right| \\
\leq & \left(L_{n,n}\left(\phi_{(x,y)}^2(u,v);x,y\right)\right)^{1/2}\cdot\left(n^2L_{n,n}\left((u-x)^4+(v-y)^4;x,y\right)\right)^{1/2} \\
= & \left(L_{n,n}\left(\phi_{(x,y)}^2(u,v);x,y\right)\right)^{1/2}\left\{n^2L_{n,n}\left((u-x)^4;x,y\right)+n^2L_{n,n}\left((v-y)^4;x,y\right)\right\}^{1/2}
\end{aligned}$$

Let $\eta_{(x,y)}(u,v)=\phi_{(x,y)}^2(u,v)$. In this case, we show that $\eta_{(x,y)}(x,y)=0$ and $\eta_{(x,y)}(.,.)\in C(D)$. From Theorem 2.1,

$$\begin{aligned}
st_{C(1,1)}^2-\lim_n L_{n,n}\left(\phi_{(x,y)}^2(u,v);x,y\right) &= st_{C(1,1)}^2-\lim_n L_{n,n}\left(\eta_{(x,y)}(u,v);x,y\right) \\
&= \eta_{(x,y)}(x,y) \qquad (15)\\
&= 0.
\end{aligned}$$

Using (15) and Lemma 3.1, we obtain from (14)

$$st_{C(1,1)}^2-\lim_n nL_{n,n}\left(\phi_{(x,y)}(u,v)\sqrt{(u-x)^4+(v-y)^4};x,y\right)=0. \qquad (16)$$

Considering (16), (14) and also $st_{C(1,1)}^2-\lim_n\gamma_{nn}=0$, we have

$$st_{C(1,1)}^2-\lim_n n\left\{L_{n,n}(f;x,y)-f(x,y)\right\}=\frac{1}{2}\left\{\left(x-x^2\right)f_{xx}(x,y)+\left(y-y^2\right)f_{yy}(x,y)\right\}.$$

So the proof is completed. □

Theorem 3.3 *For every $f\in C(D)$ such that f_x, $f_y\in C(D)$, we have*

$$\begin{aligned}
st_{C(1,1)}^2-\lim_n\frac{\partial}{\partial x}L_{n,n}(f;x,y) &= \frac{\partial f}{\partial x}(x,y),\ x\neq 0,1, \qquad (17)\\
st_{C(1,1)}^2-\lim_n\frac{\partial}{\partial y}L_{n,n}(f;x,y) &= \frac{\partial f}{\partial y}(x,y),\ y\neq 0,1. \qquad (18)
\end{aligned}$$

Proof. We shall prove only (17) because the proof of (18) is identical. Let $(x,y)\in D$ for $x\neq 0,1$ and f_x, $f_y\in C(D)$. From (9), we obtain

$$\begin{aligned}
\frac{\partial}{\partial x}L_{n,n}(f;x,y) &= \frac{-n}{1-x}(1+\gamma_{nn})\sum_{k=0}^{n}\sum_{j=0}^{m}f\left(\frac{k}{n},\frac{j}{m}\right)C_n^kx^k(1-x)^{n-k}C_m^jy^j(1-y)^{m-j} \\
&\quad+\frac{n}{x(1-x)}(1+\gamma_{nn})\sum_{k=0}^{n}\sum_{j=0}^{m}f\left(\frac{k}{n},\frac{j}{m}\right)C_n^kx^k(1-x)^{n-k}kC_m^jy^j(1-y)^{m-j} \\
&= \frac{-n}{1-x}L_{n,n}(f(u,v);x,y)+\frac{n}{x(1-x)}L_{n,n}(uf(u,v);x,y).
\end{aligned}$$

Define the function η by

$$\eta_{(x,y)}(u,v)=\begin{cases}\frac{f(u,v)-f(x,y)-f_x(u-x)-f_y(v-y)}{\sqrt{(u-x)^2+(v-y)^2}} & , \quad (u,v)\neq(x,y)\,,\\ 0 & , \quad (u,v)=(x,y)\,.\end{cases}$$

Then by assumption we get $\eta_{(x,y)}(u,v)=0$ and $\eta_{(x,y)}(.,.)\in C(D)$. By the Taylor formula for $f\in C(D)$, we have

$$\begin{aligned} f(u,v) &= f(x,y)+f_x(u-x)+f_y(v-y)\\ &\quad +\eta_{(x,y)}(u,v)\sqrt{(u-x)^2+(v-y)^2}. \end{aligned}$$

