Some properties of gr-multiplication ideals

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Abstract

In this paper, we study some of the properties of gr-multiplication ideals in a graded ring \( R \). We first characterize finitely generated gr-multiplication ideals and then give a characterization of gr-multiplication ideals by using the gr-localization of \( R \). Finally we determine the set of gr-\( P \)-primary ideals of \( R \) when \( P \) is a gr-multiplication gr-prime ideal of \( R \).

Key Words: Graded Rings, Graded Ideals, Gr-primary Ideals and Gr-multiplication Ideals.

1. Introduction

Let \( G \) be a group. A ring \((R, G)\) is called a \( G \)-graded ring if there exists a family \( \{R_g : g \in G\} \) of additive subgroups of \( R \) such that \( R = \bigoplus_{g \in G} R_g \) and \( R_g R_h \subseteq R_{gh} \) for each \( g \) and \( h \) in \( G \). For simplicity, we will denote the graded ring \((R, G)\) by \( R \). A element of a graded ring \( R \) is called homogeneous if it belongs to \( \bigcup_{g \in G} R_g \) and this set of homogeneous elements is denoted by \( h(R) \). If \( x \in R_g \) for some \( g \in G \), then we say that \( x \) is of degree \( g \). A graded ideal \( I \) of a graded ring \( R \) is an ideal verifying \( I = \bigoplus_{g \in G} (I \cap R_g) = \bigoplus_{g \in G} I_g \).

Equivalently, \( I \) is graded in \( R \) if and only if \( I \) has a homogeneous set of generators. If \( R = \bigoplus_{g \in G} R_g \) and \( R' = \bigoplus_{g \in G} R'_g \) are two graded rings, then a mapping \( \eta : R \to R' \) with \( \eta(1_R) = 1_{R'} \) is called a gr-homomorphism if \( \eta(R_g) \subseteq R'_g \) for all \( g \in G \). A graded ideal \( P \) of a graded ring \( R \) is called gr-prime if whenever \( x, y \in h(R) \) with \( xy \in P \), then \( x \in P \) or \( y \in P \). A graded ideal \( M \) of a graded ring \( R \) is called gr-maximal if it is maximal in the lattice of graded ideals of \( R \). A graded ring \( R \) is called a gr-local ring if it has unique gr-maximal ideal.

Let \( R \) be a graded ring and let \( S \subseteq h(R) \) be a multiplicatively closed subset of \( R \). Then the ring of fractions \( S^{-1}R \) is a graded ring which is called the gr-ring of fractions. Indeed, \( S^{-1}R = \bigoplus_{g \in G} (S^{-1}R)_g \) where

\[
(S^{-1}R)_g = \left\{ \frac{r}{s} : r \in R, \ s \in S \ and \ g = (deg \ s)^{-1} (deg \ r) \right\}.
\]
Consider the ring gr-homomorphism $\eta : R \rightarrow S^{-1}R$ defined by $\eta(r) = \frac{r}{s}$. For any graded ideal $I$ of $R$, the ideal of $S^{-1}R$ generated by $\eta(I)$ is denoted by $S^{-1}I$. Similar to non graded case, one can prove that

$$S^{-1}I = \left\{ \lambda \in S^{-1}R : \lambda = \frac{r}{s} \text{ for } r \in I \text{ and } s \in S \right\}$$

and that $S^{-1}I \neq S^{-1}R$ if and only if $S \cap I = \Phi$. Moreover, similar to the non graded case, we have the following properties for graded ideal $I$ and $J$ of $R$:

1. $S^{-1}(I \cap J) = S^{-1}I \cap S^{-1}J$ ,
2. $S^{-1}(IJ) = (S^{-1}I)(S^{-1}J)$ and
3. $S^{-1}(I : J) = (S^{-1}I : S^{-1}J)$ if $J$ is finitely generated.

If $\mathcal{J}$ is a graded ideal in $S^{-1}R$, then $\mathcal{J} \cap R$ will denote the graded ideal $\eta^{-1}(\mathcal{J})$ of $R$. Moreover, similar to the non graded case one can prove that $S^{-1}(\mathcal{J} \cap R) = \mathcal{J}$.

