Strong Convergence Theorems by an Extragradient Method for Solving Variational Inequalities and Equilibrium Problems in a Hilbert Space*

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Abstract

In this paper, we introduce an iterative process for finding the common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem and the set of solutions of the variational inequality for monotone, Lipschitz-continuous mappings. The iterative process is based on the so-called extragradient method. We show that the sequence converges strongly to a common element of the above three sets under some parametric controlling conditions. This main theorem extends a recent result of Yao, Liou and Yao [Y. Yao, Y. C. Liou and J.-C. Yao, “An Extragradient Method for Fixed Point Problems and Variational Inequality Problems,” Journal of Inequalities and Applications Volume 2007, Article ID 38752, 12 pages doi:10.1155/2007/38752] and many others.

Key Words: Nonexpansive mapping; Equilibrium problem; Fixed point; Lipschitz-continuous mappings; Variational inequality; Extragradient method.

1. Introduction

Let \( H \) be a real Hilbert space and let \( C \) be a nonempty closed convex subset of \( H \). Recall that a mapping \( T \) of \( H \) into itself is called nonexpansive if \( \|Tx - Ty\| \leq \|x - y\| \) for all \( x, y \in H \). Let \( F \) be a bifunction of \( C \times C \) into \( \mathbb{R} \), where \( \mathbb{R} \) is the set of real numbers. The equilibrium problem for \( F : C \times C \rightarrow \mathbb{R} \) is to find \( x \in C \) such that
\[
F(x, y) \geq 0 \quad \text{for all } y \in C.
\] (1.1)
The set of solutions of (1.1) is denoted by \( EP(F) \). Given a mapping \( T : C \rightarrow H \), let \( F(x, y) = \langle Tx, y - x \rangle \) for all \( x, y \in C \). Then \( z \in EP(F) \) if and only if \( \langle Tz, y - z \rangle \geq 0 \) for all \( y \in C \), i.e., \( z \) is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.1). In

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1997 Combettes and Hirstoaga [2] introduced an iterative scheme of finding the best approximation to initial data when $EP(F)$ is nonempty and proved a strong convergence theorem.

Let $A : C \rightarrow H$ be a mapping. The classical variational inequality, denoted by $VI(A, C)$, is to find $x^* \in C$ such that $\langle Ax^*, v - x^* \rangle \geq 0$ for all $v \in C$. The variational inequality has been extensively studied in the literature. See, e.g. [12, 15] and the references therein. A mapping $A$ of $C$ into $H$ is called monotone if

$$\langle Au - Av, u - v \rangle \geq 0,$$

(1.2)

for all $u, v \in C$. $A$ is called $k$-Lipschitz-continuous if there exists a positive constant $k$ such that for all $u, v \in C$

$$\|Au - Av\| \leq k\|u - v\|.$$  

(1.3)

We denote by $F(S)$ the set of fixed points of $S$. For finding an element of $F(S) \cap VI(A, C)$, Takahashi and Toyoda [9] introduced the iterative scheme

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n)$$

(1.4)

for every $n = 0, 1, 2, \ldots$, where $x_0 = x \in C$, $\alpha_n$ is a sequence in $(0, 1)$, and $\lambda_n$ is a sequence in $(0, 2\alpha)$. Recently, Nadezhkina and Takahashi [6] and Zeng and Yao [16] proposed some new iterative schemes for finding elements in $F(S) \cap VI(A, C)$.

The algorithm suggested by Takahashi and Toyoda [9] is based on two well-known types of methods, namely, on the projection-type methods for solving variational inequality problems and so-called hybrid or outer-approximation methods for solving fixed point problems. The idea of “hybrid” or “outer-approximation” types of methods was originally introduced by Haugazeau in 1968; see [3] for more details.

In 1976, Korpelevich [4] introduced the following so-called extragradient method:

$$\begin{aligned}
&x_0 = x \in C, \\
&\bar{x}_n = P_C(x_n - \lambda_n Ax_n), \\
&x_{n+1} = P_C(x_n - \lambda_n A\bar{x}_n)
\end{aligned}$$

(1.5)

for all $n \geq 0$, where $\lambda_n \in (0, \frac{1}{2})$, $C$ is a closed convex subset of $\mathbb{R}^n$ and $A$ is a monotone and $k$-Lipschitz continuous mapping of $C$ into $\mathbb{R}^n$. He proved that if $VI(C, A)$ is nonempty, then the sequences $\{x_n\}$ and $\{\bar{x}_n\}$, generated by (1.5), converge to the same point $z \in VI(C, A)$.

