Some properties of the first eigenvalue of the $p(x)$-laplacian on Riemannian manifolds

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Abstract

The main result of the present paper establishes a stability property of the first eigenvalue of the associated problem which deals with the $p(x)$-Laplacian on Riemannian manifolds with Dirichlet boundary condition.

Key Words: Variable exponent Lebesgue and Sobolev spaces; first eigenvalue; Riemannian manifolds; $p(x)$-Laplacian.

1. Introduction

Over the last decades the variable exponent Lebesgue spaces $L^{p(x)}$ and the corresponding Sobolev space $W^{1,p(x)}$ have been a subject of active research stimulated by development of the studies of problems in elasticity, fluid dynamics, calculus of variations, and differential equations with $p(x)$-growth (see [2], [3], [12]). We refer the reader to [5], [7], [8] for fundamental properties of these spaces.

The $p(x)$-Laplacian equations related to eigenvalue problems have been studied in [6], [9], [10], [11].

Let $G \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain with a smooth boundary. For measurable function $p(x)$ we denote the variable exponent Lebesgue space by

$$L^{p(x)}(G) = \left\{ u \text{ measurable real functions} : \int_G |u(x)|^{p(x)} \, dx < \infty \right\},$$

which is equipped with the norm, the so-called Luxemburg norm (see [5], [7])

$$|u|_{p(x)} := |u|_{L^{p(x)}(G)} = \inf \left\{ \delta > 0 : \int_G \left| \frac{u(x)}{\delta} \right|^{p(x)} \, dx \leq 1 \right\},$$

where

$$1 < \text{ess inf}_{x \in G} p(x) := p^- \leq p(x) \leq \text{ess sup}_{x \in G} p(x) := p^+ < \infty,$$

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where \( L^{p(x)} (G) \) becomes a Banach space, which is known as a variable exponent Lebesgue space. Define the variable exponent Sobolev space \( W^{1,p(x)} (G) \) by

\[
W^{1,p(x)} (G) = \{ u \in L^{p(x)} (G) : |\nabla u| \in L^{p(x)} (G) \},
\]

and equip with the norm

\[
\| u \|_{W^{1,p(x)} (G)} := \| u \|_{W^{1,p(x)} (G)} = |u|_{p(x)} + |\nabla u|_{p(x)}, \forall u \in W^{1,p(x)} (G).
\]

The space \( W^{1,p(x)}_0 (G) \) is denoted by the closure of \( C^\infty (G) \) in \( W^{1,p(x)} (G) \) which is equipped with the norm for \( u \in W^{1,p(x)}_0 (G) \)

\[
\| u \|_{W^{1,p(x)}_0 (G)} = |\nabla u|_{p(x)}, \forall u \in W^{1,p(x)}_0 (G).
\]

If \( p^- > 1 \), then the spaces \( L^{p(x)} (G), W^{1,p(x)} (G) \) and \( W^{1,p(x)}_0 (G) \) are separable and reflexive Banach spaces (see [5], [7]).

**Proposition 1.** ([5], [7]). Denote

\[
\varrho_{p(x)} (u) = \int_G |u (x)|^{p(x)} \, dx,
\]

and

\[
\varrho_{1,p(x)} (u) := \varrho_{p(x)} (\nabla u) = \int_G |\nabla u (x)|^{p(x)} \, dx, \forall u, \nabla u \in L^{p(x)} (G),
\]

then we have

\[
\min \left\{ |u|_{p^-}^{p^-}, |u|_{p^+}^{p^+} \right\} \leq \varrho_{p(x)} (u) \leq \max \left\{ |u|_{p^-}^{p^-}, |u|_{p^+}^{p^+} \right\},
\]

\[
\min \left\{ |\nabla u|_{p^-}^{p^-}, |\nabla u|_{p^+}^{p^+} \right\} \leq \varrho_{1,p(x)} (u) \leq \max \left\{ |\nabla u|_{p^-}^{p^-}, |\nabla u|_{p^+}^{p^+} \right\}.
\]