Let $P$ be any gr-prime ideal of a graded ring $R$ and consider the multiplicatively closed subset $S = h(R) - P$. We denote the graded ring of fraction $S^{-1}R$ of $R$ by $R_p$, and we call it the gr-localization of $R$. This ring is gr-local with the unique gr-maximal $S^{-1}P$ which will be denoted by $PR_p$. Moreover, for graded ideals $I$ and $J$ of $R$, if $IR_p = JR_p$ for every gr-prime (gr-maximal) ideal $P$ of $R$, then $I = J$. For a positive integer $n$ the graded ideal $(PR_p)^n \cap R$ of $R$ is denoted by $P^{(n)}$. For more definitions and theorems about gr-ring of fractions of graded rings, one can see [8].

Let $I$ be a graded ideal in a graded ring $R$. The graded radical of $I$ (denoted by $g-rad(I)$) is defined in [9] as the set of all $x \in R$ such that for each $g \in G$, there exists $n_g \geq 0$ such that $x^{n_g} \in I$.

A graded ideal $Q$ of a graded ring $R$ is called gr-primary if $Q \neq R$ and whenever $a, b \in h(R)$ with $ab \in Q$, then $a \in Q$ or $b \in g-rad(Q)$. If $Q$ is gr-primary ideal of $R$, then $g-rad(Q) = P$ is a gr-prime ideal of $R$ and we say that $Q$ is gr-$P$-primary. If $I$ is a graded ideal of $R$ with $g-rad(I) = M$, a gr-maximal ideal of $R$, then $I$ is gr-$M$-primary; see [9].

Recall that a graded ring $R$ is called gr-PIR if every graded ideal of $R$ is gr-principal, where a gr-principal ideal of a graded ring $R$ is generated by some homogeneous element in $R$. Also, recall that a graded ring $R$ is called gr-SPIR if $R$ has unique gr-prime ideal $P$ and every graded ideal of $R$ is a power of $P$. Similar to the non graded case, one can prove that if $R$ is a gr-SPIR, then $R$ is a gr-PIR and the unique gr-prime ideal of $R$ is nilpotent.

An ideal $I$ of a ring $R$ is called multiplication if whenever $J$ is an ideal of $R$ with $J \subseteq I$, then there is an ideal $K$ of $R$ such that $J = IK$. If every ideal in a ring $R$ is multiplication, then $R$ is called a multiplication ring. Multiplication ideals and rings have been studied in detail in [1], [2] and [7]. A generalization of multiplication graded ideals and rings to gr-multiplication ideals and rings have been studied in [3], [4] and [5].

In this paper, we study more properties of gr-multiplication ideals in a graded ring $R$ and give a characterization for finitely generated gr-multiplication ideals. For an ideal $I$ of a graded ring $R$, we define the graded ideal $\theta(I)$ and use it together with the gr-localization of $R$ to give a general characterization for gr-multiplication ideals. Finally, we determine the set of gr-$P$-primary ideals of a graded ring $R$ when $P$ is both gr-prime and gr-multiplication in $R$. 

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2. Properties for gr-multiplication ideals

Definition 2.1  Let $R$ be a graded ring graded by the group $G$. A graded ideal $I$ of $R$ is called a gr-multiplication ideal of $R$ if whenever $J$ is a graded ideal of $R$ with $J \subseteq I$, then there is a graded ideal $K$ of $R$ such that $J = KI$. If every graded ideal in a graded ring $R$ is gr-multiplication, then $R$ is called a gr-multiplication ring.

Clearly, any graded ideal which is multiplication is a gr-multiplication ideal. A graded ideal $I$ of a graded ring $R$ is called a gr-invertible ideal if there exists a graded ideal $J$ of $R$ such that $IJ = R$. Also, one can easily see that every gr-invertible ideal is gr-multiplication. In particular, the gr-principal ideals are gr-multiplication.

