Perturbation of Closed Range Operators

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Abstract

Let $T, A$ be operators with domains $D(T) \subseteq D(A)$ in a normed space $X$. The operator $A$ is called $T$-bounded if $\|Ax\| \leq a\|x\| + b\|Tx\|$ for some $a, b \geq 0$ and all $x \in D(T)$. If $A$ has the Hyers-Ulam stability then under some suitable assumptions we show that both $T$ and $S := A + T$ have the Hyers-Ulam stability. We also discuss the best constant of Hyers-Ulam stability for the operator $S$. Thus we establish a link between $T$-bounded operators and Hyers-Ulam stability.

Key Words: Hilbert space; perturbation; Hyers-Ulam stability; closed operator; semi-Fredholm operator.

1. Introduction and preliminaries

Let $X, Y$ be normed linear spaces and $T$ be a (not necessarily linear) mapping from $X$ into $Y$. Following [5, 6] we say that $T$ has the Hyers-Ulam stability if there exists a constant $K > 0$ with the property:

(i) For any $y$ in the range $\mathcal{R}(T)$ of $T$, $\varepsilon > 0$ and $x \in X$ with $\|T(x) - y\| \leq \varepsilon$, there exists a $x_0 \in X$ such that $T(x_0) = y$ and $\|x - x_0\| \leq K\varepsilon$.

We call such $K > 0$ a Hyers-Ulam stability constant for $T$ and denote by $K_T$ the infimum of all Hyers-Ulam stability constants for $T$. If $K_T$ is a Hyers-Ulam stability constant for $T$, then $K_T$ called the Hyers-Ulam stability constant for $T$.

If $T$ is linear then condition (i) is equivalent to:

(ii) For any $\varepsilon > 0$ and $x \in X$ with $\|Tx\| \leq \varepsilon$, there exists a $x_0 \in X$ such that $Tx_0 = 0$ and $\|x - x_0\| \leq K\varepsilon$.

If put $\mathcal{N}(T) := \{x \in X : Tx = 0\}$, condition (ii) is equivalent to

(iii) For any $x \in X$ there exists a $x_0 \in \mathcal{N}(T)$ such that $\|x - x_0\| \leq K\|Tx\|$.

We refer the interested reader for more results on the stability of various mappings to papers [10, 11, 12] and references therein, and for a comprehensive accounts of the Hyers-Ulam-Rassias stability of functional equations to the monographs [3, 8, 13].

In [6] the authors proved the following useful result.

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Theorem 1.1 Let $T$ be a closed operator from the subspace $D(T)$ of a Hilbert space $H$ into a Hilbert space $K$. The following assertions are equivalent:

(i) $T$ has the Hyers-Ulam stability;

(ii) $T$ has closed range.

Moreover, if one of the conditions above is true, then $K_T = \gamma(T)^{-1}$, where

$$\gamma(T) = \sup\{\gamma > 0 : \|Tx\| \geq \gamma\|x\|, \quad x \in D(T) \cap (N(T))^\perp\}.$$ 

(Here $\perp$ denotes the orthogonal complement in Hilbert spaces.)

Let $X$ be a Banach space and let $M, N$ be closed linear subspaces of $X$. Following [9] we define the quantity

$$\delta(M, N) := \inf\{\frac{\text{dist}(x, N)}{\text{dist}(x, M \cap N)} : x \in M, x \notin N\}(\leq 1)$$

If $M \subseteq N$, then we set $\delta(M, N) = 1$. Obviously $\delta(M, N) = 1$, if $M \supseteq N$. It is well known that $\delta(M, N)$ is not symmetric with respect to $(M, N)$. If $\delta(M, N) = \delta(N, M)$, we say that the pair $(M, N)$ is regular. It is known that any pair $(M, N)$ is regular if $X$ is a Hilbert space [9].

Let $A$ and $T$ be operators with their domains in a normed space $X$ such that $D(T) \subseteq D(A)$, and

$$\|Ax\| \leq a\|x\| + b\|Tx\| \quad (x \in D(T)),$$ (1.1)

where $a, b$ are nonnegative constants. Then we say that $A$ is relatively bounded with respect to $T$ or simply it is $T$-bounded [9].

