On $\phi$-Recurrent Kenmotsu Manifolds

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Abstract

The object of this paper is to study $\phi$-recurrent Kenmotsu manifolds. Also three-dimensional locally $\phi$-recurrent Kenmotsu manifolds have been considered. Among others it is proved that a locally $\phi$-recurrent Kenmotsu spacetime is the Robertson-Walker spacetime. Finally we give a concrete example of a three-dimensional Kenmotsu manifold.

Key Words: Kenmotsu manifolds, $\phi$-recurrent Kenmotsu manifolds, locally $\phi$-recurrent Kenmotsu manifolds.

1. Introduction

The notion of local symmetry of a Riemannian manifold has been weakened by many authors in several ways to a different extent. As a weaker version of local symmetry, T. Takahashi [16] introduced the notion of locally $\phi$-symmetry on a Sasakian manifold. Generalizing the notion of $\phi$-symmetry, one of the authors, De, [7] introduced the notion of $\phi$-recurrent Sasakian manifold. In the context of contact geometry the notion of $\phi$-symmetry is introduced and studied by Boeckx, Bueken and Vanhecke [3] with several examples.

On the other hand Kenmotsu [11] defined a type of contact metric manifold which is nowadays called Kenmotsu manifold. It may be mentioned that a Kenmotsu manifold is not a Sasakian manifold. Also, a Kenmotsu manifold is not compact because of $\text{div}\xi = 2n$. In [11], Kenmotsu showed that locally a Kenmotsu manifold is a warped product $I \times f N$ of an interval $I$ and a Kahler manifold $N$ with warping function $f(t) = se^t$, where $s$ is a nonzero constant.

The present paper is organized as follows: Section 2 is devoted to preliminaries. In section 3, we prove that a $\phi$-recurrent Kenmotsu manifold is an Einstein manifold and a locally $\phi$-recurrent Kenmotsu manifold is locally a hyperbolic space. In the next section, it is proved that a three-dimensional locally $\phi$-recurrent Kenmotsu manifold is a manifold of constant curvature. In section 5, we prove that a locally $\phi$-recurrent Kenmotsu spacetime is the Robertson-Walker spacetime. In the last section, we construct an example of a three-dimensional Kenmotsu manifold.

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2. Preliminaries

Let $M^{2n+1}(\phi, \xi, \eta, g)$ be an almost contact Riemannian manifold, where $\phi$ is a $(1, 1)$ tensor field, $\xi$ is the structure vector field, $\eta$ is a 1-form and $g$ is the Riemannian metric. It is well known that $(\phi, \xi, \eta, g)$ satisfy

$$\phi \xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1,$$

(2.1)

$$\phi^2 X = -X + \eta(X) \xi,$$

(2.2)

$$g(X, \xi) = \eta(X),$$

(2.3)

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

(2.4)

for any vector fields $X$ and $Y$ on $M$ [1], [2].

If, moreover,

$$(\nabla X \phi) Y = -\eta(Y) \phi X - g(X, \phi Y) \xi, \quad X, Y \in \chi(M),$$

(2.5)

$$\nabla X \xi = X - \eta(X) \xi,$$

(2.6)

where $\nabla$ denotes the Riemannian connection of $g$, then $(M, \phi, \xi, \eta, g)$ is called an almost Kenmotsu manifold [11].

Kenmotsu manifolds have been studied by many authors such as Binh, Tamassy, De and Tarafdar [4], Pitis [15], De and Pathak [5], Jun, De and Pathak [10], Ozgür [13], Ozgür and De [14], Dileo and Pastore [8] and many others.

In a Kenmotsu manifold the following relations hold: [11] .

$$(\nabla_X \eta)(Y) = g(X, Y) - \eta(X)\eta(Y),$$

(2.7)

$$\eta(R(X, Y) Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X),$$

(2.8)

$$R(X, Y) \xi = \eta(X) Y - \eta(Y) X,$$

(2.9)

$$R(\xi, X) Y = \eta(Y) X - g(X, Y) \xi,$$

(2.10)

$$S(X, \xi) = -2n\eta(X),$$

(2.11)

$$(\nabla_Z R)(X, Y) \xi = g(X, Z) Y - g(Y, Z) X - R(X, Y) Z,$$

(2.12)

for any vector fields $X, Y, Z$, where $R$ is the Riemannian curvature tensor and $S$ is the Ricci tensor.