The class of gr-multiplication domains has been characterized in [5] as the class of gr-Dedekind domains which is the class of graded domains in which every graded ideal is gr-invertible. In [10], we can see an example of a gr-multiplication ring which is not multiplication. Indeed, the group ring $R[Z]$, where $R$ is a Dedekind domain is gr-Dedekind domain and so it is gr-multiplication domain. On the other hand, if $R$ is not a field, then $R[Z]$ is not a Dedekind domain and so it is not a multiplication domain, see [6].

If $I$ and $J$ are two graded ideals in a graded ring $R$, then the ideal $(J:I) = \{x \in R : xI \subseteq J\}$ is a graded ideal, see [4]. In the following theorem, we can see another equivalent definition of gr-multiplication ideals.

Theorem 2.2  Let $I$ be a graded ideal in a graded ring $R$. Then $I$ is gr-multiplication iff $I \cap J = I(J:I)$ for every graded ideal $J$ of $R$.

Proof. Suppose that $J \subseteq I$ for a graded ideal $J$ of $R$. Then $J = I \cap J = I(J:I) = IJ$.

Conversely, suppose that $I$ is a gr-multiplication ideal in $R$. Let $J$ be any graded ideal of $R$. Then $I \cap J \subseteq I$ and so there is a graded ideal $K$ of $R$ with $I \cap J = IK$. Therefore, $K \subseteq (I \cap J : I) \subseteq (J : I)$ and then $I \cap J = IK \subseteq I(J : I)$. On the other hand, clearly, $I(J : I) \subseteq I \cap J$ and therefore, $I(J : I) = I \cap J$. □

The following theorem is a characterization of gr-multiplication ideals in gr-local rings; see [3].

Theorem 2.3  Let $R$ be a gr-local ring with the unique gr-maximal ideal $M$. A graded ideal $I$ of $R$ is gr-multiplication iff $I$ is gr-principal.

Proof. If $I = \langle x \rangle$ for some $x \in h(R)$, then clearly $I$ is a gr-multiplication ideal of $R$.

Conversely, suppose that $I$ is gr-multiplication in $R$. Since $I$ is graded, then it is generated by a set of homogeneous elements, say, $\{a_\alpha : \alpha \in \Lambda\}$. Now, for each $\alpha \in \Lambda$, $\langle a_\alpha \rangle \subseteq I$ and so there is a graded ideal $B_\alpha$ of $R$ such that $\langle a_\alpha \rangle = IB_\alpha$. Therefore, $I = \sum_{\alpha \in \Lambda} \langle a_\alpha \rangle = \sum_{\alpha \in \Lambda} IB_\alpha = I \sum_{\alpha \in \Lambda} B_\alpha$. If $\sum_{\alpha \in \Lambda} B_\alpha = R$, then $B_{a_\alpha} = R$ for some $a_\alpha \in \Lambda$, since otherwise if $B_\alpha \subset R$ for each $\alpha \in \Lambda$, then $B_\alpha \subseteq M$ for each $\alpha \in \Lambda$ and so $R = \sum_{\alpha \in \Lambda} B_\alpha \subseteq M$, a contradiction. Therefore, $\langle a_\alpha \rangle = IB_{a_\alpha} = I$ and $I$ is gr-principal. If $\sum_{\alpha \in \Lambda} B_\alpha \neq R$, then $\sum_{\alpha \in \Lambda} B_\alpha \subseteq M$ and then $I = I \sum_{\alpha \in \Lambda} B_\alpha \subseteq IM \subseteq I$. Therefore, $I = IM$ and then $I = 0$ by proposition 2.4 in [4]. It follows that $I$ is gr-principal. □
Theorem 2.4 If \( I \) is a gr-multiplication ideal of a graded ring \( R \) and \( S \subseteq h(R) \) is a multiplicatively closed subset of \( R \), then \( S^{-1}I \) is a gr-multiplication ideal of \( S^{-1}R \).

