Modified Szász-Mirakjan-Kantorovich Operators Preserving Linear Functions

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Abstract

In this paper, we introduce a modification of the Szász-Mirakjan-Kantorovich operators, which preserve the linear functions. This type of operator modification enables better error estimation on the interval \([1/2, +\infty)\) than the classical Szász-Mirakjan-Kantorovich operators. We also obtain a Voronovskaya-type theorem for these operators.

Key Words: Szász-Mirakjan operators, Szász-Mirakjan-Kantorovich operators, the Korovkin-type approximation theorem, modulus of continuity, Lipschitz class functionals, Voronovskaya type theorem

1. Introduction

Previous studies demonstrate that providing a better error estimation for positive linear operators plays an important role in approximation theory, which allows us to approximate much faster to the function being approximated. In [1, 6, 11], various approximation properties of the classical Szász-Mirakjan operators and Szász-Mirakjan-Kantorovich operators were investigated. Recently, in [3], by modifying the Szász-Mirakjan operators, we have showed that our modified operators have better error estimation than the classical ones. We should recall that such investigations were accomplished for Bernstein polynomials by King [7], for Meyer-König and Zeller operators by Özarslan and Duman [9] and for Szász-Mirakjan-Beta operators by Duman, Özarslan and Aktuğlu [4]. In this paper, we apply our method to the classical Szász-Mirakjan-Kantorovich operators.

Consider the Banach lattice

\[ C_{\gamma}[0, +\infty) := \{ f \in C[0, +\infty) : |f(t)| \leq M(1 + t)^{\gamma} \text{ for some } M > 0, \gamma > 0 \}. \]

Then, the classical Szász-Mirakjan operators are defined by

\[ S_n(f; x) := e^{-nx} \sum_{k=1}^{\infty} \frac{(nx)^k}{k!} f \left( \frac{k}{n} \right), \]

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where \( f \in C_\gamma[0, +\infty), \ x \geq 0 \) and \( n \in \mathbb{N} \). Various approximation properties of the Szász-Mirakjan operators and their iterates may be found in [1, 3, 4, 5, 6, 8, 10, 11, 12] and the references cited therein.

The Kantorovich version of the Szász-Mirakjan operators are defined by

\[
K_n(f; x) := ne^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \int_{I_{n,k}} f(t) dt, \tag{1.1}
\]

where \( I_{n,k} = \left[ \frac{k}{n}, \frac{k+1}{n} \right] \) and \( f \in C_\gamma[0, +\infty) \).

Now, for the Szász-Mirakjan-Kantorovich operators \( K_n \) given by (1.1), the following lemma follows from [6] immediately.

**Lemma A** [6]. Let \( e_i(x) = x^i, \ i = 0, 1, 2, 3, 4 \). Then, for each \( x \geq 0 \), and \( n > 1 \), we have

(a) \( K_n(e_0; x) = 1 \),

(b) \( K_n(e_1; x) = x + \frac{1}{2n} \),

(c) \( K_n(e_2; x) = x^2 + \frac{2x}{n} + \frac{1}{3n^2} \),

(d) \( K_n(e_3; x) = x^3 + \frac{9x^2}{2n} + \frac{7x}{2n^2} + \frac{1}{4n^3} \),

(e) \( K_n(e_4; x) = x^4 + \frac{8x^3}{n} + \frac{15x^2}{n^2} + \frac{6x}{n^3} + \frac{1}{5n^4} \).

### 2. Construction of the Operators

The set \( \{e_0, e_1, e_2\} \) is a \( K_+ \)-subset of \( C_\gamma[0, +\infty) \) for \( \gamma \geq 2 \); also the space \( C_\gamma[0, +\infty) \) is isomorphic to \( C[0,1] \). Recall that a subset \( H \) of \( C_\gamma[0, +\infty) \) is called a Korovkin subset with respect to positive linear operators or, briefly, a \( K_+ \)-subset of \( C_\gamma[0, +\infty) \) if it satisfies the following property:

if \( \{L_n\} \) is an arbitrary sequence of positive linear operators
from \( C_\gamma[0, +\infty) \) into itself such that \( \lim_{n \to \infty} L_n(h) = h \) for all \( h \in H \),
then \( \lim_{n \to \infty} L_n(f) = f \) for every \( f \in C_\gamma[0, +\infty) \)

(see [2] for details).