Let \( M \) be a compact Riemannian manifold with \( \text{dim} M = m \), and \( \Delta_{p(x)} \) is nonhomogeneous \( p(x) \)-Laplacian acting on functions on \( M \), where \( \Delta_{p(x)} u = \text{div} \left( |\nabla u|^{p(x)-2} \nabla u \right) \), and \( 1 < p(x) < \infty \). Let \( M^* \) be a compact submanifold of \( \Omega \), and \( B_\varepsilon \) the tubular neighborhood of \( M^* \) of radius \( \varepsilon > 0 \); that is,

\[
B_\varepsilon = \{ x \in M : d (x, M^*) < \varepsilon \},
\]

where \( d (\cdot, \cdot) \) is the distance function on \( M \) induced by the Riemannian metric. Denote by \( \Delta_{p(x), \varepsilon} \) the restriction of \( \Delta_{p(x)} \) to those functions on \( M \) vanishing identically in \( B_\varepsilon \). Set

\[
\Omega_\varepsilon = M \setminus B_\varepsilon \text{ and } \partial \Omega_\varepsilon = \partial B_\varepsilon.
\]

We consider the following Dirichlet problem

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2. Main results and proofs

The first eigenvalue \( \lambda_{1,p(x)}(\Omega_\varepsilon, \phi) \) of the \( p(x) \)-Laplacian is defined as the least number \( \lambda \) for which the Dirichlet problem has a nontrivial solution \( u \in W^{1,p(x)}_0(\Omega_\varepsilon) \). It can be characterized by

\[
\lambda_{1,p(\cdot)}(\Omega_\varepsilon, \phi) = \inf_{u \in W^{1,p(x)}_0(\Omega_\varepsilon) \setminus \{0\}} \frac{\int_{\Omega_\varepsilon} |\nabla u|^{p(x)} \, dv_\phi}{\int_{\Omega_\varepsilon} |u|^{p(x)} \, dv_\phi},
\]

where \( u \) runs over \( W^{1,p(x)}_0(\Omega_\varepsilon) \) and \( dv_\phi \) denotes the volume element of \( M \). It turns out \( \lambda_{1,p(x)}(\Omega_\varepsilon, \phi) > 0 \). When \( B_\varepsilon = \emptyset \), that is, \( \Omega_\varepsilon = M \), \( u \) runs over \( W^{1,p(x)}(\Omega_\varepsilon) \) and \( \int_M |u|^{p(x)} \, dv_\phi = 0 \). We can easily see that \( \lambda_{1,p(x)}(M) = 0 \). The corresponding eigenfunctions are constant functions. In the case of \( p = 2 \), many people have studied the asymptotic expansion of the eigenvalues \( \lambda_{k,2}(\varepsilon) \) \( (k = 1, 2, \ldots) \) for the 2- Laplacian of a manifold \( M \setminus B_\varepsilon \) with the Dirichlet condition on the tubular neighborhood \( B_\varepsilon \). Chavel and Feldman in [4] showed that \( \lambda_{k,2}(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \) tends to zero under the condition \( \text{codim}(M^*) \geq 2 \). Eigenvalue problems for quasilinear operators of \( p \)-Laplace type have received considerable attention in the last years (see [1]).

In the present paper, we establish conditions ensuring a stability property of the first eigenvalue of the associated problem which deals with the \( p(x) \)-Laplacian. As far as we are concerned, this is the first paper that discusses this subject. We prove that the first eigenvalues of the \( p(x) \)-Laplacian on Riemannian manifolds \( \Omega_\varepsilon \) converges to zero as \( \varepsilon \) tends to zero. In precisely we show \( \lambda_{1,p(x)}(\Omega_\varepsilon, \phi) \to \lambda_{1,p(x)}(M, \phi) = 0 \) as \( \varepsilon \to 0 \).

Moreover, the investigation of the Riemannian manifolds in the variable exponent Lebesgue space \( L^{p(x)} \) is firstly dealt with in our study.

**Lemma 1.** Suppose that the codimension \( m - k \geq p^+ \). Given any \( f \in W^{1,p(x)}(M) \), there exists a function \( f_\varepsilon \in W^{1,p(x)}_0(\Omega_\varepsilon) \) such that

\[
\lim_{\varepsilon \to 0} \|f_\varepsilon - f\|_{1,p(x)} = \lim_{\varepsilon \to 0} |f_\varepsilon - f|_{p(x)} + \lim_{\varepsilon \to 0} |\nabla(f_\varepsilon - f)|_{p(x)} = 0
\]

for sufficiently small \( \varepsilon > 0 \).