Proof. Let \( J \) be a graded ideal of \( S^{-1}R \) such that \( J \subseteq S^{-1}I \). Then \( J = S^{-1}J \) for some graded ideal \( J \) of \( R \). Now, \( I \cap J \subseteq I \) and therefore, there is a graded ideal \( K \) of \( R \) such that \( I \cap J = IK \). Thus

\[
J = S^{-1}I \cap S^{-1}J = S^{-1}(I \cap J) = S^{-1}(IK) = (S^{-1}I)(S^{-1}K).
\]

Therefore, \( S^{-1}I \) is a gr-multiplication ideal in \( S^{-1}R \). \( \square \)

Definition 2.5 A graded ideal \( I \) of a graded ring \( R \) is called locally gr-principal if \( IR_{p} \) is gr-principal for any gr-prime ideal \( P \) of \( R \).

As a corollary of theorem 2.3, we have the following.

Corollary 2.6 Any gr-multiplication ideal in a graded ring \( R \) is locally gr-principal.

In [3], it has been proved that if \( I \) is a finitely generated graded ideal of \( R \), then \( I \) is gr-multiplication if and only if \( I \) is locally gr-principal. In the following theorem, we can see another characterization of finitely generated gr-multiplication ideals. First, we have the following technical lemma.

Lemma 2.7 Let \( R \) be a gr-local ring with gr-maximal ideal \( M \) and \( I \) be a gr-principal ideal in \( R \). If \( I = \langle a_{1}, a_{2}, \ldots, a_{n} \rangle \), then \( I = \langle a_{j} \rangle \) for some \( j \in \{1, 2, \ldots, n\} \).

Proof. Suppose that \( I = \langle a \rangle \) for some \( a \in h(R) \) and suppose that \( I = \langle a_{1}, a_{2}, \ldots, a_{n} \rangle \). Then \( a = \sum_{i=1}^{n} a_{i}x_{i} \) where \( r_{i} \in R \) for all \( i \). Also for all \( i \), \( a_{i} = ax_{i} \) for some \( x_{i} \in R \). Thus, \( a(1 - \sum_{i=1}^{n} x_{i}r_{i}) = 0 \). If \( 1 - \sum_{i=1}^{n} x_{i}r_{i} \) is a unit in \( R \), then \( a = 0 \) and so \( I = \langle 0 \rangle = \langle a_{i} \rangle \) for all \( i = 1, 2, \ldots, n \) since \( a_{i} = 0 \) for all \( i \). If \( 1 - \sum_{i=1}^{n} x_{i}r_{i} \) is not a unit, then \( \sum_{i=1}^{n} x_{i}r_{i} \notin M \) and so \( \sum_{i=1}^{n} x_{i}r_{i} \) is a unit. Therefore, there is some \( j \in \{1, 2, \ldots, n\} \) such that \( x_{j}r_{j} \) is a unit and then \( x_{j} \) is also a unit. Hence, \( a = ax_{j}x_{j}^{-1} = a_{j}x_{j}^{-1} \) and \( I = \langle a \rangle \subseteq \langle a_{j} \rangle \). Hence, \( I = \langle a_{j} \rangle \) for some \( j \in \{1, 2, \ldots, n\} \). \( \square \)

Theorem 2.8 Let \( I = \langle a_{1}, a_{2}, \ldots, a_{n} \rangle \) be a finitely generated graded ideal of a graded ring \( R \). Then the following are equivalent.

1. \( I \) is gr-multiplication.
2. \( I \) is locally gr-principal.
3. \( \sum_{i=1}^{n} (a_{i}) : I_{i} = R \), where \( I_{i} = \langle a_{1}, a_{2}, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n} \rangle \).
Proof. (1) $\iff$ (2): see [3].

(2) $\Rightarrow$ (3): Suppose that $I$ is locally gr-principal. Then for each gr-prime ideal $P$ of $R$, we have $IR_P^0 = \langle a_1^+, a_2^+, \ldots, a_n^+ \rangle = \langle a_j \rangle R_P^0$ for some $j \in \{1, 2, \ldots, n\}$ by lemma 2.7. Hence, for any gr-prime ideal $P$ of $R$, $(\langle a_j \rangle R_P^0 : I_j R_P^0) = R_P^0$, and then

$$
\left( \sum_{i=1}^{n} ((a_i) : I_i) R_P^0 \right) = \sum_{i=1}^{n} ((a_i) : I_i) R_P^0 = R_P^0.
$$

since $I_i$ is finitely generated for each $i$. Therefore, $\sum_{i=1}^{n} ((a_i) : I_i) = R$.

