On Symmetric Monomial curves in $\mathbb{P}^3$

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Abstract

In this paper, we give an elementary proof of the fact that symmetric arithmetically Cohen-Macaulay monomial curves are set-theoretic complete intersections. The proof is constructive and provides the equations of the surfaces cutting out the monomial curve.

Key Words: Set-theoretic complete intersections, monomial curves

1. Introduction

Let $K$ be an algebraically closed field and $R$ be the polynomial ring $K[x_0, \ldots, x_n]$. To any irreducible curve $C$ in $\mathbb{P}^n$, one can associate a prime ideal $I(C) \subset R$ to be the set of all polynomials vanishing on $C$. The arithmetical rank of $C$, denoted by $\mu(C)$, is the least positive integer $r$ for which $I(C) = \text{rad}(f_1, \ldots, f_r)$, for some polynomials $f_1, \ldots, f_r$ or equivalently $C = H_1 \cap \cdots \cap H_r$, where $H_1, \ldots, H_r$ are the hypersurfaces defined by $f_1 = 0, \cdots, f_r = 0$, respectively. We denote by $\mu(I(C))$ the minimal number $r$ for which $I(C) = (f_1, \ldots, f_r)$, for some polynomials $f_1, \ldots, f_r \in R$. These invariants are known to be bounded below by the codimension of the curve (or height of its ideal). So, one has the following relation:

$$n - 1 \leq \mu(C) \leq \mu(I(C))$$

Although $\mu(I(C))$ has no upper bound (see [1], for an example), an upper bound for $\mu(C)$ is provided to be $n$ in [7] via commutative algebraic methods. Later in [2, 22] the equations of these $n$ hypersurfaces that cut out the curve $C$ were given explicitly by using elementary algebraic methods.

The curve $C$ is called a complete intersection if $\mu(I(C)) = n - 1$. It is called an almost complete intersection, if instead, one has $\mu(I(C)) = n$. When the arithmetical rank of $C$ takes its lower bound, that is $\mu(C) = n - 1$, the curve $C$ is called a set-theoretic complete intersection, s.t.c.i. for short. It is clear that complete intersections are set-theoretic complete intersection. The corresponding question for almost complete intersection monomial curves is answered affirmatively in a series of papers by Eto [8, 9, 10].

Determining s.t.c.i. monomial curves is a classical and longstanding problem in algebraic geometry. Even more difficult is to give explicitly the equations of the hypersurfaces involved. It is known that all monomial

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curves are s.t.c.i. in the projective $n$-space over a field of positive characteristic [3, 12, 14]. On the other hand, nobody knows whether or not the same question has an affirmative answer in the characteristic zero case. However, there are many partial results in this case [11, 13, 15, 16, 17, 19, 20, 21]. In fact, even the case of symmetric monomial curves is still mysterious.

The purpose of this note is to give an alternative proof of the fact that symmetric monomial curves in $\mathbb{P}^3$ which are arithmetically Cohen-Macaulay are s.t.c.i. by elementary algebraic methods inspired by [4]. The proof is constructive and provides the equations of the surfaces cutting out the curve.

2. Symmetric ACM Monomial Curves in $\mathbb{P}^3$

Let $p < q < r$ be some positive integers. A monomial curve $C(p, q, r)$ in $\mathbb{P}^3$ is given parametrically by

$$(w, x, y, z) = (u^r, u^{r-p}v^p, u^{r-q}v^q, v^r),$$

where $(u, v) \in \mathbb{P}^1$.

We say that the monomial curve $C(p, q, r)$ is symmetric if $p + q = r$. In this case the parametric representation of the curve $C(p, q, p + q)$ becomes $(u^p, u^q, u^{p+q})$.

It is known that all monomial curves are s.t.c.i. in $\mathbb{P}^3$, if the base field $K$ is of positive characteristic [12]. But, no one knows whether even the symmetric monomial curves are s.t.c.i. in $\mathbb{P}^3$ in the characteristic zero case. To address this case, we work with an algebraically closed field $K$ of characteristic zero throughout the paper.

