On \(\tau\)-lifting Modules and \(\tau\)-semiperfect Modules

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Abstract

Motivated by [1], we study on \(\tau\)-lifting modules (rings) and \(\tau\)-semiperfect modules (rings) for a preradical \(\tau\) and give some equivalent conditions. We prove that:

i) if \(M\) is a projective \(\tau\)-lifting module with \(\tau(M) \subseteq \delta(M)\), then \(M\) has the finite exchange property;

ii) if \(R\) is a left hereditary ring and \(\tau\) is a left exact preradical, then every \(\tau\)-semiperfect module is \(\tau\)-lifting;

iii) \(R\) is \(\tau\)-lifting if and only if every finitely generated free module is \(\tau\)-lifting if and only if every finitely generated projective module is \(\tau\)-lifting;

iv) if \(\tau(R) \subseteq \delta(R)\), then \(R\) is \(\tau\)-semiperfect if and only if every finitely generated module is \(\tau\)-semiperfect if and only if every simple \(R\)-module is \(\tau\)-semiperfect.

Key Words: \(\tau\)-lifting modules, Projective \(\tau\)-covers, \(\tau\)-supplement submodules, \(\tau\)-semiperfect modules.

1. Introduction

The concept of semiperfect rings was generalized to \(I\)-semiperfect ring for an ideal \(I\) of a ring by Yousif and Zhou in [16]. Then Nicholson and Zhou defined the concept of strongly lifting and gave some characterizations of \(I\)-semiperfect rings in [9]. A module theoretic version of \(I\)-semiperfect ring is studied in [10] and [11] by considering any fully invariant submodule of a module. Let \(M\) be an \(R\)-module. Following [10], \(M\) is said to be \(U\)-semiperfect if for any submodule \(N\) of \(M\), there is a projective direct summand \(A\) of \(M\) such that \(N = A \oplus B\) and \(B \subseteq U\) for a fully invariant submodule \(U\) of \(M\). Moreover, in [11], Özcan and Aydogdu generalized the concept of strongly lifting ideals and gave some characterization of \(U\)-semiperfect module. In [1], for a radical \(\tau\), Al-Takhman, Lomp and Wisbauer defined and studied the concept of \(\tau\)-lifting, \(\tau\)-supplement and \(\tau\)-semiperfect modules. Following [1], \(M\) is \(\tau\)-lifting if any submodule \(N\) of \(M\) has a decomposition \(N = A \oplus (B \cap N)\) such that \(M = A \oplus B\) and \(B \cap N \subseteq \tau(B)\) and also they called that \(M\) is \(\tau\)-semiperfect if for any submodule \(N\) of \(M\), \(M/N\) has a projective \(\tau\)-cover. It is clear that if \(M\) is projective, then the concepts of \(\tau(M)\)-semiperfect and \(\tau\)-lifting are coincide and if \(N\) is a submodule of \(M\) with the decomposition in the definition of \(\tau\)-lifting, then \(M/N\) has a projective \(\tau\)-cover. Motivated by [1], we study on \(\tau\)-lifting module and the relations between a projective \(\tau\)-cover and the decomposition for a preradical \(\tau\). We also give some equivalent condition for a \(\tau\)-semiperfect module and a \(\tau\)-lifting module. The remainder of our paper is organized as follows.

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In Section 2, we define the concept of quasi-strongly lifting (QSL). We call submodule $U$ is called quasi strongly lifting (QSL) in $M$ if whenever $(A + U)/U$ is a direct summand of $M/U$, $M$ has a direct summand $P$ such that $P \subseteq A$ and $P + U = A + U$. Then we prove that $\tau(L)$ is QSL in $L$ if $L$ is direct summand of $M$ and $\tau(M)$ is QSL in $M$. Also, we recall SDM submodule which is given in [3], and show that $\delta(M)$ is the sum of all SDM submodule of $M$ if $M$ is a projective module.

In Section 3, we concern with $\tau$-lifting modules and consider certain preradicals $Soc$, $Z$ and $\delta$. We show that if $M$ is $\tau$-lifting, then $M$ is refinable if and only if every submodule of $\tau(M)$ is DM in $M$ if and only if every submodule of $\tau(M)$ is QSL in $M$ and we prove that $M$ is $\delta$-lifting and $M$ has the finite exchange property whenever $M$ is a projective $\tau$-lifting and $\tau(M) \subseteq \delta(M)$. For two preradicals $\tau, \rho$, we also study the relation between a $\tau$-lifting module and $\rho$-lifting module. We also prove that if $M$ is a $\delta$-lifting projective module, $M/Soc(M)$ is lifting, but we prove the converse if $M/Soc(M)$ is projective. Moreover, we show that if $R$ is a left hereditary ring and $\tau$ is a left exact preradical, then every $\tau$-semiperfect module is $\tau$-lifting. Finally, we give some equivalent statements for $\tau$-semiperfect modules (rings) and $\tau$-lifting modules (rings) as well:

i) $R$ is $\tau$-lifting if and only if every finitely generated free module is $\tau$-lifting and only if every finitely generated projective module is $\tau$-lifting; ii) if $M$ is a finitely generated projective module with $\tau(M) \subseteq \delta(M)$, then $M$ is $\tau$-semiperfect if and only if every simple factor module of $M$ has a projective $\tau$-semiperfect; iii) if $\tau(U) \subseteq \delta(U)$, then $R$ is $\tau$-semiperfect if and only if every finitely generated module is $\tau$-semiperfect if and only if every simple $R$-module is $\tau$-semiperfect.

A functor $\tau$ from the category of the left $R$-modules to itself is called a preradical if it satisfies the following properties:

i) $\tau(M)$ is a submodule of an $R$-module $M$,

ii) If $f : M' \rightarrow M$ is an $R$-module homomorphism, then $f(\tau(M')) \subseteq \tau(M)$ and $\tau(f)$ is the restriction of $f$ to $\tau(M')$.

A preradical $\tau$ is called a left exact preradical if for any submodule $K$ of $M$, $\tau(K) = \tau(M) \cap K$. But it is well known if $K$ is a direct summand of $M$, then $\tau(K) = \tau(M) \cap K$ for a preradical. In this note, $\tau$ will be a preradical unless otherwise stated.