(3) $\Rightarrow$ (2): Suppose that $\sum_{i=1}^{n} ((a_i) : I_i) = R$. Then for any gr-prime ideal $P$ of $R$, we have

$$
\sum_{i=1}^{n} ((a_i) R_P^0 : IR_P^0) = \sum_{i=1}^{n} ((a_i) : I_i) R_P^0 = R_P^0.
$$

Therefore, there is $j \in \{1, 2, \ldots, n\}$ such that $(\langle a_j \rangle R_P^0 : IR_P^0) = R_P^0$ and then $IR_P^0 \subseteq \langle a_j \rangle R_P^0 = \langle a_j^+ \rangle$. It follows that $IR_P^0 = \langle a_j^+ \rangle$ for each gr-prime ideal $P$ of $R$ and $I$ is locally gr-principal.

\[\Box\]

If $I$ is a graded ideal in a graded ring $R$, then we define the subset $\theta^0(I)$ of $R$ as $\theta^0(I) = \sum_{x \in I \cap h(R)} ((x) : I)$. Clearly, $\theta^0(I)$ is a graded ideal of $R$.

Lemma 2.9 Let $I$ be a gr-multiplication ideal of a graded ring $R$. Then

1. $I = \theta^0(I)$;
2. $J = J \theta^0(I)$ for any graded ideal $J \subseteq I$.

Proof. (1) For $x \in I \cap h(R)$, $\langle x \rangle \subseteq I$ and so $\langle x \rangle = I ((x) : I)$. Therefore

$$
I = \sum_{x \in I \cap h(R)} \langle x \rangle = \sum_{x \in I \cap h(R)} I ((x) : I) = I \sum_{x \in I \cap h(R)} ((x) : I) = \theta^0(I).
$$

(2) Suppose that $J$ is a graded ideal with $J \subseteq I$. Then $J = IK$ for some graded ideal $K$ of $R$. Hence,

$$
J = IK = I \theta^0(I) K = I K \theta^0(I) = J \theta^0(I).
$$

\[\Box\]

Lemma 2.10 Let $I$ and $J$ be graded ideals in a graded ring $R$ and let $S \subseteq h(R)$ be a multiplicatively closed subset of $R$. Then

1. $\theta^0(I) \theta^0(J) \subseteq \theta^0(IJ)$;
2. $S^{-1}(\theta^0(I)) \subseteq \theta^0(S^{-1}I)$.
Let $\theta$ be principal iff $\theta(I) = R$. Then $\theta(I)$ is locally gr-principal.

Proof. Let $I$ be a graded ideal in a graded ring $R$. Then $I$ is gr-finitely generated and locally gr-principal iff $\theta(I) = R$.

Proof. Let $M$ be a gr-maximal ideal in $R$. Then $IR_M^g = \langle x \rangle R_M^g$ for some $x \in I \cap h(R)$. Hence, $R_M^g = \langle x \rangle R_M^g = \langle x \rangle I R_M^g$ since $I$ is gr-finitely generated. Therefore, $R_M^g = \theta^g(I) R_M^g$ and then $\theta^g(I) = R$.