Since the operator $L_{n,n}$ is linear, we obtain

$$\begin{aligned} \frac{\partial}{\partial x}L_{n,n}(f;x,y) &= \frac{-n}{1-x}\left\{(1+\gamma_{nn})f(x,y)+L_{n,n}\left(\eta_{(x,y)}(u,v)\sqrt{(u-x)^2+(v-y)^2};x,y\right)\right\}\\ &\quad +\frac{n}{x(1-x)}\left\{(1+\gamma_{nn})\left[xf(x,y)+\frac{x-x^2}{n}f_x\right]\right.\\ &\quad \left.+L_{n,n}\left(u\eta_{(x,y)}(u,v)\sqrt{(u-x)^2+(v-y)^2};x,y\right)\right\}. \end{aligned}$$

which yields

$$\begin{aligned} \frac{\partial}{\partial x}L_{n,n}(f;x,y) &= (1+\gamma_{nn})f_x\\ &\quad +\frac{n}{x(1-x)}L_{n,n}\left((u-x)\eta_{(x,y)}(u,v)\sqrt{(u-x)^2+(v-y)^2};x,y\right). \end{aligned} \tag{19}$$

By the Cauchy-Schwarz inequality, we get

$$\begin{aligned} & n\left|L_{n,n}\left((u-x)\eta_{(x,y)}(u,v)\sqrt{(u-x)^2+(v-y)^2};x,y\right)\right|\\ \leq\ & \left(L_{n,n}\left(\eta^2_{(x,y)}(u,v);x,y\right)\right)^{1/2}\cdot\left(n^2L_{n,n}\left((u-x)^4+(u-x)^2(v-y)^2;x,y\right)\right)^{1/2}\\ =\ & \left(L_{n,n}\left(\eta^2_{(x,y)}(u,v);x,y\right)\right)^{1/2}\cdot\left\{n^2L_{n,n}\left((u-x)^4;x,y\right)\right.\\ & \left.+n^2L_{n,n}\left((u-x)^2(v-y)^2;x,y\right)\right\}^{1/2}\\ =\ & \left(L_{n,n}\left(\eta^2_{(x,y)}(u,v);x,y\right)\right)^{1/2}\cdot\left\{n^2L_{n,n}\left((u-x)^4;x,y\right)\right.\\ & \left.n^2(1+\gamma_{nn})B_{nn}\left((u-x)^2;x,y\right)B_{nn}\left((v-y)^2;x,y\right)\right\}^{1/2}\\ =\ & \left(L_{n,n}\left(\eta^2_{(x,y)}(u,v);x,y\right)\right)^{1/2}\\ & \cdot\left(n^2L_{n,n}\left((u-x)^4;x,y\right)+(1+\gamma_{nn})\left(x-x^2\right)\left(y-y^2\right)\right)^{1/2}. \end{aligned} \tag{20}$$

Let $\phi_{(x,y)}(u,v) = \eta^2_{(x,y)}(u,v)$. In this case, we show that $\phi_{(x,y)}(x,y) = 0$ and $\phi_{(x,y)}(.,.) \in C(D)$. From Theorem 2.1,

$$\begin{aligned} st^2_{C(1,1)} - \lim_n L_{n,n}\left(\eta^2_{(x,y)}(u,v);x,y\right) &= st^2_{C(1,1)} - \lim_n L_{n,n}\left(\phi_{(x,y)}(u,v);x,y\right) \\ &= \phi_{(x,y)}(x,y) \\ &= 0. \end{aligned} \tag{21}$$

Using (21) and Lemma 3.1, we obtain from (20)

$$st^2_{C(1,1)} - \lim_n nL_{n,n}\left((u-x)\,\eta_{(x,y)}(u,v)\sqrt{(u-x)^2+(v-y)^2};x,y\right) = 0. \tag{22}$$

Considering (22) in (19) and also $st^2_{C(1,1)} - \lim_n \gamma_{nn} = 0$, we have

$$st^2_{C(1,1)} - \lim_n \frac{\partial}{\partial x} L_{n,n}(f;x,y) = \frac{\partial f}{\partial x}(x,y).$$

So the proof is complete. □

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Received 13.02.2008