**Proof.** By Proposition 1, we can write

\[
\varrho_{1,p(x)}(f_\varepsilon - f) = \varrho_{p(x)}(f_\varepsilon - f) + \varrho_{p(x)} \nabla(f_\varepsilon - f).
\]
Define a function \( \omega_\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+ \) as

\[
\omega_\varepsilon(r) =
\begin{cases}
1, & \varepsilon \leq r; \\
\left( \frac{\varepsilon}{\varepsilon} \right)^{\varepsilon}, & \exp(-\varepsilon^{-2}) \leq r \leq \varepsilon; \\
\left( \frac{\exp(-\varepsilon^{-2})}{\varepsilon} \right)^{\varepsilon} \left( \frac{2r}{\exp(-\varepsilon^{-2})} \right), & 2^{-1} \exp(-\varepsilon^{-2}) \leq r \leq \exp(-\varepsilon^{-2}) \\
0, & r \leq \exp(-\varepsilon^{-2}).
\end{cases}
\]

Then we have

\[
|\omega_\varepsilon'(r)|^{p(r)} =
\begin{cases}
0, & \varepsilon \leq r \\
\left( \frac{\varepsilon}{\varepsilon} \right)^{p(r)-p(r)}, & \exp(-\varepsilon^{-2}) \leq r \leq \varepsilon \\
\left( \frac{\exp(-\varepsilon^{-2})}{\varepsilon} \right)^{p(r)} \left( \frac{2r}{\exp(-\varepsilon^{-2})} \right)^{p(r)}, & 2^{-1} \exp(-\varepsilon^{-2}) \leq r \leq \exp(-\varepsilon^{-2}) \\
0, & r \leq \exp(-\varepsilon^{-2}).
\end{cases}
\]

Define \( f_\varepsilon = \omega_\varepsilon f \in W^{1,p(x)}_0(\Omega_\varepsilon) \) and \( r(x) = d(M^*,x) \). Then we have

\[
\varrho_{p(x)}(f_\varepsilon - f) = \int_M |(f_\varepsilon - f)|^{p(x)} \, dv_\phi,
\]

and

\[
\lim_{\varepsilon \to 0} \int_r^\varepsilon |f_\varepsilon - f|^{p(x)} \, dv_\phi = \lim_{\varepsilon \to 0} \int_\varepsilon^r |(\omega_\varepsilon - 1)f|^{p(x)} \, dv_\phi \to 0,
\]

and

\[
\lim_{\varepsilon \to 0} \int_\exp(-\varepsilon^{-2})^{\exp(-\varepsilon^{-2})} \left| \left( \frac{\exp(-\varepsilon^{-2})}{\varepsilon} \right)^{\varepsilon} \left( \frac{2r}{\exp(-\varepsilon^{-2})} \right) - 1 \right| f^{p(x)} \, dv_\phi \to 0,
\]

and

\[
\lim_{\varepsilon \to 0} \int_r^\varepsilon |f_\varepsilon - f|^{p(x)} \, dv_\phi \to 0;
\]

so we have

\[
\lim_{\varepsilon \to 0} \int_M |f_\varepsilon - f|^{p(x)} \, dv_\phi = 0.
\]
Hence, by Proposition 1 we can write \(|f_\varepsilon - f|_{p(x)} = 0\).

Let’s consider

\[
\varphi_{p(x)} \nabla (f_\varepsilon - f) = \int_M \nabla (f_\varepsilon - f)|^{p(x)} d\nu_\phi = \int_{T_\varepsilon,2^{-1} \exp(-\varepsilon^{-2})} |\nabla (f_\varepsilon - f)|^{p(x)} d\nu_\phi + \int_{B_{2^{-1}} \exp(-\varepsilon^{-2})} |\nabla f|^{p(x)} d\nu_\phi,
\]

where \(T_\varepsilon,2^{-1} \exp(-\varepsilon^{-2}) = B_\varepsilon \setminus B_{2^{-1}} \exp(-\varepsilon^{-2})\). Since \(f \in W^{1,p(x)}(M)\),

\[
\int_{B_{2^{-1}} \exp(-\varepsilon^{-2})} |\nabla f|^{p(x)} d\nu_\phi \to 0 \text{ as } 2^{-1} \exp(-\varepsilon^{-2}) \to 0,
\]

and

\[
\int_{T_\varepsilon,2^{-1} \exp(-\varepsilon^{-2})} |\nabla (f_\varepsilon - f)|^{p(x)} d\nu_\phi \leq 2^{p^+ - 1} \int_{T_\varepsilon,2^{-1} \exp(-\varepsilon^{-2})} |\nabla f_\varepsilon|^{p(x)} d\nu_\phi + \int_{T_\varepsilon,2^{-1} \exp(-\varepsilon^{-2})} |\nabla f|^{p(x)} |\omega_\varepsilon|^{p(x)} d\nu_\phi.
\]