**Definition 1** A Kenmotsu manifold is said to be a locally $\phi$-symmetric manifold if

$$\phi^2((\nabla_W R)(X, Y) Z) = 0,$$

(2.13)

for all vector fields $X, Y, Z, W$ orthogonal to $\xi$.

This notion was introduced for Sasakian manifolds by Takahashi [16].
Definition 2 A Kenmotsu manifold is said to be a $\phi$-recurrent manifold if there exists a non-zero 1-form $A$ such that
\[ \phi^2(\nabla W R)(X,Y)Z = A(W)R(X,Y)Z, \]
for arbitrary vector fields $X, Y, Z, W$.

If $X, Y, Z, W$ are orthogonal to $\xi$, then the manifold is called locally $\phi$-recurrent manifold.

If the 1-form $A$ vanishes, then the manifold reduces to a $\phi$-symmetric manifold.

3. $\phi$-Recurrent Kenmotsu Manifolds

To prove the main theorem of the paper we first prove the following lemma.

Lemma 1 In a $\phi$-recurrent Kenmotsu manifold $(M^{2n+1}, g)$, $n > 1$, the characteristic vector field $\xi$ and the vector field $\rho$ associated to the 1-form $A$ are co-directional and the 1-form $A$ is given by
\[ A(W) = \eta(\rho)\eta(W). \]

Proof. Two vector fields $P$ and $Q$ are said to be co-directional if $P = fQ$ where $f$ is a non-zero scalar. That is,
\[ g(P, X) = fg(Q, X) \quad \text{for all } X. \]

Let us consider a $\phi$-recurrent Kenmotsu manifold. Then by virtue of (2.2) and (2.14), we have
\[ (\nabla W R)(X,Y)Z = \eta((\nabla W R)(X,Y)Z)\xi - A(W)R(X,Y)Z. \]

From (3.16) and the Bianchi identity, we get
\[ A(W)\eta(R(X,Y)Z) + A(X)\eta(R(Y,W)Z) + A(Y)\eta(R(W,X)Z) = 0. \]

Let $\{e_i\}$, $i = 1, 2, 3, ..., 2n + 1$, be an orthonormal basis of the tangent space at any point of the manifold. Putting $Y = Z = e_i$ in (3.17) and taking summation over $i$, $1 \leq i \leq 2n + 1$, we get by virtue of (2.8)
\[ A(W)\eta(X) = A(X)\eta(W), \]
for all vector fields $X, W$. Replacing $X$ by $\xi$ in (3.18), it follows that
\[ A(W) = \eta(\rho)\eta(W), \]

where $A(X) = g(X, \rho)$ and $\rho$ is the vector field associated to the 1-form $A$. From (3.15) and (3.19) it is clear that $\xi$ and $\rho$ are co-directional.

Theorem 1 A $\phi$-recurrent Kenmotsu manifold is an Einstein manifold.

Proof. From (3.16), we have
\[ -g(\nabla W R)(X,Y)Z, U) + \eta((\nabla W R)(X,Y)Z)\eta(U) = A(W)g(R(X,Y)Z, U). \]
Putting $X = U = e_i$ in (3.20) and taking summation over $i$, $1 \leq i \leq 2n + 1$, we get
\[-(\nabla W)(Y, Z) + \sum_{i=1}^{2n+1} \eta((\nabla W R)(e_i, Y)Z)\eta(e_i) = A(W)S(Y, Z).\] (3.21)