Motivated by the idea of Korpelevichs extragradient method Zeng and Yao [16] introduced a new extragradient method for finding an element of $F(S) \cap VI(C, A)$ and proved the following strong convergence theorem.

**Theorem 1.1** ([16, Theorem 3.1]) Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $A$ be monotone and $k$-Lipschitz-continuous mapping of $C$ into $H$. Let $S$ be a nonexpansive mappings from $C$ into itself such that $F(S) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in $C$ defined as follows:

$$\begin{aligned}
&x_0 = x \in C, \\
y_n = P_C(x_n - \lambda_n Ax_n), \\
z_n = \alpha_n x_0 + (1 - \alpha_n)SP_C(x_n - \lambda_n Ay_n), \quad \forall n \geq 0,
\end{aligned}$$

(1.6)
where \( \{\lambda_n\} \) and \( \{\alpha_n\} \) satisfy the conditions

(i) \( \lambda_n k \subset (0, 1 - \delta) \) for some \( \delta \in (0, 1) \);

(ii) \( \alpha_n \subset (0, 1), \sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n \to \infty} \alpha_n = 0 \).

Then the sequence \( \{x_n\} \) and \( \{y_n\} \) converges strongly to the same point \( P_{F(S) \cap VI(C, A)} x_0 \) provided that \( \lim_{n \to \infty} \|x_{n+1} - x_n\| = 0 \).

In 2007, Yao, Liou and Yao [14] introduced the following iterative scheme: Let \( C \) be a closed convex subset of real Hilbert space \( H \). Let \( A \) be a monotone \( k \)-Lipschitz-continuous mapping of \( C \) into \( H \) and let \( S \) be a nonexpansive mapping of \( C \) into itself such that \( F(S) \cap VI(A, C) \neq \emptyset \). Suppose \( x_1 = u \in C \) and \( \{x_n\}, \{y_n\} \) are given by

\[
\begin{aligned}
  y_n &= P_C(x_n - \lambda_n Ax_n) \\
  x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n SP_C(x_n - \lambda_n Ay_n),
\end{aligned}
\]  

where \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \) are three sequences in \([0, 1]\). They proved that the sequence \( \{x_n\} \) defined by (1.7) converges strongly to common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for a monotone \( k \)-Lipschitz-continuous mapping under some parameters controlling conditions.

Recently, Takahashi and Takahashi [10] introduced an iterative scheme:

\[
\begin{aligned}
  y_n &= P_C(x_n - \lambda_n Ax_n) \\
  x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n) Ty_n, \quad n \geq 1
\end{aligned}
\]

for approximating a common element of the set of fixed points of a non-self nonexpansive mapping and the set of solutions of the equilibrium problem under some parameters controlling conditions.

In this paper, motivated and inspired by the above results, we introduce a new iterative scheme by the extragradient method as follows: For \( x_1 = u \in C \) and \( \{x_n\}, \{y_n\} \) and \( \{u_n\} \) are given by

\[
\begin{aligned}
  F(y_n, u) + \frac{1}{\alpha_n} \langle u - y_n, y_n - x_n \rangle &\geq 0, \quad \forall u \in C; \\
  x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n) Ty_n, \quad n \geq 1
\end{aligned}
\]  

for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem, and the solution set of the variational inequality problem for a monotone \( k \)-Lipschitz-continuous mapping in a real Hilbert space. Moreover, we obtain a strong convergence theorem which is connected with Yao, Liou and Yao’s result [14], Takahashi and Tada’s result [9] and Zeng and Yao’s result [16].
2. Preliminaries

Let \( H \) be a real Hilbert space with norm \( \| \cdot \| \) and inner product \( \langle \cdot, \cdot \rangle \) and let \( C \) be a closed convex subset of \( H \). Let \( H \) be a real Hilbert space. Then

\[
\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle
\]

(2.1)

and

\[
\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2
\]

(2.2)

for all \( x, y \in H \) and \( \lambda \in [0, 1] \). For every point \( x \in H \), there exists a unique nearest point in \( C \), denoted by \( P_C x \), such that

\[
\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.
\]

\( P_C \) is called the metric projection of \( H \) onto \( C \). It is well known that \( P_C \) is a nonexpansive mapping of \( H \) onto \( C \) and satisfies

\[
\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2
\]

(2.3)

for every \( x, y \in H \). Moreover, \( P_C x \) is characterized by the following properties: \( P_C x \in C \) and

\[
\langle x - P_C x, y - P_C x \rangle \leq 0,
\]