Throughout this paper, $R$ denotes an associative ring with an identity and modules are an unital left $R$-modules. We write $Rad(M)$, $Soc(M)$ and $Z(M)$ for Jacobson radical, the socle, the singular submodule, respectively.

2. Strongly Lifting

Let $U$ be a submodule of an $R$-module $M$. $U$ is called strongly lifting in $M$ if whenever $M/U = (A + U)/U \oplus (B + U)/U$, then $M$ has a decomposition $M = P \oplus Q$ such that $P \subseteq A$, $(A + U)/U = (P + U)/U$ and $(B + U)/U = (Q + U)/U$ in [11]. By removing the condition on $B$, we may extend the definition; the submodule $U$ is called quasi strongly lifting (QSL) in $M$ if whenever $(A + U)/U$ is a direct summand of $M/U$, $M$ has a direct summand $P$ such that $P \subseteq A$ and $P + U = A + U$.

Lemma 2.1 Let $U$ be a submodule of a projective module $M$. If $U$ is QSL then $U$ is strongly lifting in $M$. 
Let $M = (A + U)/U \oplus (B + U)/U$ for submodules $A, B$ of $M$. Then there is a decomposition $M = P \oplus Q$ such that $P + U = A + U$ and $P \subseteq A$. Then $M = A + U + B = P + (U + B)$ and since $M$ is projective, $M = P \oplus P'$ for a submodule $P' \subseteq U + B$. Then $M/U = (P + U)/U \oplus (P' + U)/U = (P + U)/U \oplus (B + U)/U$ and so $(P' + U)/U = (B + U)/U$. □

By using a similar proof of Theorem 2.3 in [11], we have the following lemma

**Lemma 2.2** Let $M$ be a module and $A$ be a direct summand of $M$ such that $M/A$ is projective then $A$ is QSL in $M$.

**Proof.** Let $M/A = (X_1 + A)/A \oplus (X_2 + A)/A$ for submodules $X_1$ and $X_2$. Assume that $M = A \oplus B$ and $\alpha$ is an isomorphism from $B$ to $M/A$ and so for submodules $B_1$ and $B_2$ of $B$, we have that $\alpha(B_i) = (X_i + A)/A$ and so $(B_i + A)/A = (X_i + A)/A$ for $i = 1, 2$. Then $B_1 \cap B_2 \subseteq (B_1 + A) \cap (B_2 + A) = A$ and so $B_1 \cap B_2 = 0$.

Now we claim that $B = B_1 + B_2$. Let $b \in B$ and so $b = b_1 + b_2 + a$ where $b_i \in B_i$ and $a \in A$ for $i = 1, 2$. Then since $A \cap B = 0$, it follows that $a = 0$. Then $M = A \oplus B_1 \oplus B_2$ and so $B_i$ are projective. On the other hand, since $A \oplus B_i = A + X_i$, we have $A \oplus B_i = A \oplus Y_i$ where $Y_i \subseteq X_i$ by [7, 4.47]. Then $A$ is QSL in $M$. □

**Proposition 2.3** Let $M$ be a module such that $\tau(M)$ is QSL in $M$. If $L$ is a direct summand of $M$, then $\tau(L)$ is QSL in $L$.

**Proof.** Let $M = L \oplus K$ and $L/\tau(L) = [A + \tau(L)]/\tau(L) \oplus B/\tau(L)$ for submodules $A, B$ of $L$. Then $[A + \tau(M)]/\tau(M) \oplus [B + K + \tau(M)]/\tau(M) = M/\tau(M)$ and so there is a decomposition $M = P \oplus Q$ such that $P \subseteq A$, $A + \tau(M) = P + \tau(M)$. Hence

$$A + \tau(M) = (A + \tau(L)) \oplus \tau(K) = (P + \tau(L)) \oplus \tau(K)$$

and so $A + \tau(L) = P + \tau(L)$. This completes the proof. □

**Proposition 2.4** Let $M$ be projective. Then the following are equivalent:

i) $\tau(M)$ is QSL in $M$.

ii) If $M/\tau(M) = (M_1 + \tau(M))/\tau(M) \oplus \ldots \oplus (M_t + \tau(M))/\tau(M)$ for any positive integer $t$, then $M = A_1 \oplus \ldots \oplus A_t$, where $A_i \subseteq M_1$ and $A_i + \tau(M) = M_i + \tau(M)$ for all $i$.

**Proof.** It is enough to show that $i) \implies ii)$. Let $M/\tau(M) = ([M_1 + \tau(M)]/\tau(M)) \oplus \ldots \oplus ([M_t + \tau(M)]/\tau(M))$ for any positive integer $t$ then $M/\tau(M) = ([M_1 + \tau(M)]/\tau(M)) \oplus ([M_2 + \ldots + M_t + \tau(M)]/\tau(M))$. There is a direct summand $A_1$ of $M$ such that $A_1 \subseteq M_1$ and $A_1 + \tau(M) = M_1 + \tau(M)$. Since $M$ is projective, there is a decomposition $M = A_1 \oplus B$ such that $B \subseteq M_2 + \ldots + M_t + \tau(M)$ and so $B + \tau(M) = M_2 + \ldots + M_t + \tau(M)$. Then there are submodules $N_i$ of $B$ such that $N_i + \tau(B) + \tau(A) = M_i + \tau(M)$ and $B/\tau(B) = (N_2 + \tau(B))/\tau(B) \oplus \ldots \oplus (N_t + \tau(B))/\tau(B)$; and since $\tau(B)$ is QSL in $B$, there is a decomposition $B = A_2 \oplus B_2$ such that $A_2 \subseteq N_2$, $B_2 \subseteq N_3 + \ldots + N_t + \tau(B)$
and $A_2 + \tau(B) = N_2 + \tau(B)$ and so $A_2 + \tau(M) = N_2 + \tau(M)$. Then $M = A_1 \oplus B = A_1 \oplus A_2 \oplus B_2$. And so after finite steps, we have the decomposition $M = A_1 \oplus \ldots \oplus A_i$ where $A_i \subseteq M_i$ and $A_i + \tau(M) = M_i + \tau(M)$. □

Let $K$ be a submodule of a module $M$. Following [15], $K$ is called $\delta$-small in $M$ if $K + L \neq M$ for any proper submodule $L$ of $M$ with $M/L$ singular. Zhou also defined the fully invariant submodule $\delta(M) = \cap \{K \leq M : M/K \text{ is singular simple in } \text{mod} \} = \sum \{K : K \text{ is } \delta\text{-small in } M\}$.