Let \( \{r_n(x)\} \) be a sequence of real-valued continuous functions defined on \( [0, +\infty) \) with \( 0 \leq r_n(x) < +\infty \).

Then we have

\[
K_n(f; r_n(x)) := ne^{-nr_n(x)} \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{k!} \int_{I_{n,k}} f(t) dt.
\]
Now, if we replace \( r_n(x) \) by \( r_n^*(x) \) defined as
\[
r_n^*(x) := x - \frac{1}{2n}, \quad x \geq \frac{1}{2} \text{ and } n \in \mathbb{N},
\] then we get the following positive linear operators:
\[
K_n^*(f; x) := ne^{-\frac{1}{2n}} \sum_{k=0}^{\infty} \frac{(2nx - 1)^k}{2^k k!} \int_{1_n,k} f(t) dt,
\] where \( f \in C_{\gamma}[0, +\infty), \gamma > 0 \) and \( x \geq 1/2 \). Observe that if \( x \in [1/2, +\infty) \), then \( r_n^*(x) \) given by (2.2) belongs to the interval \([0, +\infty)\).

On the other hand, from Lemma A we obtain the following result at once.

**Lemma 2.1** For each \( x \geq 1/2 \), we have

(a) \( K_n^*(e_0; x) = 1 \),

(b) \( K_n^*(e_1; x) = x \),

(c) \( K_n^*(e_2; x) = x^2 + \frac{x}{n} - \frac{5}{12n^2} \),

(d) \( K_n^*(e_3; x) = x^3 + \frac{3x^2}{n} - \frac{x}{4n^2} - \frac{1}{2n^3} \),

(e) \( K_n^*(e_4; x) = x^4 + \frac{6x^3}{n} + \frac{9x^2}{2n^2} - \frac{7x}{2n^3} - \frac{1}{80n^4} \).

By Lemma 2.1, it is clear that the positive linear operators \( K_n^* \) given by (2.3) preserve the linear functions, that is, for \( h(t) = ct + d \) (\( c \) and \( d \) are any real numbers), \( K_n^*(h; x) = h(x) \) for all \( x \geq 1/2 \) and \( n \in \mathbb{N} \).

Now, fix \( b > 1/2 \) and consider the lattice homomorphism \( T_b : C[0, +\infty) \to C[0, b] \) defined by \( T_b(f) := f|_{[0,b]} \) for every \( f \in C[0, +\infty) \), where \( f|_{[0,b]} \) denotes the restriction of the domain of \( f \) to the interval \([0, b]\). In this case, we see that, for each \( i = 0, 1, 2, \ldots \),
\[
\lim_{n \to \infty} T_b(K_n^*(e_i)) = T_b(e_i) \quad \text{uniformly on } [1/2, b].
\] (2.4)

Thus, by using (2.4) and with the universal Korovkin-type property with respect to monotone operators (see Theorem 4.1.4 (vi) of [2, p. 199]) we have the following Korovkin-type approximation result.

**Theorem 2.2** \( \lim_{n \to \infty} K_n^*(f; x) = f(x) \) uniformly with respect to \( x \in [1/2, b] \) provided \( f \in C_{\gamma}[0, +\infty), \gamma \geq 2 \) and \( b > 1/2 \).

In order to get uniform convergence on \([1/2, +\infty)\) of the sequence \( \{K_n^*(f)\} \) we consider the following subspace \( E \) of \( C_{\gamma}[0, +\infty) \):
\[
E := \left\{ f \in C[0, +\infty) : \lim_{t \to +\infty} f(t) \text{ is finite} \right\}
\]
endowed with the sup-norm.