(2.4)

\[
\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2
\]

(2.5)

for all \( x \in H, y \in C \). It is easy to see that the following is true:

\[
u \in VI(A, C) \iff u = P_C(u - \lambda Au), \lambda > 0.
\]

(2.6)

We also have that, for all \( u, v \in C \) and \( \lambda > 0 \),

\[
\|(I - \lambda A)u - (I - \lambda A)v\|^2 = \|(u - v) - \lambda(Au - Av)\|^2
\]

\[
= \|u - v\|^2 - 2\lambda\langle u - v, Au - Av \rangle + \lambda^2\|Au - Av\|^2
\]

\[
\leq \|u - v\|^2 + \lambda(\lambda - 2\alpha)\|Au - Av\|^2.
\]

(2.7)

So, if \( \lambda \leq 2\alpha \), then \( I - \lambda A \) is a nonexpansive mapping from \( C \) to \( H \).

The following lemmas will be useful for proving the convergence result of this paper.

**Lemma 2.1** (See Osilike and Igboke [7].) Let \((E, \langle \cdot, \cdot \rangle)\) be an inner product space. Then for all \( x, y, z \in E \) and \( \alpha, \beta, \gamma \in [0, 1] \) with \( \alpha + \beta + \gamma = 1 \), we have

\[
\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2.
\]

**Lemma 2.2** (See Suzuki [8].) Let \( \{x_n\} \) and \( \{y_n\} \) be bounded sequences in a Banach space \( X \) and let \( \{\beta_n\} \) be a sequence in \([0, 1]\) with \( 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1 \). Suppose \( x_{n+1} = (1 - \beta_n) y_n + \beta_n x_n \) for all integers \( n \geq 0 \) and \( \limsup_{n \to \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0 \). Then, \( \lim_{n \to \infty} \|y_n - x_n\| = 0 \).
Lemma 2.3 (Demiclosedness Principle; cf. Goebel and Kirk [5]). Let $H$ be a Hilbert space, $C$ a closed convex subset of $H$, and $T : C \rightarrow C$ a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in $C$ weakly converging to $x \in C$ and if $\{(I - T)x_n\}$ converges strongly to $y$, then $(I - T)x = y$.

Lemma 2.4 (See Xu [11]). Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, \quad n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that:

1. $\sum_{n=1}^{\infty} \alpha_n = \infty$,
2. $\limsup_{n \rightarrow \infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

For solving the equilibrium problem for a bifunction $F : C \times C \rightarrow \mathbb{R}$, let us assume that $F$ satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) $F$ is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$, $\lim_{t \rightarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$;
(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [1].

Lemma 2.5 (See Blum and Oettli [1]) Let $C$ be a nonempty closed convex subset of $H$ and let $F$ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$F(z, y) + \frac{1}{r}(y - z, z - x) \geq 0 \text{ for all } y \in C.$$ 

The following lemma was also given in [2].

Lemma 2.6 (See Combettes and Hirstoaga [2].) Assume that $F : C \times C \rightarrow \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \rightarrow C$ as follows:

$$T_r(x) = \{z \in C : F(z, y) + \frac{1}{r}(y - z, z - x) \geq 0, \forall y \in C\}$$

for all $z \in H$. Then, the following hold:

1. $T_r$ is single-valued;
2. $T_r$ is firmly nonexpansive, i.e., for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq (T_r x - T_r y, x - y)$;
3. $F(T_r) = EP(F)$;
4. $EP(F)$ is closed and convex.
3. Main Results

In this section, we introduce an iterative process by the extragradient method for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem, and the solution set of the variational inequality problem for a monotone $k$-Lipschitz-continuous mapping in a real Hilbert space. We prove that the iterative sequences converges strongly to a common element of the above three sets.

**Theorem 3.1** Let $C$ be a closed convex subset of a real Hilbert space $H$. Let $F$ be a bifunction from $C \times C \rightarrow \mathbb{R}$ satisfying (A1)--(A4) and let $A$ be a monotone $k$-Lipschitz continuous mapping of $C$ into $H$ and let $S$ be a nonexpansive mapping of $C$ into itself such that $F(S) \cap VI(A,C) \cap EP(F) \neq \emptyset$. Suppose $x_1 = u \in C$ and $\{x_n\}, \{y_n\}$ and $\{u_n\}$ are given by

\[
\begin{aligned}
F(u_n, y) + \frac{1}{r_n}(y - u_n, u_n - x_n) &\geq 0, \quad \forall y \in C; \\
y_n &= P_C(u_n - \lambda_n Au_n) \\
x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n SP_C(x_n - \lambda_n Ay_n),
\end{aligned}
\]  

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0,1]$, $\{\lambda_n\} \subset [a,b]$ for some $a, b \in (0, \frac{1}{k})$ and $\{r_n\} \subset (0, \infty)$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < \lim \inf_{n \rightarrow \infty} \beta_n \leq \lim \sup_{n \rightarrow \infty} \beta_n < 1$,

(iv) $\lim \inf_{n \rightarrow \infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$,

(v) $\lim_{n \rightarrow \infty} (\lambda_{n+1} - \lambda_n) = 0$.