Conversely, suppose $\theta^g(I) = R$. Then there exist $x_1, x_2, ..., x_n \in I \cap h(R)$ such that $R = \theta^g(I) = \langle x_1 \rangle I + \langle x_2 \rangle I + ... + \langle x_n \rangle I$. Thus,

$$I = \theta^g(I) = \langle x_1 \rangle I + \langle x_2 \rangle I + ... + \langle x_n \rangle I \subseteq \langle x_1 \rangle + \langle x_2 \rangle + ... + \langle x_n \rangle.$$

So, $I = \langle x_1, x_2, ..., x_n \rangle$ is gr-finitely generated. Now, let $M$ be a gr-maximal ideal of $R$. Since $\theta^g(I) = R$, there is $x \in I \cap h(R)$ with $\langle x \rangle I \not\subseteq M$. Therefore, there exists $r \in R - M$ with $rI \not\subseteq \langle x \rangle$ and then $r IR_M^g = \langle r \rangle R_M^g$. Hence, $IR_M^g = \langle x \rangle R_M^g$ for any gr-maximal ideal $M$ of $R$ and so $I$ is locally gr-principal.

Definition 2.12 A graded ideal $I$ of a graded ring $R$ is called meet-gr-principal if $JI \cap K = (J \cap (K : I))I$ for all graded ideals $J$ and $K$ of $R$.

We are ready now for the following characterization of gr-multiplication ideals similar to that in the non graded case; see [1].

Theorem 2.13 Let $I$ be a graded ideal in a graded ring $R$. Then the following are equivalent:

1. $I$ is meet-gr-principal.
2. $I$ is gr-multiplication.
3. $IR_M^g = 0$ for any gr-maximal ideal $M$ of $R$ with $M \supseteq \theta^g(I)$. 

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Proof. (1) $\Rightarrow$ (2): Let $J$ be a graded ideal of $R$ with $J \subseteq I$. Then $J = RI \cap J = (R \cap (J : I))I = (J : I)I$ and then $I$ is a gr-multiplication ideal by theorem 2.2.

(2) $\Rightarrow$ (3): Suppose that $I$ is a gr-multiplication ideal. Let $M$ be any gr-maximal ideal of $R$ such that $\theta^g(I) \subseteq M$. Let $x \in I \cap h(R)$. Then $\langle x \rangle$ is a graded ideal and $\langle x \rangle = \langle x \theta^g(I) \rangle$ by lemma 2.9 and so, $\langle x \rangle R^g_M = \langle x \rangle R^g_M \theta^g(I) R^g_M$. By proposition 2.4 in [4], we see that $\langle x \rangle R^g_M = \{0\} R^g_M$, and so $I R^g_M = \{0\} R^g_M$.

(3) $\Rightarrow$ (1): Let $J$ and $K$ be graded ideals of $R$. We prove that $JI \cap K = (J \cap (K : I))I$. Clearly, $JI \cap K \supseteq (J \cap (K : I))I$ is always true. We prove the other containment locally. Let $M$ be a gr-maximal ideal of $R$. If $\theta^g(I) \subseteq M$, then $I R^g_M = \{0\} R^g_M$ by assumption and so $(J \cap (K : I))I R^g_M = \{0\} R^g_M = (JI \cap K) R^g_M$.

Suppose that $\theta^g(I) \not\subseteq M$. Then $(\langle x \rangle : I) \not\subseteq M$ for some $x \in I \cap h(R)$ and so there is $r \in R$ such that $rI \subseteq \langle x \rangle$ and $r \not\in M$. Let $b = y_1z_1 + y_2z_2 + \ldots + y_nz_n \in JI \cap K$ where $y_k \in J$ and $z_k \in I$ for $k = 1, 2, \ldots, n$. Then there exist $r_1, r_2, \ldots, r_n \in R$ such that

$$rb = r(y_1z_1 + y_2z_2 + \ldots + y_nz_n) = y_1(r_1z_1) + y_2(r_2z_2) + \ldots + y_n(r_nz_n) = y_1(r_1x) + y_2(r_2x) + \ldots + y_n(r_nx) = (y_1r_1 + y_2r_2 + \ldots + y_nr_n)x.$$

where the third equality holds since $rI \subseteq \langle x \rangle$. Now,

$$(y_1r_1 + y_2r_2 + \ldots + y_nr_n)rI \subseteq (y_1r_1 + y_2r_2 + \ldots + y_nr_n)\langle x \rangle = \langle rb \rangle \subseteq K.$$