The second term of (3.21) by putting $Z = \xi$ takes the form
\[\eta((\nabla W R)(e_i, Y)\xi)\eta(e_i) = g((\nabla W R)(e_i, Y)\xi, \xi)g(\xi, \xi),\] (3.22)
which is denoted by $E$. In this case $E$ vanishes. Namely, we have
\[g((\nabla W R)(e_i, Y)\xi, \xi) = g(\nabla W R(e_i, Y)\xi, \xi) - g(R(e_i, Y)\nabla W \xi, \xi)\] (3.23)
at $p \in M$. In local coordinates $\nabla X e_i = X^j \Gamma^h_{ji}e_h$, where $\Gamma^h_{ji}$ are the Christoffel symbols. Since $\{e_i\}$ is an orthonormal basis, the metric tensor $g_{ij} = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker delta and hence the Christoffel symbols are zero. Therefore, $\nabla X e_i = 0$. Also we have
\[g(R(e_i, \nabla W Y)\xi, \xi) = 0,\] (3.24)
since $R$ is skew-symmetric. Using (3.24) and $\nabla X e_i = 0$ in (3.23), we obtain
\[g((\nabla W R)(e_i, Y)\xi, \xi) = g(\nabla W R(e_i, Y)\xi, \xi) - g(R(e_i, Y)\nabla W \xi, \xi).\]

By virtue of $g(R(e_i, Y)\xi, e_i) = 0$, we have
\[g(\nabla W R(e_i, Y)\xi, \xi) + g(R(e_i, Y)\xi, \nabla W \xi) = 0,\] (3.25)
which implies
\[g((\nabla W R)(e_i, Y)\xi, \xi) = -g(R(e_i, Y)\xi, \nabla W \xi) - g(R(e_i, Y)\nabla W \xi, \xi).\]

Since $R$ is skew-symmetric
\[g((\nabla W R)(e_i, Y)\xi, \xi) = 0.\] (3.26)

Using (3.26) from (3.21), we get
\[(\nabla W S)(Y, \xi) = -A(W)S(Y, \xi).\] (3.27)

We know that
\[(\nabla W S)(Y, \xi) = \nabla W S(Y, \xi) - S(\nabla W Y, \xi) - S(Y, \nabla W \xi).\]

Again using (2.6), (2.7) and (2.11), we get
\[(\nabla W S)(Y, \xi) = -2n g(Y, W) - S(Y, W).\] (3.28)

Now using (3.28) in (3.27), we obtain
\[S(Y, W) = -2n A(W)\eta(Y) - 2n g(Y, W).\] (3.29)

Applying Lemma 1, equation (3.29) reduces to
\[S(Y, W) = -2n g(Y, W) - 2n \eta(\rho)\eta(Y)\eta(W),\]
which implies that the manifold is an $\eta$-Einstein manifold.
In Corollary 9 of Proposition 8 of [11], it is proved that if a Kenmotsu manifold is an η-Einstein manifold of type \( S = ag + b\eta \otimes \eta \) and if \( b = \text{constant} \) (or \( a = \text{constant} \)) then \( M \) is an Einstein manifold. Hence by the above result a \( \phi \)-recurrent Kenmotsu manifold is an Einstein manifold.

**Theorem 2** A locally \( \phi \)-recurrent Kenmotsu manifold \((M^{2n+1}, g), n > 1\), is a manifold of constant curvature \(-1\), i.e., it is locally a hyperbolic space.

**Proof.** From (2.12), we have

\[
(\nabla_W R)(X, Y)\xi = g(W, X)Y - g(W, Y)X - R(X, Y)W. \tag{3.30}
\]

By virtue of (2.8), it follows from (3.30) that

\[
\eta((\nabla_W R)(X, Y)\xi) = 0. \tag{3.31}
\]

In view of (3.30) and (3.31), we obtain from (3.16)

\[-(\nabla_W R)(X, Y)\xi = A(W)R(X, Y)\xi, \tag{3.32}
\]

from which by using (2.12), it follows that

\[-g(X, W)Y + g(Y, W)X + R(X, Y)W = A(W)R(X, Y)\xi. \]

Hence if \( X \) and \( Y \) are orthogonal to \( \xi \), then we get from (2.9)

\[R(X, Y)\xi = 0.
\]

Thus, we obtain

\[R(X, Y)W = -[g(Y, W)X - g(X, W)Y],
\]

for all \( X, Y, W \). \( \square \)

**Remark.** It may be mentioned that a semi-symmetric \((R(X, Y)\cdot R = 0)\) Kenmotsu manifold and a conformally flat Kenmotsu manifold of dimension \( > 3 \) are of constant sectional curvature [11]. Also De and Pathak [5] proved that three dimensional Ricci semi-symmetric \((R(X, Y) \cdot S = 0)\) Kenmotsu manifold is of constant sectional curvature.