The first term on the right hand side tends to 0 as \(\varepsilon \to 0\) because of \(f \in W^{1,p(x)}(M)\). By the definition of \(f_\varepsilon\), we have

\[
\int_{T_\varepsilon,2^{-1} \exp(-\varepsilon^{-2})} |\nabla f|^{p(x)} d\nu_\phi \leq |\nabla \omega_\varepsilon|^{p(x)} d\nu_\phi \leq 2^{p^+ - 1} \int_{T_\varepsilon,2^{-1} \exp(-\varepsilon^{-2})} |\nabla f|^{p(x)} |\omega_\varepsilon|^{p(x)} d\nu_\phi.
\]

Since \(f \in W^{1,p(x)}(M)\), the term \(\int_{T_\varepsilon,2^{-1} \exp(-\varepsilon^{-2})} |\nabla f|^{p(x)} |\omega_\varepsilon|^{p(x)} d\nu_\phi \to 0 \text{ as } \varepsilon \to 0\).

Next we shall show

\[
\int_{T_\varepsilon,2^{-1} \exp(-\varepsilon^{-2})} |f|^{p(x)} |\nabla \omega_\varepsilon|^{p(x)} d\nu_\phi \to 0 \text{ as } \varepsilon \to 0.
\]

From \(\dim M^* = k\) and \(m - k \geq p^+\), that is, the codimension of \(M^*\) and \(M\) is greater than \(p^+\) or equal to \(p^+\). We consider the fact

\[
p^+ = \inf_{x \in (\varepsilon, \exp(-\varepsilon^{-2}))} p(x) \text{ and } \sup_{x \in (\varepsilon, \exp(-\varepsilon^{-2}))} p(x) \leq p^+,
\]

with which we can estimate

\[
\int_{T_\varepsilon,2^{-1} \exp(-\varepsilon^{-2})} |f|^{p(x)} |\nabla \omega_\varepsilon|^{p(x)} d\nu_\phi \leq C p^+ \int_{T_\varepsilon,2^{-1} \exp(-\varepsilon^{-2})} |\nabla \omega_\varepsilon|^{p(x)} d\nu_\phi
\]

\[
\leq 2^{p^+ - 1} C p^+ \left( \int_{\varepsilon}^{\exp(-\varepsilon^{-2})} \left( \frac{\varepsilon}{\varepsilon} \right)^{p(r)-p(r)} x^{m-k-1} dr + \int_{2^{-1} \exp(-\varepsilon^{-2})} \left( \frac{\exp(-\varepsilon^{-2})}{\varepsilon} \right)^{p(r)} \right)
\]

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\begin{equation}
\times \left( \frac{2}{\exp(-\varepsilon^{-2})} \right)^{p(r)} r^{m-k-1} dr \leq C_1 (p^+) \int_{\exp(-\varepsilon^{-2})}^{\varepsilon} \left( \frac{1}{\varepsilon} \right)^{ep^- - p^+} r^{m-k-1} dr
\end{equation}

\begin{equation}
+ \frac{C_2 (p^+)}{\varepsilon^{p^+}} \exp \left( - \frac{(e^{p^+} - p^+)^2}{\varepsilon^2} \right) \int_{2^{-1} \exp(-\varepsilon^{-2})}^{\exp(-\varepsilon^{-2})} r^{m-k-1} dr = C_1 (p^+) \left[ \frac{r^{ep^- - p^+ + m-k}}{\varepsilon^{p^+}} \exp(-\varepsilon^{-2}) \right]
\end{equation}

\begin{equation}
+ \frac{C_2 (p^+, m, k)}{\varepsilon^{p^+}} \left\{ \exp(-\varepsilon^{-2}(e^{p^+} - p^+ + m-k)) \right\}
\end{equation}

\begin{equation}
\leq \frac{C_1 (p^+)}{p^- \varepsilon^{p^+} - p^+ + 1} \left\{ \exp(-\varepsilon^{-2}(e^{p+} - p^+ + m-k)) \right\}
\end{equation}

\begin{equation}
+ \frac{C_2 (p^+, m, k)}{\varepsilon^{p^+}} \exp(-\varepsilon^{-2}(e^{p^+} - p^+ + m-k))
\end{equation}

The right hand side tends to 0 by using

\begin{equation}
\lim_{\varepsilon \to 0} \varepsilon^{m-k-1} = 0 \text{ (because of } m-k \geq p^+ - 1 > 0), \quad \lim_{\varepsilon \to 0} \varepsilon^{p^+ - 1} \exp(-\varepsilon^{-2}(e^{p^+} - p^+ + m-k)) = 0
\end{equation}

(because of \( \lim_{\varepsilon \to 0} \varepsilon = 1 \), \( \lim_{\varepsilon \to 0} \varepsilon^{p^+} \exp(-\varepsilon^{-2}(e^{p^+} - p^+ + m-k)) = 0 \). This completes the proof of Lemma 1.