A bounded operator $A$ is clearly $T$-bounded for any $T$ with $D(T) \subseteq D(A)$.

In this paper, we show that if a $T$-bounded operator $A$ has the Hyers-Ulam stability then under some suitable assumptions the operator $T$ and the perturbation $S := A + T$ have the Hyers-Ulam stability. We also discuss the best constant of Hyers-Ulam stability for the operator $S$. Thus we establish a link between $T$-bounded operators and the Hyers-Ulam stability.

2. Main Results

Throughout this section $H$ and $K$ denote Hilbert spaces and $A$ and $T$ are operators having their domains in $H$ and their images in $K$. We start our work with the following theorem.

Theorem 2.1 Suppose that $A$ is a $T$-bounded operator with a $T$-bound smaller than 1. If $T$ is a closed operator and $S := T + A$, then the following assertions are equivalent:

(i) $S$ has the Hyers-Ulam stability;

(ii) $S$ has closed range.

Moreover, if $A$ is closed and the operators $A$ and $T$ have the Hyers-Ulam stability and $R(S) = R(A) + R(T)$ then conditions (i) and (ii) are equivalent with the following assertions:

(iii) $\delta(M, N) > 0$, where $M = R(A)$ and $N = R(T)$;

(iv) $\delta(M, N) > 0$, $M = R(A)$ and $N = R(T)$. 

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Ulam stability, and Corollary 2.4

Suppose that \( S \)

Therefore operator \( S = T + A \) has the Hyers-Ulam stability.

Remark 2.2 If \( A \) and \( T \) are closed operators as in the above theorem, the operators \( A \) and \( T \) have the Hyers-Ulam stability, \( S := T + A, \) \( \mathcal{R}(S) = \mathcal{R}(A) + \mathcal{R}(T) \) and we have \( \mathcal{R}(A) \subseteq \mathcal{R}(T) \) or \( \mathcal{R}(T) \subseteq \mathcal{R}(A) \) then \( \delta(\mathcal{R}(A), \mathcal{R}(T)) > 0. \) Hence the operator \( S \) has the Hyers-Ulam stability and therefore its range is closed.

Corollary 2.3 Suppose that \( A \) is a \( T \)-bounded operator with a \( T \)-bound smaller than \( 1. \) Let \( A \) and \( T \) be closed, \( S := A + T \) and let \( A \) and \( T \) have the Hyers-Ulam stability. Suppose that at least one of the spaces \( \mathcal{R}(A) \) or \( \mathcal{R}(T) \) is finite dimensional and assume that \( \mathcal{R}(S) = \mathcal{R}(A) + \mathcal{R}(T). \) Then operator \( S \) has the Hyers-Ulam stability and so it has closed range.

Proof. Without loss of generality assume that \( \mathcal{R}(A) \) is finite dimensional. It is known that there exists \( u \in \mathcal{R}(T) \) such that \( \text{dist}(u, \mathcal{R}(A)) = \|u\| \) (see [2]). Hence

\[
\delta(\mathcal{R}(A), \mathcal{R}(T)) = \delta(\mathcal{R}(T), \mathcal{R}(A)) > 0.
\]

Therefore operator \( S = T + A \) has the Hyers-Ulam stability.

Corollary 2.4 Suppose that \( A \) is a \( T \)-bounded operator with a \( T \)-bound smaller than \( 1. \) Let \( A \) and \( T \) be closed, \( S := A + T \) and let \( A, T \) and \( S \) have the Hyers-Ulam stability. If \( \mathcal{R}(A) \cap \mathcal{R}(T) = \emptyset, \) then \( \delta(\mathcal{R}(T), \mathcal{R}(A)) = 1 \) and

\[
K_S \leq \min\left\{\frac{1}{\gamma(T)}, \frac{1}{\gamma(A)}\right\}.
\]