In [3], it is called that a proper submodule $N$ of $M$ is SDM (resp., DM) in $M$ if there is a direct summand $S$ of $M$ such that $S \leq N$ and $M = S \oplus X$ (resp., $M = S + X$) whenever $N + X = M$ for a submodule $X$ of $M$.

It is clear that a $\delta$-small submodule of a module and any direct summand of a module is DM, but there is a SDM-submodule which is not $\delta$-small (see Example 3.25).

We note the following lemma.

**Lemma 2.5** [15, Lemma 1.2] Let $K$ be a submodule of a module $M$. Then $K$ is $\delta$-small if and only if $M = X \oplus Y$ for a projective semisimple submodule $Y$ with $Y \leq K$ whenever $X + K = M$.

Let $S(M)$ denote the sum of all SDM submodules of a module $M$. It is clear that $S(M)$ contains $\text{Soc}(M)$ and $\delta(M)$.

**Lemma 2.6** Let $A, B$ be SDM submodule of a module $M$. Then $A + B$ is SDM in $M$.

**Proof.** Let $A + B + K = M$ for a submodule $K$. Since $A$ is SDM in $M$, there is a submodule $S$ of $A$ such that $S \oplus (B + K) = M$ and so $B + (S \oplus K) = M$. Then similarly $M = Q \oplus (S \oplus K)$ for a submodule $Q$ of $B$. Then $A + B$ is SDM in $M$. □

**Theorem 2.7** Let $M$ be a a projective module. Then

i) $Rx$ is SDM in $M$ where $x \in S(M)$.

ii) $S(M) = \delta(M)$ and every finitely generated SDM submodule of $M$ is $\delta$-small.

**Proof.** i) Let $x \in S(M)$ and $Rx + K = M$ for a submodule $K$. Then $x \in \sum_{i=1}^n K_i$ where $n \in \mathbb{Z}$ and $K_i$ is SDM in $M$ and $\sum_{i=1}^n K_i$ is SDM in $M$. Then $(\sum_{i=1}^n K_i) + K = M$ and so for a submodule $S$, we have that $S \oplus K = M$. Then since $M$ is projective and $K$ is a direct summand, we have $M = A \oplus K$ for a submodule $A$ of $Rx$. Hence $Rx$ is SDM in $M$.

ii) Since $M$ is projective, $\delta(RM)$ is the intersection of all essential maximal submodules of $M$. Take $x \in S(M)$ and assume that $x \notin L$ for an essential maximal submodule $L$. Since $x \in S(M)$, we get that $S \oplus L = M$ for a submodule $S$ of $Rx$, a contradiction. Hence $S(M) = \delta(M)$. □

3. $\tau$-lifting

We concern with $\tau$-lifting modules and consider certain preradicals $\text{Soc}$, $Z$ and $\delta$. We state [1, Proposition 2.8] for a preradical $\tau$.
Proposition 3.1 For a submodule $S$ of a module $M$, the following are equivalent:

i) there is a decomposition $M = X \oplus X'$ such that $X \subseteq S$ and $X' \cap S \subseteq \tau(X')$,

ii) there is a decomposition $S = A \oplus T$ with $A \subseteq \perp M$ and $T \subseteq \tau(M)$,

iii) there exists a direct summand $A$ of $M$ such that $A \subseteq S$ and $S/A \subseteq \tau(M/A)$,

iv) there exists an idempotent homomorphism $\gamma$ from $M$ to $M$ such that $(1 - \gamma)(S) \subseteq \tau(M)$ and $\gamma(M) \subseteq S$.

For a submodule $S$ of a module $M$, in [1], Al-Takhman, Lomp and Wisbauer say that $S$ contains a $\tau$-dense direct summand if $S$ satisfies one of the conditions of Proposition 3.1 and also $M$ is called $\tau$-lifting if every submodule of $M$ contains a $\tau$-dense direct summand. In [11], $\tau$-dense direct summand is named as $\tau(M)$ respects $S$.

Following [1], (i) a submodule $K \subseteq M$ is called a $\tau$-supplement provided there exists some $U \subseteq M$ such that $U + K = M$ and $U \cap K \subseteq \tau(K)$; (ii) $M$ is said to be $\tau$-supplemented if every submodule $K \subseteq M$ has a $\tau$-supplement in $M$; (iii) it is called amply $\tau$-supplemented if for any submodules $K, V \subseteq M$ such that $M = K + V$, there is a $\tau$-supplement $U$ for $K$ with $U \subseteq V$. It is clear that a $\tau$-lifting module is $\tau$-supplemented.

Lemma 3.2 Let $M$ be a projective $\tau$-supplemented module and assume that every $\tau$-supplement submodule is a direct summand of $M$. Then $M$ is $\tau$-lifting.

Proof. Let $U$ be a submodule of $M$. Then there is a submodule $K$ of $M$ such that $U \cap K \subseteq \tau(K)$ and $M = K + U$. Hence $K$ is a direct summand of $M$ and since $M = K + U$ and $M$ is projective, it follows that $M = K \oplus A$ such that $A \subseteq U$. Then $U = A \oplus (K \cap U)$ and $U$ is a $\tau$-dense direct summand. \hfill \Box

Now we give relations between a $\tau$-lifting module and an amply $\tau$-supplemented module.

Lemma 3.3 Let $M$ be an amply $\tau$-supplemented module and assume that every $\tau$-supplement submodule is a direct summand of $M$. Then $M$ is $\tau$-lifting.