Hence,

$$(y_1r_1 + y_2r_2 + \ldots + y_nr_n)r \in J \cap (K : I),$$

and then

$$\frac{y_1r_1 + y_2r_2 + \ldots + y_nr_n}{1} \in (J \cap (K : I)) R^g_M.$$

Now,

$$\frac{b}{1} = \left(\frac{r}{1}\right)^{-1} (\frac{rb}{1}) = \left(\frac{r}{1}\right)^{-1} \frac{y_1r_1 + y_2r_2 + \ldots + y_nr_n}{1} \in (J \cap (K : I)) R^g_M I R^g_M.$$

Therefore, $(JI \cap K) R^g_M = (J \cap (K : I)) R^g_M I R^g_M$ for any gr-maximal ideal $M$ of $R$ and so $JI \cap K \subseteq (J \cap (K : I))I$.

We have the following as a corollary of the previous theorem and lemma 2.10.

Corollary 2.14 If $I$ and $J$ are gr-multiplication ideals of a graded ring $R$, then $IJ$ is gr-multiplication.

Proof. Let $M$ be a gr-maximal ideal of $R$ such that $\theta^g(IJ) \subseteq M$. Then $\theta^g(I)\theta^g(J) \subseteq \theta^g(IJ) \subseteq M$ and so either $\theta^g(I) \subseteq M$ or $\theta^g(J) \subseteq M$. Hence, by theorem 2.13, either $IR_M^g = 0R_M^g$ or $JR_M^g = 0R_M^g$. In both cases, $(IJ)R_M^g = 0R_M^g$ and then $IJ$ is a gr-multiplication ideal of $R$ again by theorem 2.13.

3. Gr-primary ideals with gr-multiplication gr-radicals

Definition 3.1 Let $P$ be a gr-prime ideal in a graded ring $R$. Then we define the graded rank of $P$ (denoted by $gr-rank(P)$) as the supremum of the lengths of all chains of distinct proper gr-prime ideals of $R$ having $P$. 211
as last term. The gr-dimension of a graded ring $R$ is defined as the supremum of the lengths of all chains of distinct gr-prime ideals of $R$ and is denoted by $\operatorname{gr-dim}(R)$.

Now, any gr-prime ideal in the graded ring $R^p_t$ is of the form $P' R^p_t$, where $P'$ is a gr-prime ideal of $R$ with $P' \subseteq P$. Therefore we conclude that $\operatorname{gr-dim}(R^p_t) = \operatorname{gr-rank}(P)$. Recall that a gr-prime ideal $P$ of $R$ is called minimal gr-prime over a graded ideal $I$ if there is no gr-prime ideal $Q$ of $R$ such that $I \subseteq Q \subseteq P$.

**Definition 3.2** Let $I$ be a graded ideal in a graded ring $R$. Then the graded rank of $I$ (denoted by $\operatorname{gr-rank}(I)$) is defined as the infimum of the values of $\operatorname{gr-rank}(P)$ as $P$ runs over all of the minimal gr-prime ideals of $I$.

**Theorem 3.3** Let $I$ be a gr-multiplication ideal in a graded ring $R$. If $\operatorname{gr-rank}(I) \geq 0$, then $I$ is gr-finitely generated.

**Proof.** Suppose that $I$ is not gr-finitely generated, then by theorem 2.11, $\theta^i(I) \neq R$ and, therefore, $\theta^i(I) \subseteq M$ for some gr-maximal ideal $M$ of $R$. Hence, $IR^0_M = 0R^0_M$ by theorem 2.13 and so $\operatorname{gr-rank}(I) \leq \operatorname{gr-rank}(IR^0_M) = 0$, a contradiction.

Now, in the following main theorem, we determine the set of all gr-$P$-primary ideals of a graded ring $R$ where $P$ is any gr-prime ideal of $R$ that is gr-multiplication. First, we have the following lemma.

**Lemma 3.4** If $I$ is a graded ideal of a graded ring $R$ such that $g$-rad$(I)$ is gr-finitely generated, then there exists a positive integer $t$ such that $(g$-rad$(I))^t \subseteq I$.