Proof. Each \( z \in \mathcal{R}(S) \) has a unique expression as \( z = x + y \) in which \( y \in \mathcal{R}(T) \) and \( x \in \mathcal{R}(A). \) Consider the projection \( P \) of \( \mathcal{R}(S) \) onto \( \mathcal{R}(T) \) along \( \mathcal{R}(A). \) Now we have

\[
1 = \|P\| = \sup_{z \in \mathcal{R}(S)} \frac{\|Pz\|}{\|z\|} = \sup_{y \in \mathcal{R}(T), x \in \mathcal{R}(A)} \frac{\|y\|}{\|x + y\|} = \sup_{y \in \mathcal{R}(T)} \frac{\|y\|}{\text{dist}(y, \mathcal{R}(A))} = \delta(\mathcal{R}(A), \mathcal{R}(T))^{-1}.
\]

By the definition of \( \gamma(T), \) we have \( \|Tv\| \geq \gamma(T)\|v\|. \) Hence \( \|P\|\|Tv + Av\| \geq \|P(Tv + Av)\| \geq \gamma(T)\|v\|. \) So \( \|Se\| \geq \gamma(T)\|v\|. \) Since \( \gamma(S) \geq \gamma(T), \) by [6, Theorem 3.1], we have \( K_S \leq \frac{1}{\gamma(T)}. \) We can analogously show that \( K_S \leq \frac{1}{\gamma(A)}. \) Thus \( K_S \leq \min\left\{\frac{1}{\gamma(T)}, \frac{1}{\gamma(A)}\right\}. \)

Recall that if \( x, y \) are elements of the Hilbert space \( \mathcal{H}, \) then the bounded operator \( x \otimes y \) defined on \( \mathcal{H} \) by \( (x \otimes y)(z) = \langle z, y \rangle x \) is rank one if \( x, y \) are not zero. Let \( x_1, x_2, y \) be elements of \( \mathcal{H} \) such that \( \|x_1\| \leq \frac{\|x_2\|}{2}. \)

If \( A = x_1 \otimes y, T = x_2 \otimes y \) and \( S = A + T, \) then \( \mathcal{N}(A) = \mathcal{N}(T) \) and \( \|Ax\| \leq \frac{\|x_2\|}{2}. \) It is clear that \( A, T \) and
$S$ have the Hyers-Ulam stability (note that they have closed range). This motivates us toward the following theorem.

**Theorem 2.5** Suppose that $A$ is a $T$-bounded operator with a $T$-bound $b$ and a constant $a$ and $A$ has the Hyers-Ulam stability.

If $a = 0$ and $\mathcal{N}(A) = \mathcal{N}(T)$, then $T$ has also the Hyers-Ulam stability.

**Proof.** There exists a constant $K_0 > 0$ such that for every $x \in \mathcal{D}(A)$ there exists $x_0 \in \mathcal{N}(A) = \mathcal{N}(T)$ such that $\|x - x_0\| \leq K_0 \|Ax\| \leq K_0 b \|Tx\|$. Thus operator $T$ has the Hyers-Ulam stability. $\square$

Now we show that conditions $\mathcal{N}(A) = \mathcal{N}(T)$ and $a = 0$ in Theorem 2.5 are necessary.

**Example 2.6** Consider the operators $A, T : \ell^2 \to \ell^2$ defined by

$$A(x_1, x_2, \cdots) = (x_1, 0, 0, \cdots), \quad (x_1, x_2, \cdots) \in \ell^2$$

and

$$T(x_1, x_2, \cdots) = \left( x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots \right), \quad (x_1, x_2, \cdots) \in \ell^2.$$  

It is clear that the operator $A$ is $T$-bounded with constant $a = 0$. Then $\mathcal{R}(A)$ is of finite dimension. Hence the operator $A$ has closed range. Hence $A$ has the Hyers-Ulam stability and $\mathcal{N}(A) \neq \mathcal{N}(T)$. If we take $a_n$ to be

$$a_n = \begin{cases} 1 & i \leq n \\ 0 & i > n \end{cases}$$

then

$$(T a_n)(i) = \begin{cases} 1/i & i \leq n \\ 0 & i > n \end{cases}$$

and $(T a_n)$ converges to $b = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$ which does not belong to the range of $T$. Therefore $\mathcal{R}(T)$ is not closed, i.e, operator $T$ does not have the Hyers-Ulam stability.