Proof. By hypothesis, a submodule $A$ of $M$ has a $\tau$-supplement $B$ and so $B$ has a $\tau$-supplement submodule $B'$ such that $B' \subseteq A$ and $B = B' \oplus B''$ for some $B''$. Then $M = B' + B''$ and so $A = B' + (A \cap B) = B' \oplus (A \cap B'')$. Let $\pi$ denote the projection map from $M$ to $B''$. Then $A \cap B'' = \pi(A) = \pi(A \cap B)$. Since $B$ is a $\tau$-supplement of $A$, it follows that $A \cap B \subseteq \tau(B)$ and so $A \cap B'' \subseteq \tau(B'')$. \hfill \Box

Lemma 3.4 Let $\tau$ be a left exact preradical and $M$ be a $\tau$-lifting module. Then $M$ is an amply $\tau$-supplemented module.

Proof. Let $X$ and $S$ be submodules of $M$ such that $M = X + S$. We show that $S$ contains a $\tau$-supplement of $X$. By assumption, write $S = Y \oplus T$ where $M = Y \oplus Y'$ for submodules $Y'$, $Y$ and $T = S \cap Y' \subseteq \tau(Y')$. Then $M = X + Y + T$ and also there is a decomposition $M = Y_1 \oplus Y_1'$ such that $(X + T) \cap Y = Y_1 \oplus T_1$ and $T_1 = (X + T) \cap Y \cap Y_1' \subseteq \tau(Y_1')$ and so $T_1 \subseteq \tau(Y_1') \cap Y = \tau(Y_1' \cap Y)$. Then $Y = Y_1 \oplus (Y_1' \cap Y)$ and so $M = X + T + (Y_1' \cap Y)$. Let $L = T + (Y_1' \cap Y)$ and so $L \subseteq S$.

\[ X \cap L \subseteq [T \cap (X + (Y_1' \cap Y))] + [(Y_1' \cap Y) \cap (T + X)] \subseteq T + \tau(Y_1' \cap Y) \]
Since \( \tau \) is left exact, we have \( T + \tau(Y_1 \cap Y) \subseteq \tau(T) + \tau(Y_1 \cap Y) \subseteq \tau(T + (Y_1 \cap Y)) \) and so \( L \) is a \( \tau \)-supplement submodule of \( X \) in \( S \). This completes the proof. \( \square \)

**Lemma 3.5** Let \( M \) be a \( \tau \)-lifting module. Then \( \tau(M) \) is QSL in \( M \).

**Proof.** Let \( M/\tau(M) = [K + \tau(M)/\tau(M)] \oplus L/\tau(M) \) for submodules \( K, L \). Since \( M \) is \( \tau \)-lifting, there is a decomposition \( M = A \oplus B \) such that \( K = A \oplus (B \cap K) \) and \( B \cap K \subseteq \tau(M) \) and so \( A + \tau(M) = K + \tau(M) \). \( \square \)

**Proposition 3.6** Let \( M \) be a module. Then the following statements are equivalent:

i) \( M \) is \( \tau \)-lifting,

ii) \( M \) is \( \tau \)-supplemented and \( \tau(M) \) is QSL,

iii) \( M/\tau(M) \) is semisimple and \( \tau(M) \) is QSL.

**Proof.** i) \( \Rightarrow \) ii) \( \Rightarrow \) iii) Obvious.

iii) \( \Rightarrow \) i) Let \( U \) be a submodule of \( M \). Then we have that \( M/\tau(M) = [U + \tau(M)/\tau(M)] \oplus [K/\tau(M)] \) for a submodule \( K \) and so there is a decomposition \( M = A \oplus B \) such that \( A \subseteq U \), \( A + \tau(M) = U + \tau(M) \).

Since \( \tau(M) = \tau(A) \oplus \tau(B) \), it follows that \( U \cap B \subseteq (U + \tau(M)) \cap (B + \tau(M)) = (A + \tau(M)) \cap (B + \tau(M)) = [(A + \tau(B)) \cap B] + \tau(A) = \tau(M) \). Hence, \( U \cap B \subseteq \tau(M) \cap B \subset \tau(B) \) and so \( U \) contains a \( \tau \)-dense direct summand.

A module \( M \) is called **refinable** if whenever \( M = A + B \) for submodules \( A, B \), there is a direct summand \( C \) of \( M \) such that \( C \subseteq A \) and \( M = C + B \) (see [6]). Then we have the following theorem

**Theorem 3.7** Let \( M \) be a module. Consider the following conditions:

i) \( M \) is refinable,

ii) every submodule of \( \tau(M) \) is QSL in \( M \),

iii) every submodule of \( \tau(M) \) is DM in \( M \).

Then i) \( \Rightarrow \) ii) \( \Rightarrow \) iii). If \( M \) is \( \tau \)-lifting then iii) \( \Rightarrow \) i).

**Proof.** i) \( \Rightarrow \) ii) Let \( N \) be a submodule of \( \tau(M) \) and \( (L + N)/N \oplus K/N = M/N \) for submodules \( L, K \). Then \( L + K = M \) and so there is a direct summand \( S \) of \( M \) such that \( S + K = M \) and \( S \subseteq L \). Hence \( (S + N)/N \oplus K/N = (L + N)/N \oplus K/N \) and so \( S + N = L + N \).

ii) \( \Rightarrow \) iii) Let \( K \) be a submodule of \( \tau(M) \) such that \( M = K + L \) for a submodule \( L \) and \( N := K \cap L \). Then \( K/N \) is a direct summand of \( M/N \). Then there is a direct summand \( S \) of \( M \) such that \( S \subseteq K \) and \( S + N = K \). Then \( S + L = M \) and so \( K \) is DM in \( M \).

iii) \( \Rightarrow \) i) Assume every submodule of \( \tau(M) \) is DM. Let \( M = K + L \) for submodules \( L \) and \( K \). Then \( K = A \oplus (K \cap B) \) such that \( M = A \oplus B \) and \( K \cap B \subseteq \tau(B) \). It follows that \( M = A + (K \cap B) + L \) and so \( B = (K \cap B) + [(A + L) \cap B] \). Since every submodule of \( \tau(B) \) is DM in \( B \) by [3, Lemma 3.2], there is a direct summand \( C \) of \( B \) such that \( B = [(A + L) \cap B] + C \) and \( C \subseteq K \cap B \) and so \( A \oplus C \) is a direct summand of \( M \) and \( M = (A + C) + L \). Then \( K \) is DM in \( M \). \( \square \)
Corollary 3.10 Let $M$ be a projective module and $\tau = \text{Rad}$ or $\tau = Z$. Then $M$ is $\tau$-lifting if and only if $\tau(M) = \text{Rad}(M)$ is small and $M$ is lifting.