Then $\{x_n\}$ converges strongly to $P_{F(S) \cap VI(A,C) \cap EP(F)} u$.

**Proof.** For all $x, y \in C$, we note that

\[
\begin{aligned}
\|(I - \lambda_n A)x - (I - \lambda_n A)y\|^2 &= \|(x - y) - \lambda_n (Ax - Ay)\|^2 \\
&= \|x - y\|^2 - 2\lambda_n \langle x - y, Ax - Ay \rangle + \lambda_n^2 \|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 + \lambda_n^2 k^2 \|x - y\|^2 = (1 + \lambda_n^2 k^2)\|x - y\|^2,
\end{aligned}
\]  

which implies that

\[
\|(I - \lambda_n A)x - (I - \lambda_n A)y\| \leq (1 + \lambda_n^2)\|x - y\|. 
\]  

Let $x^* \in F(S) \cap VI(A,C) \cap EP(F)$, and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.6 and $u_n = T_{r_n} x_n$. Then $x^* = P_C(x^* - \lambda_n Ax^*) = T_{r_n} x^*$. Put $v_n = P_C(x_n - \lambda_n Ay_n)$. For any $n \in \mathbb{N}$, we get

\[\|u_n - x^*\| = \|T_{r_n} x_n - T_{r_n} x^*\| \leq \|x_n - x^*\|.\]
From (2.5) and the monotonicity of $A$, we have
\[
\|v_n - x^*\|^2 \leq \|x_n - \lambda_nAy_n - x^*\|^2 - \|x_n - \lambda_nAy_n - v_n\|^2
\]
\[
= \|x_n - x^*\|^2 - \|x_n - v_n\|^2 + 2\lambda_n\langle Ay_n, u - v_n \rangle
\]
\[
= \|x_n - x^*\|^2 - \|x_n - v_n\|^2 + 2\lambda_n\langle Ay_n, y_n - v_n \rangle
\]
\[
= \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - 2\langle x_n - y_n, y_n - v_n \rangle - \|y_n - v_n\|^2 + 2\lambda_n\langle Ay_n, y_n - v_n \rangle
\]
\[
= \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - v_n\|^2 + 2\lambda_n\|x_n - y_n\|^2 + \|x_n - \lambda_nAy_n - y_n, v_n - y_n\|^2.
\]
Since $A$ is $k$-Lipschitz-continuous, it follows that
\[
\langle x_n - \lambda_nAy_n - y_n, v_n - y_n \rangle = \langle x_n - \lambda_nAx_n - y_n, v_n - y_n \rangle + \langle \lambda_nAx_n - \lambda_nAy_n, v_n - y_n \rangle
\]
\[
\leq \lambda_nk\|x_n - y_n\|\|v_n - y_n\|.
\]
Thus, we have
\[
\|v_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - v_n\|^2 + 2\lambda_nk\|x_n - y_n\|\|v_n - y_n\|
\]
\[
\leq \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - v_n\|^2 + \lambda_n^2k^2(\|x_n - y_n\|^2 + \|v_n - y_n\|^2)
\]
\[
= \|x_n - x^*\|^2 + (\lambda_n^2k^2 - 1)\|x_n - y_n\|^2 + (\lambda_n^2k^2 - 1)\|y_n - v_n\|^2
\]
\[
\leq \|x_n - x^*\|^2.
\]
Then, we have also
\[
\|x_{n+1} - x^*\| = \|\alpha_nu + \beta_nx_n + \gamma_nSv_n - x^*\|
\]
\[
\leq \alpha_n\|u - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|v_n - x^*\|
\]
\[
\leq \alpha_n\|u - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|x_n - x^*\|
\]
\[
\leq \alpha_n\|u - x^*\| + (1 - \alpha_n)\|x_n - x^*\|
\]
\[
\leq \max\{\|u - x^*\|, \|x_n - x^*\|\}
\]
Therefore $\{x_n\}$ is bounded. Consequently, the sets $\{u_n\}$ and $\{v_n\}$ are also bounded. Moreover, we observe that
\[
\|v_{n+1} - v_n\| = \|P_C(x_{n+1} - \lambda_n+1Ay_{n+1}) - P_C(x_n - \lambda_nAy_n)\|
\]
\[
\leq \|(x_{n+1} - \lambda_n+1Ay_{n+1}) - (x_n - \lambda_nAy_n)\|
\]
\[
= \|(x_{n+1} - x_n) - \lambda_n+1(Ay_{n+1} - Ay_n) - (\lambda_n+1 - \lambda_n)Ay_n\|
\]
\[
\leq \|x_{n+1} - x_n\| + \lambda_n+1\|y_{n+1} - y_n\| + |\lambda_n+1 - \lambda_n|\|Ay_n\|
\]
\[
\leq \|x_{n+1} - x_n\| + \lambda_n+1\|u_{n+1} - u_n\| + |\lambda_n+1 - \lambda_n|\|Ay_n\|.
\]
On the other hand, from $u_n = T_{\alpha_n}x_n$ and $u_{n+1} = T_{\alpha_{n+1}}x_{n+1}$, we have
\[
F(u_n, y) + \frac{1}{r_n}(y - u_n, u_n - x_n) \geq 0 \text{ for all } y \in C
\]
and