**Example 2.7** Consider the operators $A, T : \ell^2 \to \ell^2$ defined by

$$A(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots), \quad (x_1, x_2, \cdots) \in \ell^2$$

and

$$T(x_1, x_2, \cdots) = \left( x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots \right), \quad (x_1, x_2, \cdots) \in \ell^2.$$  

The operator $A$ is $T$-bounded with a nonzero constant $a$. Since $\gamma(A) > 0$, the operator $A$ has closed range and $\mathcal{N}(A) = \mathcal{N}(T)$. The space $\mathcal{R}(T)$ is not closed, i.e, operator $T$ does not have the Hyers-Ulam stability.
Let \( x_1, x_2, y \) be elements of \( \mathcal{H} \) such that \( x_1 \perp x_2 \). If \( A = x_1 \otimes y, T = x_2 \otimes y \) and \( S = A + T \), then
\[
\gamma(A) = \|x_1\|\|y\|, \quad \gamma(T) = \|x_2\|\|y\| \quad \text{and} \quad \gamma(S) = \gamma(A) + \gamma(T),
\]
therefore \( K_S = \gamma(S)^{-1} = \frac{1}{\gamma(A) + \gamma(T)} \). This motivates us toward the following result.

**Corollary 2.8** Suppose that \( A \) is a \( T \)-bounded operator with a \( T \)-bound \( b \) smaller than 1 and constant \( a = 0 \), \( \mathcal{N}(A) = \mathcal{N}(T) \) and \( A \) has the Hyers-Ulam stability. Then \( S := T + A \) has the Hyers-Ulam stability, if \( \mathcal{R}(A) \perp \mathcal{R}(T) \). Moreover, if \( T \) is a closed operator then \( \mathcal{R}(S) \) is closed and \( K_S = \frac{1}{\gamma(T) + \gamma(A)} \).

**Proof.** Suppose that \( K \) is a Hyers-Ulam stability constant for \( A \). By Theorem 2.5, \( K' = Kb \) is a Hyers-Ulam stability constant for \( T \). In fact, for each \( v \in \mathcal{D}(T) \) there exists \( v_0 \in \mathcal{N}(T) \) such that
\[
\|v - v_0\| \leq (Kb)\|Tv\| \leq K\|Tv\|
\]
since \( b \) is smaller than 1.

Hence for \( x \in \mathcal{D}(S) = \mathcal{D}(T) \) there exists \( x_0 \in \mathcal{N}(T) = \mathcal{N}(A) \) such that
\[
\|x - x_0\| \leq K(\|Ax\| + \|Tx\|) = K\|Ax + Tx\|.
\]

Now we show that \( \mathcal{N}(S) = \mathcal{N}(T) \). If \( x \in \mathcal{N}(S) - \mathcal{N}(T) \), then \(-Ax = Tx\) and so \( \|Tx\| = \|Ax\| \leq b\|Tx\| \). Hence \( b \geq 1 \) which is a contradiction. Thus \( \mathcal{N}(S) \subseteq \mathcal{N}(T) \) since \( \mathcal{N}(A) = \mathcal{N}(T) \) and \( \mathcal{N}(T) \subseteq \mathcal{N}(S) \). Therefore \( \mathcal{N}(S) = \mathcal{N}(T) \). Thus \( S \) has the Hyers-Ulam stability.

Assume that \( T \) is a closed operator. Then so is \( S \). Hence \( \mathcal{R}(S) \) is closed. Since \( \frac{\|Sx\|}{\|x\|} = \frac{\|Tx + Ax\|}{\|x\|} = \frac{\|Tx\|}{\|x\|} + \frac{\|Ax\|}{\|x\|} \), and \( \mathcal{N}(T) = \mathcal{N}(S) \) we have \( \gamma(S) = \gamma(T) + \gamma(A) \). Hence, by [6, Theorem 3.1], \( K_S = \frac{1}{\gamma(T) + \gamma(A)} \). \( \square \)

The following result can be regarded as a special case of [1, Theorem 2.2] with a Hyers-Ulam stability approach.