A minimal system of generators for the ideal of symmetric monomial curves in $\mathbb{P}^3$ is given in [6] as follows:

$$f = xy - wz \quad \text{and} \quad F_i = w^{q-i}y^{p+i} - x^{q-i}z^i, \quad \text{for all} \quad 0 \leq i \leq q - p.$$ 

Recall that a monomial curve $C(p, q, r) \subset \mathbb{P}^3$ is called Arithmetically Cohen-Macaulay (ACM) if its projective coordinate ring is Cohen-Macaulay. In the same article [6], it is also proven that a monomial curve in $\mathbb{P}^3$ is ACM if and only if its ideal is generated by at most 3 polynomials. Now, if the ideal of a symmetric monomial curve $C(p, q, p + q)$ is generated by two polynomials it would follow that $p = q$. But, this contradicts with the assumption that $p < q < r$. So, the ideal of an ACM symmetric monomial curve $C(p, q, p + q)$ is generated by three polynomials and hence $p = q - 1$, where necessarily $q > 1$. Thus, all symmetric ACM monomial curves in $\mathbb{P}^3$ are of the form $C(q - 1, q, 2q - 1)$ and their defining ideals are generated minimally by the following three polynomials:

$$f = xy - zw,$$
$$g = -F_1 = x^{q-1}z - y^q,$$
$$h = -F_0 = x^q - y^{q-1}w.$$ 

The fact that $C(q - 1, q, 2q - 1)$ is a s.t.c.i. curve was shown in [17], but the equation of the second surface was not given. Here, we give an alternative proof that constructs the polynomial $G$ such that the symmetric ACM monomial curve is the intersection of the surface $G = 0$ and a binomial surface defined by one of $f, g$.
and $h$. We construct $G$ by adding $x^q g$ to the $q$-th power of $f$ and dividing the sum by $z$. Hence we get the following theorem.

**Theorem.** Any symmetric Arithmetically Cohen-Macaulay monomial curve in $\mathbb{P}^3$, which is given by $C(q-1, q, 2q-1)$ for some $q > 1$, is a set-theoretic complete intersection of the following two surfaces:

$$g = x^{q-1} z - y^q = 0$$

and

$$G = x^{2q-1} + \sum_{k=1}^{q} \frac{q!}{(q-k)!} x^{q-k} y^{q-k} z^{k-1} w^k = 0.$$

**Proof.** Note first that $zG = f^q + x^q g$. Take a point $(w_0, x_0, y_0, z_0)$ from $Z(f, g, h)$. Then, by $z_0 G(w_0, x_0, y_0, z_0) = f^q w_0 x_0 y_0 z_0 + x_0^q g(w_0, x_0, y_0, z_0) = 0$ we observe that either $G(w_0, x_0, y_0, z_0) = 0$ or $z_0 = 0$.

If $G(w_0, x_0, y_0, z_0) = 0$ then $(w_0, x_0, y_0, z_0) \in Z(g, G)$. If $z_0 = 0$ then by $g(w_0, x_0, y_0, z_0) = 0$ we get $y_0 = 0$, and by $h(w_0, x_0, y_0, z_0) = 0$ we get $x_0 = 0$. Thus $(w_0, x_0, y_0, z_0) = (1, 0, 0, 0)$ which is in $Z(g, G)$.

Let us now take a point $(w_0, x_0, y_0, z_0) \in Z(g, G)$. Then either $z_0 = 0$ or we can assume $z_0 = 1$. If $z_0 = 0$ then by $g(w_0, x_0, y_0, z_0) = 0$ we get $y_0 = 0$, and by $G(w_0, x_0, y_0, z_0) = 0$ we obtain $x_0 = 0$ in this case. Thus $(w_0, x_0, y_0, z_0) = (1, 0, 0, 0)$ which is in $Z(f, g, h)$. On the other hand, if $z_0 = 1$ then by $G = f^q + x_0^q g$ we see that $f(w_0, x_0, y_0, z_0) = 0$. Moreover, we have $x_0 y_0 = w_0$ and $x_0^{q-1} = y_0^q$ in this case. Hence we obtain the following $x_0^q = x_0 x_0^{q-1} = x_0 y_0^q = x_0 y_0 y_0^{q-1} = w_0 y_0^{q-1}$, meaning that $h(w_0, x_0, y_0, z_0) = 0$. □

**Remark.** Note that the symmetric ACM monomial curves above are s.t.c.i. on the binomial surface $g = 0$. This is not true for symmetric non-ACM monomial curves; that is, they can never be a s.t.c.i. on a binomial surface [18, Theorem 5.1]. Thus it is very difficult to construct surfaces on which symmetric non-ACM monomial curves in $\mathbb{P}^3$ are s.t.c.i. with the simplest open case being the Macaulay’s quartic curve $C(1, 3, 4)$.

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**References**


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