### 4. Three-Dimensional Kenmotsu Manifolds

It is known that in a three-dimensional Kenmotsu manifold the curvature tensor has the following form [5]

\[
R(X, Y)Z = \frac{(r + 4)}{2}[g(Y, Z)X - g(X, Z)Y] \\
-\frac{(r + 6)}{2}[g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \tag{4.33}
\]
Taking the covariant differentiation of the equation (4.33), we have

$$(\nabla_W R)(X, Y)Z = \frac{dr(W)}{2}[g(Y, Z)X - g(X, Z)Y] - \frac{dr(W)}{2}[g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi$$

\[+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y - \left(\frac{r + 6}{2}\right)[g(Y, Z)(\nabla_W \eta)(X)\xi + g(Y, Z)\eta(X)\nabla_W \xi - g(X, Z)(\nabla_W \eta)(Y)\xi + (\nabla_W \eta)(Y)\eta(Z)X + \eta(Y)(\nabla_W \eta)(Z)X$$

\[-(\nabla_W \eta)(X)\eta(Z)Y - \eta(X)(\nabla_W \eta)(Z)Y].$$

(4.34)

Now applying $\phi^2$ to the both sides of (4.34), we obtain

$$\phi^2(\nabla_W R)(X, Y)Z = -\frac{dr(W)}{2}[g(Y, Z)X - g(X, Z)Y - g(Y, Z)\eta(X)\xi + g(X, Z)\eta(Y)\xi + \eta(X)\eta(Z)Y$$

\[-\eta(Y)\eta(Z)X + \left(\frac{r + 6}{2}\right)(\nabla_W \eta)(Y)\eta(Z)X + \eta(Y)(\nabla_W \eta)(Z)X - (\nabla_W \eta)(X)\eta(Z)Y$$

\[-\eta(X)(\nabla_W \eta)(Z)Y - (\nabla_W \eta)(Y)\eta(Z)X + (\nabla_W \eta)(X)\eta(Y)\xi].$$

(4.35)

Taking $X, Y, Z, W$ orthogonal to $\xi$ and using (2.14), we finally get from (4.35)

$$A(W)R(X, Y)Z = -\frac{dr(W)}{2}[g(Y, Z)X - g(X, Z)Y].$$

(4.36)

Putting $W = \{e_i\}$ in (4.36), where $\{e_i\}, i = 1, 2, 3,$ is an orthonormal basis of the tangent space at any point of the manifold and taking summation over $i$, $1 \leq i \leq 3$, we obtain

$$R(X, Y)Z = \lambda[g(Y, Z)X - g(X, Z)Y],$$

where $\lambda = -\frac{dr(e_i)}{2\lambda(e_i)}$ is a scalar, since $A$ is a non-zero 1-form. Then by Schur’s theorem $\lambda$ will be a constant on the manifold. Therefore, $M^3$ is of constant curvature $\lambda$. Thus we get the following theorem.

**Theorem 3** A three-dimensional locally $\phi$-recurrent Kenmotsu manifold is of constant curvature.

5. **Locally $\phi$-Recurrent Kenmotsu Spacetime**

In this section we consider locally $\phi$-recurrent Kenmotsu spacetime. By a spacetime, we mean a 4-dimensional semi-Riemannian manifold endowed with Lorentzian metric of signature $(-+++)$. In a recent paper one of the authors De and Pathak [6] prove that the characteristic vector field $\xi$ in a Kenmotsu manifold is a concircular vector field [18]. Also from Theorem 2, we can easily prove that a locally $\phi$-recurrent Kenmotsu manifold is conformally flat. Hence $\text{div}C = 0$, where $C$ denotes the conformal curvature tensor and “$\text{div}$” denotes divergence.