\[ F(u_{n+1}, y) + \frac{1}{r_{n+1}}(y - u_{n+1}, u_{n+1} - x_{n+1}) \geq 0 \text{ for all } y \in C. \]  

(3.7)

Putting \( y = u_{n+1} \) in (3.6) and \( y = u_n \) in (3.7), we obtain

\[ F(u_n, u_{n+1}) + \frac{1}{r_n}(u_{n+1} - u_n, u_n - x_n) \geq 0 \]

and

\[ F(u_{n+1}, u_n) + \frac{1}{r_{n+1}}(u_n - u_{n+1}, u_{n+1} - x_{n+1}) \geq 0. \]

It follows from (A2) that

\[ \langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \rangle \geq 0 \]

and hence

\[ \langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) \rangle \geq 0. \]

Since \( \liminf_{n \to \infty} r_n > 0 \), without loss of generality, let us assume that there exists a real number \( c \) such that \( r_n > c > 0 \) for all \( n \in \mathbb{N} \). Then, we have

\[
\|u_{n+1} - u_n\|^2 \leq \langle u_{n+1} - u_n, x_{n+1} - x_n + (1 - \frac{r_n}{r_{n+1}})(u_{n+1} - x_{n+1}) \rangle
\]

\[
\leq \|u_{n+1} - u_n\| \left\{ \|x_{n+1} - x_n\| + |1 - \frac{r_n}{r_{n+1}}| \|u_{n+1} - x_{n+1}\| \right\}
\]

and hence

\[
\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\|
\]

\[
\leq \|x_{n+1} - x_n\| + \frac{L}{c}|r_{n+1} - r_n|, \tag{3.8}
\]

where \( L = \sup\{||u_n - x_n| : n \in \mathbb{N}\} \). Substituting (3.8) into (3.5), we have

\[
\|e_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + k\lambda_n + \frac{1}{c}|r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\|
\]

\[
\leq (1 + k\lambda_n)\|x_{n+1} - x_n\| + k\lambda_n \frac{L}{c}|r_{n+1} - r_n| + |\lambda_n - \lambda_{n+1}| \|Ay_n\|. \tag{3.9}
\]

Let \( x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n \). Thus, we get

\[
z_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} = \frac{\alpha_n u + \gamma_n SP_C(x_n - \lambda_n Ay_n)}{1 - \beta_n} = \frac{\alpha_n u + \gamma_n Sv_n}{1 - \beta_n}
\]
and hence we have
\[ z_{n+1} - z_n = \frac{\alpha_{n+1} u + \gamma_{n+1} Sv_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n u + \gamma_n Sv_n}{1 - \beta_n} \]
\[ = \frac{\alpha_{n+1} u + \gamma_{n+1} Sv_{n+1} - \alpha_n u - \gamma_n Sv_n}{1 - \beta_{n+1}} + \gamma_{n+1} (Sv_{n+1} - Sv_n) \]
\[ = \left( \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right) u + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} (Sv_{n+1} - Sv_n) + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n} Sv_n. \] (3.10)