For a given $\lambda > 0$, consider the function $f_\lambda(t) := e^{-\lambda t}$, $(t \geq 0)$. Then, for every $x \geq 1/2$ and $n \in \mathbb{N}$, we have

\[
K_n^*(f_\lambda; x) = ne^{-\frac{\lambda}{2n^2}} \sum_{k=0}^{\infty} \frac{(2nx - 1)^k}{2^k k!} \int_{I_{n,k}} e^{-\lambda t} dt
\]

\[
= \frac{n (1 - \exp(-\lambda/n))}{\lambda} \times \exp\left(\frac{-n(x-1/2n)}{1 - \exp(-\lambda/n)}\right) \sum_{k=0}^{\infty} \frac{(n(x-1/2n)e^{-\lambda/n})^k}{k!}
\]

\[
= \frac{n (1 - \exp(-\lambda/n))}{\lambda} \times \exp\left\{-n\left(x - \frac{1}{2n}\right)(1 - \exp(-\lambda/n))\right\}.
\]

Since $\lim_{n \to \infty} n (1 - \exp(-\lambda/n)) = \lambda$, we conclude that

\[
\lim_{n \to \infty} K_n^*(f_\lambda) = f_\lambda \text{ uniformly on } [1/2, +\infty).
\]

Hence using this limit and applying Proposition 4.2.5-(7) of [2, p. 215] one can obtain the next result at once.

**Theorem 2.3** $\lim_{n \to \infty} K_n^*(f) = f$ uniformly on $[1/2, +\infty)$ provided $f \in E$.

We can also give an $L_p$-approximation for the operators $K_n^*(f; x)$ by using Proposition 4.2.5-(2) of [2, p. 215] as follows.

**Corollary 2.4** Let $1 \leq p < +\infty$. Then, for all $f \in L_p[0, +\infty)$, $\lim_{n \to \infty} K_n^*(f; x) = f(x)$ uniformly with respect to $x \in [1/2, +\infty)$.

3. Better Error Estimation

In this section we compute the rate of convergence of the operators $K_n^*$ defined by (2.3). Then, we will show that our operators have a better error estimation on the interval $[1/2, +\infty)$ than the Szász-Mirakjan-Kantorovich operators $K_n$ given by (1.1). To achieve this we use the modulus of continuity and the elements of Lipschitz class functionals.

If we define the function $\psi_x$, $(x \geq 0)$, by $\psi_x(t) = t - x$, then by Lemma 2.1 one can get the following result, immediately.

**Lemma 3.1** For every $x \geq 1/2$, we have

(a) $K_n^*(\psi_x; x) = 0$, 
(b) $K_n^*(\psi_x^2; x) = \frac{x}{n} - \frac{5}{12n^2}$, 
(c) $K_n^*(\psi_x^3; x) = \frac{x}{n^2} - \frac{1}{2n^3}$.
Proof. Now, applying Lemma 3.1 (Remark. For the Szász-Mirakjan-Kantorovich operators given by (1.1) we may write that, for every \( \delta > 0 \), the inequality

\[
\omega(f, \delta) = \sup_{x - \delta \leq t \leq x + \delta, \ t \in [0, +\infty)} |f(t) - f(x)|.
\]

This guarantees that

\[
\alpha_n(x) = \frac{x}{n} \quad \text{and} \quad \alpha_n(x) = \frac{x}{n} + \frac{1}{3n^2}.
\]

Then we have the following theorem.

**Theorem 3.2** For every \( f \in C_B[0, +\infty) \), \( x \geq 1/2 \) and \( n \in \mathbb{N} \), we have

\[
|K_n^*(f; x) - f(x)| \leq 2 \omega(f, \delta_{n,x}),
\]

where \( \delta_{n,x} := \sqrt{\frac{x}{n} - \frac{5}{12n^2}} \).