Proof. Let $M$ be $\tau$-lifting. If $L + Z(M) = M$, then $L = A \oplus (B \cap L)$ where $M = A \oplus B$ and $B \cap L \subseteq Z(M)$. Since $M/A$ is singular, it follows that $A$ is essential and so $A = M = L$. Hence $Z(M)$ is small and since $N$ is a projective $\tau$-lifting module with $\text{Rad}(M) \subseteq \tau(M)$, it follows that $M$ is lifting and $\tau(M) = \text{Rad}(M)$. □

If $M$ is a $\tau$-lifting module, then by the same argument of [1, 2.2], there is a decomposition $M = L \oplus B$ such that $L$ is semisimple and $\tau(M)$ is an essential submodule of $B$. Then we have
Lemma 3.11 Let $M$ be a module. Then we have

i) If $M$ is Soc-lifting, then $\text{Soc}(M)$ is essential in $M$.

ii) If $M$ is projective $\delta$-lifting module, then $Z(M) \subseteq \text{Rad}(M) \subseteq \delta(M)$ and $\delta(M)$ is essential in $M$.

Proof. i) If $M$ is Soc-lifting then by [1, 2.2], $\text{Soc}(M)$ is essential in $M$.

ii) If $M$ is a projective, then $\text{Soc}(M) \subseteq \delta(M)$ and so by [1, 2.2], $\delta(M)$ is essential in $M$.

Let $x \in Z(M)$ and so $Rx = A \oplus (B \cap Rx)$ where $M = A \oplus B$ and $B \cap Rx \subseteq \delta(M)$. Then $A$ is singular and projective and so $A = 0$. Hence $Z(M) \subseteq \delta(M)$.

Let $x \in Z(M)$ and let $L$ be a submodule with $Rx + L = M$. Since $Rx$ is $\delta$-small in $M$, there is a semisimple projective submodule $S \subseteq Rx \subseteq Z(M)$ such that $S \oplus L = M$. Hence $L = M$ and so $Z(M) \subseteq \text{Rad}(M)$. □

Proposition 3.12 Let $\tau$ and $\rho$ be preradicals and $M$ be a $\tau$-lifting module such that $\tau(M) + L = M$ and $\tau(M) \cap L \subseteq \rho(L)$ for a submodule $L$ of $M$. Then there is a decomposition $M = A \oplus B$ such that $A$ is $\rho$-lifting and $B \subseteq \tau(M)$.

Proof. Let $M$ be $\tau$-lifting. Then there is a decomposition $M = A \oplus B$ such that $L = A \oplus (B \cap L)$ and $B \cap L \subseteq \tau(B)$ and so $B \cap L \subseteq \tau(M) \cap L \subseteq \rho(L)$.

Now we show that $A$ is $\rho$-lifting and $B \subseteq \tau(M)$. Let $K$ be a submodule of $A$. Since $A$ is a direct summand of $M$, it also $\tau$-lifting. Then there is a decomposition $A = X \oplus Y$ such that $K = X \oplus Y \cap K$ and $Y \cap K \subseteq \tau(Y)$. Also $Y \cap K \subseteq \tau(Y) \cap L \subseteq \rho(M) \cap Y = \rho(Y)$ since $Y$ is a direct summand of $M$. Then $A$ is $\rho$-lifting.

Since $\tau(M) = \tau(A) \oplus \tau(B)$, we get $M = \tau(M) + L = \tau(A) + \tau(B) = A + B \cap L = A \oplus \tau(B)$ and so $\tau(B) = B \subseteq \tau(M)$. □

Corollary 3.13 Let $M$ be a $\tau$-lifting projective module such that $\tau(M) + L = M$ and $\tau(M) \cap L \subseteq \rho(L)$ for a submodule $L$ of $M$ where $\tau$ and $\rho$ are elements of the set $P = \{\delta, \text{Soc}, \text{Rad}\}$. Then $M$ is $\rho$-lifting.

Proof. By Proposition 3.12, there is a decomposition $M = A \oplus T$ such that $A$ is $\rho$-lifting and $T \subseteq \tau(M)$. If $\tau = Z$, then $T = 0$. If $\tau \in \{\delta, \text{Soc}, \text{Rad}\}$, then by Proposition 3.9, $\tau(M)$ is $\delta$-small in $M$ and so does $T$. Then $T$ is semisimple and so $T$ is $\rho$-lifting. Hence, by [10, Proposition 2.13], $M$ is $\rho$-lifting. □

Let $\tau, \rho$ and $\sigma$ be preradicals and $M$ be a module. Then we say that $M$ has *-property for $\{\tau, \rho, \sigma\}$ if $\sigma(N/\rho(N)) = \tau(N)/\rho(N)$ for any direct summand $N$ of $M$. For example, if $M$ is a projective module, then by [10, Proposition 2.13], $\text{Rad}(M/\text{Soc}(M)) = \delta(M)/\text{Soc}(M)$. Then we have the following proposition, which is a generalization of [16, Theorem 1.4].

Proposition 3.14 Let $M$ be a module with *-property for $\{\tau, \rho, \sigma\}$. If $M$ is $\tau$-lifting, then $M/\rho(M)$ is $\sigma$-lifting.

In particular, the converse holds whenever $\rho(M)$ is QSL in $M$ and $M$ is projective.
Proposition 3.15 Let $M$ be a projective module. If $M$ is $\delta$-semiperfect, then $M/Soc(M)$ is lifting.

In particular, the converse holds whenever $M/Soc(M)$ is projective.

Proof. Let $\overline{M}$ denote $M/Soc(M)$ and $\overline{L}$ denote $L/Soc(M)$ for a submodule $L$ of $M$.