Combining (3.9) and (3.10), we obtain
\[
\| z_{n+1} - z_n \| - \| x_{n+1} - x_n \| \leq \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| \| u \| + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \| v_{n+1} - v_n \|
\]
\[ + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n} \| Sv_n \| - \| x_{n+1} - x_n \|
\]
\[ \leq \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| \| u \| + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} (1 + \lambda_{n+1} k) \| x_{n+1} - x_n \|
\]
\[ + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \frac{L}{c} \lambda_{n+1} k \| r_{n+1} - r_n \| + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \| Sv_n \| - \| x_{n+1} - x_n \|
\]
\[ \leq \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| \| u \| + \| Sv_n \| + \frac{\gamma_{n+1} \lambda_{n+1} k - \alpha_n}{1 - \beta_{n+1}} \| x_{n+1} - x_n \|
\]
\[ + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \left\{ \lambda_{n+1} k \frac{L}{c} \| r_{n+1} - r_n \| + \| \lambda_n - \lambda_{n+1} \| \| Ay_n \| \right\}. \]

This together with (ii), (iv) and (v) imply that
\[ \limsup_{n \to \infty} (\| z_{n+1} - z_n \| - \| x_{n+1} - x_n \|) \leq 0. \]

Hence, by Lemma 2.2, we have
\[ \lim_{n \to \infty} \| z_n - x_n \| = 0. \] (3.11)

Consequently,
\[ \lim_{n \to \infty} \| x_{n+1} - x_n \| = \lim_{n \to \infty} (1 - \beta_n) \| z_n - x_n \| = 0. \] (3.12)

From (iv), (v), (3.5) and (3.8), we also have \( \| v_{n+1} - v_n \| \to 0, \| u_{n+1} - u_n \| \to 0 \) and \( \| y_{n+1} - y_n \| \to 0 \) as \( n \to \infty \). Since
\[ x_{n+1} - x_n = \alpha_n u + \beta_n x_n + \gamma_n Sv_n - x_n = \alpha_n (u - x_n) + \gamma_n (Sv_n - x_n), \]
it follows by (ii) and (3.12) that
\[ \lim_{n \to \infty} \| x_n - Sv_n \| = 0. \] (3.13)
We note that
\[ \|y_n - v_n\| \leq \|PC(u_n - \lambda_n Au_n) - PC(x_n - \lambda_n Ay_n)\| \]
\[ \leq \|(u_n - \lambda_n Au_n) - (x_n - \lambda_n Ay_n)\| \]
\[ \leq \|u_n - x_n\| + \lambda_n\|Au_n - Ay_n\| \]
\[ \leq \|u_n - x_n\| + \lambda_n\|u_n - y_n\| \]
\[ \leq \|u_n - x_n\|. \]

since \(\lambda_n \leq 1\), hence we also have
\[ \|y_n - v_n\|^2 \leq \|u_n - x_n\|^2. \quad (3.14) \]

From this and by (3.4) and (3.14) we obtain when \(n \geq N\) that
\[ \|v_n - x^*\|^2 \leq \|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2 + (\lambda_n^2 k^2 - 1)\|v_n - y_n\|^2 \]
\[ \leq \|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|y_n - v_n\|^2 \]
\[ \leq \|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|u_n - x_n\|^2. \]

So, from this, we get
\[ \|x_{n+1} - x^*\|^2 = \|\alpha_n u + \beta_n x_n + \gamma_n Sv_n - x^*\|^2 \leq \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|Sv_n - x^*\|^2 \]
\[ \leq \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|v_n - x^*\|^2 \]
\[ \leq \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\{\|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|u_n - x_n\|^2\} \]
\[ = \alpha_n\|u - x^*\|^2 + (1 - \alpha_n)\|x_n - x^*\|^2 + \gamma_n(\lambda_n^2 k^2 - 1)\|u_n - x_n\|^2 \]
\[ \leq \alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|u_n - x_n\|^2, \]

it follows that
\[ (1 - \lambda_n^2 k^2)\|x_n - u_n\|^2 \leq \alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2 \]
\[ \leq \alpha_n\|u - x^*\|^2 + \|x_{n+1} - x_n\|^2 + \|x_{n+1} - x^*\|^2 \]
\[ \leq \alpha_n\|u - x^*\|^2 + \|x_{n+1} - x_n\|\|x_n - x^*\| - \|x_{n+1} - x^*\|). \]

Since \(\alpha_n \to 0\), \(\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{k})\) and \(\|x_{n+1} - x_n\| \to 0\), imply that
\[ \lim_{n \to \infty} \|x_n - u_n\| = 0. \quad (3.15) \]