\textbf{Theorem 2.} Let \((M, \phi)\) be a compact Riemannian manifold with \( \dim M = m \), and \( M^* \) a closed submanifold in \( M \) with \( \dim M^* = k \). Suppose that the codimension \( m-k \geq p^+ \) of \( M^* \) in \( M \) is greater than or equal to \( p^+ \). Let \( \lambda_{1,p(x)} (\Omega, \phi) \) be the first eigenvalue of the \( p(x) \)-Laplacian on \( \Omega \). We have

\begin{equation}
\lim_{\varepsilon \to 0} \lambda_{1,p(x)} (\Omega, \phi) = \lambda_{1,p(x)} (M, \phi) = 0.
\end{equation}

The corresponding eigenfunctions \( \phi_\varepsilon \) in \( \Omega \) converge to a constant function \( \phi_1 \) in \( L^{p(x)} (M) \).

\textbf{Proof.} Take the eigenfunction \( f \) for \( \lambda_{1,p(x)} (M, \phi) \) such that \( \int_M |f|^{p(x)} d\nu_\phi = 1 \). Then

\begin{equation}
\lambda_{1,p(x)} (M) = \int_M |\nabla f|^{p(x)} d\nu_\phi.
\end{equation}
From Lemma 1, there exists \( f_{\varepsilon} \in W^{1,p(x)}_0(\Omega_{\varepsilon}) \) such that
\[
\lim_{\varepsilon \to 0} \int_{\Omega_{\varepsilon}} |f_{\varepsilon} - f|^{p(x)} \, dv_{\phi} \to 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \int_{\Omega_{\varepsilon}} |\nabla (f_{\varepsilon} - f)|^{p(x)} \, dv_{\phi} \to 0.
\]
Thus
\[
\lambda_{1,p(x)}(\Omega_{\varepsilon}, \phi) \leq \frac{\int_{\Omega_{\varepsilon}} |\nabla f_{\varepsilon}|^{p(x)} \, dv_{\phi}}{\int_{\Omega_{\varepsilon}} |f_{\varepsilon}|^{p(x)} \, dv_{\phi}} \to \frac{\int_{M} |\nabla f|^{p(x)} \, dv_{\phi}}{\int_{M} |f|^{p(x)} \, dv_{\phi}} = \lambda_{1,p(x)}(M, \phi) \quad (\varepsilon \to 0). \tag{1}
\]

Now let \( \phi_{\varepsilon} \) be the first eigenvalue for the \( p(x) \)-Laplacian on \( \Omega_{\varepsilon} \). By formula (1), \( \phi_{\varepsilon} \) is uniformly bounded. \( \phi_{\varepsilon} \) has a strongly convergence limit \( \phi_1 \in L^{p(x)}(M) \) in \( L^{p(x)}(M) \). By formula (1), and the compactness of \( \phi_{\varepsilon} \), we have \( \int_{M} |\phi_1|^{p(x)} \, dv_{\phi} = 1 \) and
\[
\lambda_{1,p(x)}(M, \phi) \leq \frac{\int_{M} |\nabla \phi_1|^{p(x)} \, dv_{\phi}}{\int_{M} |\phi_1|^{p(x)} \, dv_{\phi}} \leq \lim_{\varepsilon \to 0} \int_{M} |\nabla \phi_{\varepsilon}|^{p(x)} \, dv_{\phi} \leq \lambda_{1,p(x)}(M, \phi).
\]
This shows that \( \lambda_{1,p(x)}(\Omega_{\varepsilon}, \phi) \to \lambda_{1,p(x)}(M, \phi) \) as \( \varepsilon \to 0 \), and the limit function \( \phi_1 \in L^{p(x)}(M) \) is the first eigenfunction for \( p(x) \)-Laplacian on \( M \).

This completes the proof of Theorem.

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