**Theorem 2.9** Suppose that \( A \) is a \( T \)-bounded operator with a \( T \)-bound \( b \) smaller than 1 and constant \( a = 0 \), and \( \mathcal{N}(A) = \mathcal{N}(T) \). Assume that \( A \) has the Hyers-Ulam stability and that \( T \) is a closed operator. Then \( S := T + A \) is a closed operator, \( S \) has the Hyers-Ulam stability and
\[
\frac{1}{\gamma(A) + \gamma(T)} \leq K_S \leq \frac{1}{(1 - b)\gamma(T)}.
\]

**Proof.** By Theorem 2.5 the operator \( T \) has the Hyers-Ulam stability. Hence it has closed range and so \( \gamma(T) > 0 \). Since the operator \( A \) is \( T \)-bounded with a \( T \)-bound smaller than 1 and since by [9, Theorem 1.1] \( T \) is a closed operator, we deduce that the operator \( S \) is closed. In view of \( \|Ax\| \leq b\|Tx\| \), we get
\[
\|Tx\| - \|Sx\| \leq \|Ax + Tx - Tx\| \leq b\|Tx\| \quad (x \in \mathcal{D}(T)).
\]

Hence \((1 - b)\|Tx\| \leq \|Sx\|\). Thus
\[
(1 - b)\frac{\|Tx\|}{\|x\|} \leq \frac{\|Sx\|}{\|x\|} \quad x \in (\mathcal{D}(T) - \{0\}).
\]
Since $N(T) = N(S)$ we have $0 < (1 - b)\gamma(T) \leq \gamma(S)$, therefore $S$ has closed range [9, Theorem 5.2]. Thus $S$ has the Hyers-Ulam stability and $K_S = \gamma(S)^{-1} \leq \frac{1}{(1 - b)\gamma(T)}$. Clearly $\gamma(S) \leq \gamma(A) + \gamma(T)$. Therefore $\frac{1}{\gamma(A) + \gamma(T)} \leq K_S$. \qed

Recall that a closed operator $A$ from $H$ into $K$ is called left semi-Fredholm if $\dim N(A) < \infty$ and $\mathcal{R}(A)$ is closed. It is called right semi-Fredholm if $\text{codim} \mathcal{R}(A) < \infty$ and $\mathcal{R}(A)$ is closed. We say a closed operator $A$ is semi-Fredholm if it is left or right semi-Fredholm.

**Remark 2.10** Suppose that $A$ is a $T$-bounded operator with a $T$-bound $b$ smaller than 1 and constant $a = 0$, and $N(A) = N(T)$. If $T$ is a closed operator and has the Hyers-Ulam stability. Then, by Theorem 2.9, the operator $S := A + T$ is closed and has the Hyers-Ulam stability. So that $\mathcal{R}(S)$ is closed.

The conclusion that $S$ is closed has already obtained in [4, Theorem V.3.6] under the different assumption that the operator $T$ is semi-Fredholm.

**Corollary 2.11** Suppose that $A$ is a left semi-Fredholm and $T$-bounded operator with constant $a = 0$ and a $T$-bound $b$ smaller than 1, and $T$ is a closed operator such that $N(A) = N(T)$. Then $S := T + A$ is a left semi-Fredholm operator.

**Theorem 2.12** Suppose that $A$ is a $T$-bounded operator with a $T$-bound $b$ smaller than 1 and constant $a = 0$, and $N(A) = N(T)$. If $S = T + A$ has the Hyers-Ulam stability then $T$ has the Hyers-Ulam stability. Moreover if $S$ is a closed operator then $\mathcal{R}(S)$ and $\mathcal{R}(T)$ are closed.

**Proof.** The operator $S$ has the Hyers-Ulam stability thus there exists a constant $K > 0$ with the following property:

For any $x \in \mathcal{D}(S) = \mathcal{D}(T)$ there exists a $x_0 \in \mathcal{N}(S)$ such that $\|x - x_0\| \leq K\|Sw\|$.

Since $A$ is a $T$-bounded operator and, by the proof of Corollary 2.8, $N(T) = N(S)$, we have

$$||x - x_0|| \leq K||Sw|| \leq K(||Ax|| + ||Tx||) \leq K(b + 1)||Tx||.$$  

Therefore $T$ has the Hyers-Ulam stability.

Now assume that $S$ is a closed operator. Then so is $T$. In view of $S$ and $T$ having the Hyers-Ulam stability, $\mathcal{R}(S)$ and $\mathcal{R}(T)$ are closed. \qed

References


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