Since \(\lim\inf_{n \to \infty} r_n > 0\), we get
\[ \lim_{n \to \infty} \frac{\|x_n - u_n\|}{r_n} = \lim_{n \to \infty} \frac{1}{r_n} \|x_n - u_n\| = 0. \quad (3.16) \]

By (3.4), we note that
\[ \|v_n - x^*\|^2 \leq \|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2. \quad (3.17) \]
Thus, from Lemma 2.1 and (3.17), we get
\[
\|x_{n+1} - x^*\|^2 \leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|Sv_n - x^*\|^2
\]
\[
\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|v_n - x^*\|^2
\]
\[
\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \{\|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2\}
\]
\[
\leq \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2.
\] (3.18)

Therefore, we have
\[
(1 - \lambda_n^2 k^2)\|x_n - y_n\|^2 \leq \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2
\]
\[
= \alpha_n \|u - x^*\|^2 + \|x_{n+1} - x_n\|\{\|x_n - x^*\| + \|x_{n+1} - x^*\}\}.
\] (3.19)

Since \(\alpha_n \to 0\) and \(\|x_n - x_{n+1}\| \to 0\), as \(n \to \infty\), we obtain
\[
\lim_{n \to \infty} \|x_n - y_n\| = 0.
\] (3.20)

We note that
\[
\|v_n - y_n\| = \|PC(x_n - \lambda_n Ay_n) - PC(u_n - \lambda_n Au_n)\|
\]
\[
= \|(x_n - \lambda_n Ay_n) - (u_n - \lambda_n Au_n)\|
\]
\[
\leq \|x_n - u_n\| + \lambda_n\|Au_n - Ay_n\|
\]
\[
\leq \|x_n - u_n\| + \lambda_n k\|u_n - y_n\|
\]
\[
\leq \|x_n - u_n\| + \lambda_n k\{\|u_n - x_n\| + \|x_n - y_n\|\}
\]
\[
\leq (1 + \lambda_n k)\|u_n - x_n\| + \lambda_n k\|x_n - y_n\|
\]
since (3.15) and (3.20), we have
\[
\lim_{n \to \infty} \|v_n - y_n\| = 0.
\] (3.21)

Since
\[
\|Sv_n - v_n\| \leq \|Sv_n - x_n\| + \|x_n - y_n\| + \|y_n - v_n\|
\]
and hence
\[
\lim_{n \to \infty} \|Sv_n - v_n\| = 0.
\] (3.22)

Next, we show that
\[
\limsup_{n \to \infty} \langle u - z_0, x_n - z_0 \rangle \leq 0,
\]
where \(z_0 = P_{F(S) \cap VI(A,C) \cap EP(F)}(u)\). To show this inequality, we choose a subsequence \(\{v_n\}\) of \(\{v_n\}\) such that
\[
\limsup_{n \to \infty} \langle u - z_0, Sv_n - z_0 \rangle = \lim_{i \to \infty} \langle u - z_0, Sv_{i} - z_0 \rangle.
\]
Since \( \{v_{n_i}\} \) is bounded, there exists a subsequence \( \{v_{n_{i_j}}\} \) of \( \{v_{n_i}\} \) which converges weakly to \( z \). Without loss of generality, we can assume that \( v_{n_i} \rightharpoonup z \). From \( \|Sv_n - v_n\| \to 0 \), we obtain \( Sv_{n_i} \rightharpoonup z \). Let us show \( z \in EP(F) \). Since \( u_n = T_{v_n}x_n \), we have
\[
F(u_n, y) + \frac{1}{r_n}(y - u_n, u_n - x_n) \geq 0, \forall y \in C.
\]
From (A2), we also have
\[
\frac{1}{r_n}(y - u_n, u_n - x_n) \geq F(y, u_n)
\]
and hence
\[
\langle y - u_{n_i}, u_{n_i} - x_{n_i} \rangle \geq \frac{1}{r_{n_i}} F(y, u_{n_i}).
\]
From \( \|u_n - x_n\| \to 0, \|x_n - Sv_n\| \to 0 \), and \( \|Sv_n - v_n\| \to 0 \), we get \( u_{n_i} \rightharpoonup z \). Since \( \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0 \), it follows by (A4) that \( 0 \geq F(y, z) \) for all \( y \in C \). For \( t \) with \( 0 < t \leq 1 \) and \( y \in C \), let \( y_t = ty + (1-t)z \). Since \( y \in C \) and \( z \in C \), we have \( y_t \in C \) and hence \( F(y_t, z) \leq 0 \). So, from (A1) and (A4) we have
\[
0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, z) \leq tF(y, y)
\]
and hence \( 0 \leq F(y, y) \). From (A3), we have \( 0 \leq F(z, y) \) for all \( y \in C \) and hence \( z \in EP(F) \). By the opial’s condition, we obtain \( z \in F(S) \). Finally, by the same argument as that in the proof of [9, Theorem 3.1, p. 197-198], we can show that \( z \in VI(A, C) \). Hence \( z \in F(S) \cap VI(A, C) \cap EP(F) \).