**Proof.** Now, let \( f \in C_B[0, +\infty) \) and \( x \geq 0 \). Using linearity and monotonicity of \( K_n^* \) we easily get, for \( \delta > 0 \) and \( n \in \mathbb{N} \), that

\[
|K_n^*(f; x) - f(x)| \leq \omega(f, \delta) \left\{ 1 + \frac{1}{\delta} \sqrt{K_n^*(\psi_2^4; x)} \right\}.
\]

Now applying Lemma 3.1 (b) and choosing \( \delta = \delta_{n,x} \), the proof is complete. \( \square \)

**Remark.** For the Szász-Mirakjan-Kantorovich operators given by (1.1) we may write that, for every \( f \in C_B[0, +\infty) \), \( x \geq 0 \) and \( n \in \mathbb{N} \),

\[
|K_n(f; x) - f(x)| \leq 2 \omega(f, \alpha_{n,x}),
\]

where \( \alpha_{n,x} := \sqrt{\frac{x}{n} + \frac{1}{3n^2}} \) (see [5, 6]).

Now we claim that the error estimation in Theorem 3.2 is better than that of (3.5) provided \( f \in C_B[0, +\infty) \) and \( x \geq 1/2 \). Indeed, for \( x \geq 1/2 \) and \( n \in \mathbb{N} \), it is clear that

\[
\frac{x}{n} - \frac{5}{12n^2} \leq \frac{x}{n} + \frac{1}{3n^2}.
\]

This guarantees that \( \delta_{n,x} \leq \alpha_{n,x} \) for \( x \geq 1/2 \) and \( n \in \mathbb{N} \).

Now we can also compute the rate of convergence of the operators \( K_n^* \) by means of the elements of the Lipschitz class \( Lip_M(\alpha) \), \( (\alpha \in (0, 1]) \). As usual, we say that a function \( f \in C_B[0, +\infty) \) belongs to \( Lip_M(\alpha) \) if the inequality

\[
|f(t) - f(x)| \leq M |t - x|^\alpha
\]

holds for all \( t \in [0, +\infty) \) and \( x \in [1/2, +\infty) \).

**Theorem 3.3** For every \( f \in Lip_M(\alpha) \), \( x \geq 1/2 \) and \( n \in \mathbb{N} \), we have

\[
|K_n^*(f; x) - f(x)| \leq M \left\{ \frac{x}{n} - \frac{5}{12n^2} \right\}^{\frac{\alpha}{2}}.
\]
Proof. Since $f \in \text{Lip}_M(\alpha)$ and $x \geq 0$, using inequality (3.7) and then applying the Hölder inequality with $p = \frac{2}{\alpha}$, $q = \frac{2}{2-\alpha}$ we get

$$|K^*_n(f; x) - f(x)| \leq K^*_n(|f(t) - f(x)|; x) \leq M K^*_n(|t-x|^\alpha; x) \leq M \left\{\frac{x}{n} - \frac{5}{12n^2}\right\}^{\frac{2}{\alpha}},$$

whence the result. □

Notice that as in the proof of Theorem 3.2, since $K_n(\psi^2_{4}; x) = \frac{x^2}{n} + \frac{1}{3n^2}$, the Szász-Mirakjan-Kantorovich operators defined by (1.1) satisfy

$$|K_n(f; x) - f(x)| \leq M \left\{\frac{x}{n} + \frac{1}{3n^2}\right\}^{\frac{2}{\alpha}}$$

for every $f \in \text{Lip}_M(\alpha)$, $x \geq 1/2$ and $n \in \mathbb{N}$. So, it follows from (3.6) that the above claim also holds for Theorem 3.2, i.e., the rate of convergence of the operators $K^*_n$ by means of the elements of the Lipschitz class functionals is better than the ordinary error estimation given by (3.8) whenever $x \geq 1/2$ and $n \in \mathbb{N}$.