**Proof.** Suppose that $g$-rad$(I) = \langle a_1, a_2, ..., a_n \rangle$ for $a_1, a_2, ..., a_n \in h(R)$. Then there exist $t_1, t_2, ..., t_n \in \mathbb{N}$ such that $a_i^{t_i} \in I$. Let $t = 1 + \sum_{i=1}^{n} (t_i - 1)$. Then $(g$-rad$(I))^t$ is a graded ideal generated by

$$L = \left\{ a_1^{k_1} a_2^{k_2} ... a_n^{k_n} : k_1, k_2, ..., k_n \in \mathbb{N}, \sum_{i=1}^{n} k_i = t \right\}.$$  

If $k_i \leq t_i$ for all $i = 1, 2, ..., n$, then $\sum_{i=1}^{n} k_i \leq \sum_{i=1}^{n} (t_i - 1) = t$, a contradiction. Therefore, there exists $j$, $1 \leq j \leq n$ such that $k_j \geq t_j$ and then $a_1^{k_1} a_2^{k_2} ... a_i^{k_i} ... a_n^{k_n} \in I$. Hence, $L \subseteq I$ and then $(g$-rad$(I))^t \subseteq I$.

**Theorem 3.5** Let $P$ be a gr-prime ideal of a graded ring $R$ that is gr-multiplication. If $\operatorname{gr-rank}(P) \geq 0$, then \{${P^n}_{n=1}^{\infty}$\} is the set of gr-$P$-primary ideals of $R$. If $\operatorname{gr-rank}(P) = 0$, then there is a least positive integer $m$ with $(PR^0_P)^m = 0R^0_P$ and in this case \{${P^n}_{n=1}^{\infty}$\} is the set of gr-$P$-primary ideals of $R$.

**Proof.** Suppose that $\operatorname{gr-rank}(P) \geq 0$, then $P$ is gr-finitely generated by theorem 3.3. Let $Q$ be a gr-$P$-primary ideal in $R$. Then $P^t \subseteq Q$ for some positive integer $t$ by lemma 3.4. By passing to the graded ring $R/P^t$, we have, $Q/P^t$ is gr-$P/P^t$-primary ideal and since clearly, $\operatorname{gr-rank}(P/P^t) = 0$, it is enough to consider the case where $\operatorname{gr-rank}(P) = 0$. Since $P$ is gr-multiplication, then $PR^0_P$ is gr-principal by theorem 2.3 and since $\operatorname{gr-rank}(P) = 0$, then $PR^0_P$ is the only gr-prime ideal of $R^0_P$ and each graded ideal of $R^0_P$ is a power...
of $PR_P^g$. Hence, $R_P^g$ is a gr-SPIR and so there is a least positive integer $m$ such that $(PR_P^g)^m = \langle 0 \rangle R_P^g$ and the only graded ideals (which are gr-$PR_P^g$-primary) of $R_P^g$ are $PR_P^g, (PR_P^g)^2, \ldots, (PR_P^g)^m$. Therefore, the only gr-$P$-primary ideals of $R$ are $P^{(1)}_g, P^{(2)}_g, \ldots, P^{(m)}_g$. Now, for a fixed $i$, $1 \leq i \leq m$, we have $P_i \subseteq P^{(i)}_g$. Suppose $k$ is the largest integer with $P_i \subseteq P^k$. Since by corollary 2.14, $P_k$ is gr-multiplication, there is a graded ideal $A$ of $R$ such that $P^{(i)}_g = AP^k$ where $A \not\subseteq P$. Since $P^{(i)}_g$ is gr-$P$-primary, $P^k \subseteq P^{(i)}_g$ and so, $P^k = P^{(i)}_g$. Now, $(P^{(k)}_g) R_P^g = P^k R_P^g = P^{(i)}_g R_P^g = P^i R_P^g$ and therefore, $i = k$ and $P^{(i)}_g = P^i$. It follows that $P, P^2, \ldots, P^m$ are the only gr-$P$-primary ideals of $R$. \hfill \Box

References


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