Assume that $M$ is $\delta$-lifting and $\overline{N}$ is a submodule of $\overline{M}$. Then $N = A \oplus (B \cap N)$ where $M = A \oplus B$ and $B \cap N \subseteq \tau(B)$. On the other hand, we have

\[(A + \rho(M)) \cap (B + \rho(M)) = (A + \rho(B)) \cap (B + \rho(A))
= \rho(B) + [A \cap (B + \rho(A))]
= \rho(A) + \rho(B)\]

Thus $A \oplus B = \overline{M}$ and it is enough to show that $B \cap N \subseteq \sigma(B)$. Then we get that $B \cap N = B \cap N$ and by $\sigma(B/\rho(B)) = \sigma(B)/\rho(B)$ $[B \cap N + \rho(M)]/\rho(M) \subseteq [\tau(B) + \rho(M)]/\rho(M) \subseteq \sigma([B + \rho(M)]/\rho(M))$. Hence $M/\rho(M)$ is $\sigma$-lifting.

For the converse, assume that $L$ is a submodule of $M$. Then there is a decomposition $\overline{M} = \overline{C} \oplus \overline{D}$ such that $\overline{L} = \overline{C} \oplus \overline{D}$ and $\overline{D} \cap \overline{L} \subseteq \sigma(\overline{D})$. Since $\rho(M)$ is QSL in $M$, there is a decomposition $M = A \oplus B$ such that $A \subseteq L$, $\overline{A} = \overline{C}$ and $\overline{B} = \overline{D}$. Then it is enough to show that $L \subseteq \sigma(B)$ since $L = A \oplus (B \cap L)$. Then $B \cap L = B \cap D = D \cap L \subseteq \sigma(D) = \sigma(B)$ and so $B \cap L \subseteq \rho(M) \subseteq \tau(M)$ and $B \cap L \subseteq \tau(M) \cap B = \tau(B)$ since $B$ is direct summand. \hspace*{1cm} $\square$

In [1], it is said that a module $M$ has a projective $\tau$-cover if there is an epimorphism $f$ from a projective module $P$ to $M$ such that $Ker f \subseteq \tau(P)$ and an $R$-module $M$ is called $\tau$-semiperfect if every factor module of $M$ has a projective $\tau$-cover. Now we give some properties of a $\tau$-semiperfect module.

Lemma 3.16 Let $M$ be a $\tau$-semiperfect module. Then we have that
Theorem 3.18

Let $\tau$ be a left exact preradical and $R$ be a left hereditary ring. Then a projective $\tau$-semiperfect module is $\tau$-lifting.

\[ \text{Proof.} \quad \text{First, we observe the following for an element } x \text{ of } M. \text{ Let } f \text{ be an epimorphism from a projective module } P \text{ to } M/Rx \text{ such that } Kerf \subseteq \tau(P). \text{ Let } \pi : M \to M/Rx \text{ be a canonical epimorphism. Since } P \text{ is projective, it follows that there is a homomorphism } \alpha \text{ from } P \to M \text{ such that } \pi \alpha = f. \text{ Hence } M = \alpha(P) + Rx. \]

Let $K := \alpha(P)$ and take $y \in K \cap Rx$. Then $y = \alpha(t)$ for some $t \in P$ and $f(t) = \pi \alpha(t) = \pi(y) = 0$ and so $t \in Kerf \subseteq \tau(P)$. Hence $y \in \tau(M) \cap K$.

i) If $x \in Rad(M)$ then $K = M$ and so $K \cap Rx = Rx$ and $\tau(M) \cap K = \tau(M)$. This means $x \in \tau(M)$.

Take a submodule $U/\tau(M)$ of $M/\tau(M)$ to show that $M/\tau(M)$ is semisimple. Then $M/U$ has a projective $\tau$-cover $f$ from $P$ to $M/U$ such that $Kerf \subseteq \tau(P)$. Let $\pi$ be a canonical epimorphism from $M$ to $M/U$. Then $\pi \alpha = f$ for some $\alpha \in \text{Hom}(P,M)$ since $P$ is projective and so $M = U + \alpha(P)$. Let $u = \alpha(p) \in U \cap \alpha(P)$. Then $f(p) = \pi \alpha(p) = 0$ and so $p \in Kerf \subseteq \tau(P)$. Hence $u = \alpha(p) \in \tau(M)$ and so, $U \cap \alpha(P) \subseteq \tau(M)$. Then $U/\tau(M)$ is a direct summand of $M/\tau(M)$.

ii) If $x \in \delta(M)$, then $Rx$ is $\delta$-small and so there is a semisimple projective submodule $S$ of $Rx$ such that $M = K \oplus S$ and so $Rx = (K \cap Rx) \oplus S$. If $S \subseteq \tau(M)$, then $Rx \subseteq \tau(M)$.

iii) Clear.

iv) If $M$ is projective, then $\text{Soc}(M) \subseteq \delta(M)$ and so by ii), $\text{Soc}(M) = \delta(M)$.

If $M$ is $\text{Soc}$-semiperfect, then $M$ is $\text{Soc}$-lifting and so by [2, Corollary 4.7], $Z(M) \subseteq \text{Rad}(M) \subseteq \text{Soc}(M) = \delta(M)$.

v) Let $U$ be DM in $M$ and $f$ be an epimorphism from a projective module $P$ to $M/U$ such that $Kerf \subseteq \tau(P)$ and there is an homomorphism $\alpha$ from $P$ to $M$ such that $\pi \alpha = f$ where $\pi$ is the canonical epimorphism from $M$ to $M/U$. Then $M = U + \alpha(P)$ and so $M = S + \alpha(P)$ for a direct summand $S$ of $M$ in $U$. Since $M$ is projective, $M = S \oplus Q$ for a submodule $Q$ of $\alpha(P)$. Take $x \in \alpha(P) \cap U$ and so $x = \alpha(t)$ for some $t \in P$. Since $f(t) = \pi \alpha(t) = 0$, it follows that $t \in Kerf \subseteq \tau(P)$ and $x \in \tau(M)$. Therefore, $U$ is a $\tau$-dense direct summand. \]

By the argument of the proof of Lemma 3.16, we have the following corollary.

Corollary 3.17

Let $M$ be a finitely generated module and assume that every simple factor module of $M$ has projective $\tau$-cover. Then $M/\tau(M)$ is semisimple.