Now from (2.4), we have
\[
\limsup_{n \to \infty} \langle u - z_0, x_n - z_0 \rangle = \limsup_{n \to \infty} \langle u - z_0, Sv_n - z_0 \rangle = \lim_{i \to \infty} \langle u - z_0, Sv_{n_i} - z_0 \rangle
\]
\[
= \langle u - z_0, z - z_0 \rangle \leq 0.
\]
\[\text{(3.23)}\]

Therefore,
\[
\|x_{n+1} - z_0\|^2 = \langle \alpha_n u + \beta_n x_n + \gamma_n Sv_n - z_0, x_{n+1} - z_0 \rangle
\]
\[
= \alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle + \beta_n \langle x_n - z_0, x_{n+1} - z_0 \rangle + \gamma_n \langle Sv_n - z_0, x_{n+1} - z_0 \rangle
\]
\[
\leq \frac{1}{2} \beta_n \|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2 + \alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle + \frac{1}{2} \gamma_n \|v_n - z_0\|^2 + \|x_{n+1} - z_0\|^2
\]
\[
\leq \frac{1}{2} \beta_n \|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2 + \alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle + \frac{1}{2} \gamma_n \|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2
\]
\[
= \frac{1}{2} (1 - \alpha_n) \|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2 + \alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle
\]
\[
\leq \frac{1}{2} (1 - \alpha_n) \|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2 + \alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle
\]
which implies that
\[
\|x_{n+1} - z_0\|^2 \leq (1 - \alpha_n) \|x_n - z_0\|^2 + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle.
\]
Finally by (3.23) and Lemma 2.4, we get that \( \{x_n\} \) converges to \( z_0 \), where \( z_0 = P_{F(S) \cap VI(A,C) \cap EP(F)}(u) \). This completes the proof.

Using Theorem 3.1, we can prove the following result.

**Theorem 3.2** (Yao Liou and Yao [14, Theorem 3.1]) Let \( C \) be a closed convex subset of a real Hilbert space \( H \). Let \( A \) be a monotone \( k \)-Lipschitz-continuous mapping of \( C \) into \( H \) and let \( S \) be a nonexpansive mapping of \( C \) into itself such that \( F(S) \cap VI(A,C) \neq \emptyset \). For fixed \( u \in H \) and give \( x_0 \in H \) arbitrary, let the sequence \( \{x_n\}, \{y_n\} \) be generated by

\[
\begin{align*}
  y_n &= P_C(x_n - \lambda_n Ax_n) \\
  x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n SP_C(x_n - \lambda_n Ay_n),
\end{align*}
\]

where \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \) are three sequences in \([0,1]\) and \( \{\lambda_n\} \) is a sequence in \([0,\frac{1}{k}]\). If \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \) and \( \{\lambda_n\} \) are chosen so that \( \lambda_n \in [a,b] \) for some \( a,b \) with \( 0 < a < b < \frac{1}{k} \) and

(i) \( \alpha_n + \beta_n + \gamma_n = 1 \),

(ii) \( \lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty \),

(iii) \( 0 < \lim \inf_{n \to \infty} \beta_n \leq \lim \sup_{n \to \infty} \beta_n < 1 \),

(iv) \( \lim_{n \to \infty} (\lambda_{n+1} - \lambda_n) = 0 \),

then \( \{x_n\} \) converges strongly to \( P_{F(S) \cap VI(A,C)}x_0 \).

**Proof.** Put \( F(x,y) = 0 \) for all \( x,y \in C \) and \( r_n = 1 \) for all \( n \in \mathbb{N} \) in Theorem 3.1. Then, we have \( u_n = P_C x_n = x_n \). So, from Theorem 3.1 the sequence \( \{x_n\} \) generated in Theorem 3.2 converges strongly to \( P_{F(S) \cap VI(A,C)}u \).

**Remark 3.3** In Theorem 3.2, we also obtain Yao et al.’s theorem [14].

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