4. A Voronovskaya-Type Theorem

In this section, we prove a Voronovskaya-type theorem for the operators $K^*_n$ given by (2.3).

We first need the following lemma.

Lemma 4.1 \lim_{n \to \infty} n^2 K^*_n(\psi^4_{4}; x) = 3x^2 uniformly with respect to $x \in [1/2, b]$ ($b > 1/2$).

Proof. Then, by Lemma 3.1 (d), we may write that

$$n^2 K^*_n(\psi^4_{4}; x) = 3x^2 - \frac{3x}{2n} - \frac{1}{80n^2}.$$

Now taking limit as $n \to \infty$ on the both sides of the above equality the proof is complete. □

Theorem 4.2 For every $f \in C_{\gamma}[0, +\infty)$ such that $f'$, $f'' \in C_{\gamma}[0, +\infty)$, $\gamma \geq 4$, we have

$$\lim_{n \to \infty} n \{K^*_n(f; x) - f(x)\} = \frac{1}{2} x f''(x)$$

uniformly with respect to $x \in [1/2, b]$ ($b > 1/2$).

Proof. Let $f$, $f'$, $f'' \in C_{\gamma}[0, +\infty)$ and $x \geq 1/2$. Define

$$\Psi(t, x) = \begin{cases} \frac{f(t) - f(x) - (t-x)f'(x) - \frac{1}{2}(t-x)^2f''(x)}{(t-x)^2}, & \text{if } t \neq x \\ 0, & \text{if } t = x. \end{cases}$$
Then by assumption we have \( \Psi(x, x) = 0 \) and the function \( \Psi(\cdot, x) \) belongs to \( C_\gamma[0, +\infty) \). Hence, by Taylor’s theorem we get
\[
f(t) = f(x) + (t - x)f'(x) + \frac{(t - x)^2}{2} f''(x) + (t - x)^2 \Psi(t, x).
\]
Now from Lemma 3.1 \((a) - (b)\)
\[
n \{ K_n^*(f; x) - f(x) \} = \frac{n}{2} \left( \frac{x}{n} - \frac{5}{12n^2} \right) f''(x) + n K_n^*(\psi^2(t) \Psi(t, x); x).
\]
If we apply the Cauchy-Schwarz inequality for the second term on the right-hand side of (4.9), then we conclude that
\[
n |K_n^*(\psi^2(t) \Psi(t, x); x)| \leq \left( n^2 K_n^*(\psi^4(t); x) \right)^{\frac{1}{2}} \left( K_n^*(\psi^2(t, x)) \right)^{\frac{1}{2}}.
\]
Let \( \eta(t, x) := \Psi^2(t, x) \). In this case, observe that \( \eta(x, x) = 0 \) and \( \eta(\cdot, x) \in C_\gamma[0, +\infty) \). Then it follows from Theorem 2.2 that
\[
\lim_{n \to \infty} K_n^*(\psi^2(t) \Psi(t, x); x) = \lim_{n \to \infty} K_n^*(\eta(t, x); x) = \eta(x, x) = 0
\]
uniformly with respect to \( x \in [1/2, b] \) \((b > 1/2)\). Now considering (4.10) and (4.11), and also using Lemma 4.1, we immediately see that
\[
\lim_{n \to \infty} n K_n^*(\psi^2(t) \Psi(t, x); x) = 0
\]
uniformly with respect to \( x \in [1/2, b] \). On the other hand, observe now that, by (3.6),
\[
\lim_{n \to \infty} \frac{n}{2} \left( \frac{x}{n} - \frac{5}{12n^2} \right) = \frac{1}{2} x.
\]
Then, taking limit as \( n \to \infty \) in (4.9) and using (4.12) and (4.13) we have
\[
\lim_{n \to \infty} n \{ K_n^*(f; x) - f(x) \} = \frac{1}{2} x f''(x)
\]
uniformly with respect to \( x \in [1/2, b] \) with \( b > 1/2 \). So, the proof is completed.

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