Observe that a projective $\tau$-lifting module is $\tau$-semiperfect. If $\tau = \text{Soc}$, then a projective $\tau$-semiperfect is $\tau$-lifting by [10, Lemma 2.22]. However, we don’t know whether or not a projective $\tau$-semiperfect module is $\tau$-lifting. Now, under some conditions which are given below, we prove that a projective $\tau$-semiperfect module is $\tau$-lifting.

Theorem 3.18

Let $\tau$ be a left exact preradical and $R$ be a left hereditary ring. Then a projective $\tau$-semiperfect module is $\tau$-lifting.
Theorem 3.19 Let $U$ be a submodule of $M$. Assume $f$ is an epimorphism from a projective module $Q$ to $M/U$ such that $Ker f \subseteq \tau(Q)$. Let $\pi$ be the canonical epimorphism from $M$ to $M/U$. Since $M$ is projective, there is a homomorphism $H$ from $M$ to $Q$ such that $fh = \pi$. Let $H := h(M)$ and so since $R$ is a left hereditary ring, it follows that $H$ is projective. Then there is a homomorphism $\alpha$ from $H$ to $M$ such that $\alpha h = 1_H$ and so $M = Kerh \oplus \alpha(H)$. Let $a \in Ker h$ and so $fha(a) = \pi(a) = 0$ and so $Ker h \subseteq U$. On the other hand, if $x \in \alpha(H) \cap U$ then $x = \alpha(t)$ for $t \in H$ and so $f(t) = fha(t) = \pi \alpha(t) = 0$. Then $t \in Kerf \subseteq \tau(Q)$ and so $t \in \tau(Q) \cap H = \tau(H)$ and so $\alpha(t) \in \tau(\alpha(H))$. Therefore, $U$ is a $\tau$-dense direct summand and so $M$ is $\tau$-lifting.

\begin{proof}
\ \ \ Let $M$ be a projective $\tau$-semiperfect module and $U$ be a submodule of $M$. Assume $f$ is an epimorphism from a projective module $Q$ to $M/U$ such that $Ker f \subseteq \tau(Q)$. Let $\pi$ be the canonical epimorphism from $M$ to $M/U$. Since $M$ is projective, there is a homomorphism $H$ from $M$ to $Q$ such that $fh = \pi$. Let $H := h(M)$ and so since $R$ is a left hereditary ring, it follows that $H$ is projective. Then there is a homomorphism $\alpha$ from $H$ to $M$ such that $\alpha h = 1_H$ and so $M = Ker h \oplus \alpha(H)$. Let $a \in Ker h$ and so $fha(a) = \pi(a) = 0$ and so $Ker h \subseteq U$. On the other hand, if $x \in \alpha(H) \cap U$ then $x = \alpha(t)$ for $t \in H$ and so $f(t) = fha(t) = \pi \alpha(t) = 0$. Then $t \in Kerf \subseteq \tau(Q)$ and so $t \in \tau(Q) \cap H = \tau(H)$ and so $\alpha(t) \in \tau(\alpha(H))$. Therefore, $U$ is a $\tau$-dense direct summand and so $M$ is $\tau$-lifting.
\end{proof}

Theorem 3.19 Let $M$ be a finitely generated module. Consider the following statements:

\begin{enumerate}
  \item $M$ is $\tau$-semiperfect and $\tau(M)$ is QSL.
  \item Every simple factor module of $M$ has a projective $\tau$-cover and $\tau(M)$ is QSL.
  \item $M$ is $\tau$-lifting.
\end{enumerate}

Then we have $i) \Rightarrow ii) \Rightarrow iii)$. If $M$ is projective then $iii) \Rightarrow i$.

\begin{proof}
\ \ \ $i) \Rightarrow ii)$. Obvious.

\ \ \ ii) $\Rightarrow iii)$. Let $L$ be a submodule of $M$. Since $M/\tau(M)$ is semisimple by Corollary 3.17, it follows that $M/\tau(M) + L = \oplus_{i \in K} S_i$ where $S_i$ is simple. Let $f_i : P_i \rightarrow S_i$ be a projective $\tau$-cover of $S_i$. Put $P := \oplus_{i \in K} P_i$ and $f := \oplus_{i \in K} f_i$. Then $f : P \rightarrow M/\tau(M) + L$ is a projective $\tau$-cover of $M/\tau(M) + L$ by [1, 2.13]. Let $\pi$ be a canonical epimorphism from $M$ to $M/\tau(M) + L$. Then there is a homomorphism $\alpha$ from $P$ to $M$ such that $\pi \alpha = f$ and so $M = \alpha(P) + [\tau(M) + L]$. Let $X := \alpha(P)$.

Let $x = \alpha(p) \in [L + \tau(M)] \cap X$ for $p \in P$. Since $f(p) = \pi \alpha(p) = \pi(x) = 0$ and $Ker f \subseteq \tau(P)$, we have $x \in \tau(M)$ and so $(L + \tau(M)) \cap X \subseteq \tau(M)$. Then

$$[X + \tau(M)] \cap [L + \tau(M)] = ((X + \tau(M)) \cap L) + \tau(M) \subseteq [(X + L) \cap \tau(M)] + [\tau(M) + L] \cap \tau(M) \subseteq \tau(M)$$

Hence $M/\tau(M) = [X + \tau(M)]/\tau(M) \oplus [L + \tau(M)]/\tau(M)$ and by hypothesis, there is a decomposition $M = A \oplus B$ such that $A \subseteq L$ and $A + \tau(M) = L + \tau(M)$. Then $M/\tau(M) = [A + \tau(M)]/\tau(M) \oplus [B + \tau(M)]/\tau(M) = [L + \tau(M)]/\tau(M) \oplus [B + \tau(M)]/\tau(M)$ and so $(L + \tau(M)) \cap (B + \tau(M)) = \tau(M)$. It follows that $B \cap L \subseteq \tau(M)$ and so $B \cap L \subseteq \tau(B)$. Therefore, $L$ contains a $\tau$-dense direct summand.