Hence, we have

$$(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = \frac{1}{2(n - 1)}[g(Y, Z)dr(X) - g(X, Z)dr(Y)].$$

(5.37)
Yano [17], prove that, in order that a Riemannian space admits a concircular vector field, it is necessary and sufficient that there exists a coordinate system with respect to which the fundamental quadratic differential form may be written in the form

\[ ds^2 = (dx^1)^2 + e^q g_{\alpha\beta}^* dx^\alpha dx^\beta, \]

where \( g_{\alpha\beta}^* = g_{\alpha\beta}(x') \) are the functions of \( x' \) only (\( \alpha, \beta, r = 2, 3, ..., n \)) and \( q = q(x') \neq \text{constant} \) is a function of \( x^1 \) only. In the semi-Riemannian space, we can prove that

\[ ds^2 = -(dx^1)^2 + e^q g_{\alpha\beta}^* dx^\alpha dx^\beta. \]

Thus a Kenmotsu spacetime can be expressed as a warped product \(-I \times_{e^q} M^*\), where \( M^* \) is a three-dimensional Riemannian manifold. But Gebarowski [9] prove that warped product \(-I \times_{e^q} M^*\) satisfies (5.37) if and only if \( M^* \) is an Einstein manifold. Thus a locally \( \phi \)-recurrent Kenmotsu spacetime must be warped product \(-I \times_{e^q} M^*\), where \( M^* \) is an Einstein manifold. It is known that a three-dimensional Einstein manifold is a manifold of constant curvature. Hence a locally \( \phi \)-recurrent Kenmotsu spacetime is the warped product \(-I \times_{e^q} M^*\), where \( M^* \) is a manifold of constant curvature. But such a warped product is the Robertson-Walker spacetime [12].

Thus we have the following theorem.

**Theorem 4** A locally \( \phi \)-recurrent Kenmotsu spacetime is the Robertson-Walker spacetime.

6. Example of a Three-Dimensional Kenmotsu Manifold

We consider the three-dimensional manifold \( M = \{(x, y, z) \in \mathbb{R}^3 \}, \ z \neq 0 \) where \((x, y, z)\) are the standard coordinates of \( \mathbb{R}^3 \). The vector fields

\[ e_1 = z \frac{\partial}{\partial x}, \quad e_2 = z \frac{\partial}{\partial y}, \quad e_3 = -z \frac{\partial}{\partial z}, \]

are linearly independent at each point of \( M \). Let \( g \) be the Riemannian metric defined by

\[ g(e_1, e_1) = g(e_1, e_2) = g(e_2, e_3) = 0, \]
\[ g(e_3, e_3) = 1. \]

That is, the form of the metric becomes

\[ g = \frac{(dx^2 + dy^2 + dz^2)}{z^2}. \]

Let \( \eta \) be the 1-form defined by \( \eta(Z) = g(Z, e_3) \) for any \( Z \in \chi(M) \). Let \( \phi \) be the (1,1)-tensor field defined by

\[ \phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0. \]

Then using the linearity of \( \phi \) and \( g \), we have

\[ \eta(e_3) = 1, \]
\[ \phi^2 Z = -Z + \eta(Z)e_3, \]
\[ g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W), \]

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for any $Z, W \in \chi(M)$. Then for $e_3 = \xi$, the structure $(\phi, \xi, \eta, g)$ defines an almost contact metric structure on $M$.

Let $\nabla$ be the Levi-Civita connection with respect to the metric $g$. Then we have

$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = e_2.$$ 

The Riemannian connection $\nabla$ of the metric $g$ is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]),$$

which is known as Koszul’s formula. Using this formula we obtain

\[
\begin{align*}
\nabla_{e_1} e_3 &= e_1, & \nabla_{e_2} e_3 &= e_2, \\
\nabla_{e_1} e_2 &= 0, & \nabla_{e_2} e_2 &= -e_3, \\
\nabla_{e_2} e_2 &= -e_3, & \nabla_{e_2} e_2 &= -e_3, \\
\nabla_{e_1} e_1 &= -e_3, & \nabla_{e_2} e_1 &= 0, \\
\nabla_{e_2} e_1 &= 0, & \nabla_{e_2} e_2 &= 0, \\
\nabla_{e_3} e_3 &= 0.
\end{align*}
\]

Thus (2.6) is satisfied. It is straightforward computation to verify that the manifold under consideration is a three-dimensional Kenmotsu manifold.

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