\ \ \ iii) $\Rightarrow i)$. If $M$ is projective, then $M$ is $\tau$-semiperfect. Also by Lemma 3.5, $\tau(M)$ is QSL.
\end{proof}

Theorem 3.20 The following statements are equivalent for a ring $R$:

\begin{enumerate}
  \item $R_R$ is $\tau$-lifting.
  \item Every finitely generated free $R$-module is $\tau$-lifting.
  \item Every finitely generated projective $R$-module is $\tau$-lifting.
  \item If $F$ is a finitely generated free $R$-module and $N$ is a fully invariant submodule, then $F/N$ is $\tau$-lifting.
\end{enumerate}
Moreover, $K = \alpha(\tau(P))$ such that $K = A \oplus (B \cap K)$ and $B \cap K \subseteq \tau(B)$. Then $F/N = (A + N)/N \oplus (B + N)/N$ and $(A + N)/N \subseteq K/N$. Moreover, $(B + N)/N \cap K/N = (B \cap K + N)/N \subseteq \tau(B + N/N)$. Hence $M$ is $\tau$-lifting.

**Proof.** Let $K/N$ be a submodule of $F/N$. Then there is a decomposition $F = A \oplus B$ such that $K = A \oplus (B \cap K)$ and $B \cap K \subseteq \tau(B)$. Then $F/N = (A + N)/N \oplus (B + N)/N$ and $(A + N)/N \subseteq K/N$. Moreover, $(B + N)/N \cap K/N = (B \cap K + N)/N \subseteq \tau(B + N/N)$. Hence $M$ is $\tau$-lifting.

**Corollary 3.21** Let a ring $R$ be $\tau$-lifting. Then for a finitely generated projective module $M$, $\tau(M)$ is QSL.

**Theorem 3.22** Let $M$ be a finitely generated module with a $\delta$-small submodule $\tau(M)$. Then $M$ is $\tau$-semiperfect if and only if every simple factor module of $M$ has a projective $\tau$-cover.

**Proof.** Let every simple factor module of $M$ have projective $\tau$-cover. Then $M/\tau(M)$ is semisimple by Corollary 3.17. Let $U$ be a submodule of $M$ and so $M/(U + \tau(M))$ is semisimple. Then there is a homomorphism $f$ from a projective module $P$ to $M/(U + \tau(M))$ such that $Ker(f) \subseteq \tau(P)$. Let $\pi$ be a map from $M/U$ to $M/(U + \tau(M))$ such that $\pi(m + U) = m + (U + \tau(M))$. Then there is a homomorphism $\alpha$ from $P$ to $M/U$ such that $\pi\alpha = f$ and so $M/U = \alpha(P) + (U + \tau(M))/U$ and $Ker(\alpha) \subseteq \tau(P)$. On the other hand, $(U + \tau(M))/U$ is $\delta$-small in $M/U$ as $\tau(M)$ is $\delta$-small. Hence, $M/U = \alpha(P) \oplus S$ for a semisimple projective submodule $S$ of $(U + \tau(M))/U$. Then $P \oplus S$ is projective and also we define and epimorphism $h$ from $P \oplus S$ to $M/U$ such that $h(p, s) = \alpha(p) + s$. Take an element $(p, s) \in Ker(h)$ and so $h(p, s) = \alpha(p) + s = 0$. Then $(p, s) \in Ker(\alpha) \subseteq \tau(P) \oplus 0 \subseteq \tau(P \oplus S)$. Therefore, $M/U$ has a projective $\tau$-cover.

**Theorem 3.23** Let $\tau(R) \subseteq \delta(RR)$ then the following statements are equivalent for a ring $R$:

i) $\tau(R)$ is $\tau$-semiperfect,

ii) Every finitely generated $R$-module $M$ is $\tau$-semiperfect,

iii) Every simple $R$-module has a projective $\tau$-cover.

**Proof.** i) $\Rightarrow$ ii) Let $M$ be a finitely generated module and $L$ be a submodule of $M$. Then $M/(L + \tau(RR))$ is a finitely generated $R/\tau(RR)$-module. Since $R/\tau(RR)$ is semisimple by Lemma 3.16, we get that $M/(L + \tau(RR))$ is a finitely generated $R/\tau(RR)$-module and so it is a semisimple $R$-module. Hence there are simple $R$-modules $S_i$ such that $M/(L + \tau(RR)) = S_1 \oplus \ldots \oplus S_n$ and so $S_i = Ra_i$ is isomorphic to $R/I$ for some left ideal $I$. Then $S_i$ has a projective $\tau$-cover and so does $M/(L + \tau(RR))$. Let $f$ be an epimorphism from a projective module $P$ to $M/(L + \tau(RR))$ with $Ker(f) \subseteq \tau(P)$ and $\pi$ be a natural map from $M/L$ to $M/(L + \tau(RR))$. Since $P$ is projective, there is an homomorphism $g$ from $P$ to $M/L$ such that $g\pi = f$ and so $M/L = g(P) + [(L + \tau(RR))/L]$. Then since $(L + \tau(RR))/L$ is $\delta$-small in $M/L$ and by Lemma 2.5, it follows that $M/L = g(P) \oplus K$ for a semisimple projective submodule $K$ of $M/L$. Since $g$ is a projective $\tau$-cover from $P$ to $g(P)$, we get that $M/L$ has a projective $\tau$-cover. Hence $M$ is $\tau$-semiperfect.

ii) $\Rightarrow$ iii) Clear.

iii) $\Rightarrow$ i) By Corollary 3.17, $R/\tau(RR)$ is semisimple and so by the argument of i) $\Rightarrow$ ii), $R$ is $\tau$-semiperfect.

\[\Box\]
Since $\text{Soc}_R R$ is strongly lifting, we have the following corollary.

**Corollary 3.24** The following statements are equivalent for a ring $R$;

1. $R$ is $\text{Soc}$-lifting,
2. $R$ is $\text{Soc}$-semiperfect,
3. $R/\text{Soc}(R) R$ is semisimple,
4. $R$ is $\text{Soc}$-supplemented.

**Example 3.25** [3] Let $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ be the ring of upper triangular matrices over a field $F$. Then $N = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$ is a projective left ideal, $L = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$ is a maximal left ideal and $I = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$ is an ideal of $R$. Consider the $R$-module $M = N \oplus R/L$. Then $\text{Soc}(R M) = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix} \oplus R/L$ is SDM but not $\delta$-small because $0 \oplus R/L$ is not $\delta$-small in $M$.

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