A Condition for Warped Product Semi-Invariant Submanifolds to be Riemannian Product Semi-Invariant Submanifolds in Locally Riemannian Product Manifolds

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Abstract

In this article, we give a necessary and sufficient condition for warped product semi-invariant submanifolds to be Riemannian product semi-invariant submanifolds in a locally Riemannian product manifold whose factor manifolds are real space form.

Key Words: Real space form, Riemannian product and Warped product.

1. Introduction

It is well-known that the notion of warped products plays some important role in differential geometry as well as physics. The geometry of warped product was introduced by B.Y. Chen and it has been studied in the different manifold types by many geometers[see references].

Recently, B.Y. Chen have introduced the notion of CR-warped product in Kaehlerian manifolds and showed that there exist no warped product CR-submanifolds in the form \( M = M_\perp \times_f M_\tau \) in Kaehlerian manifolds. Therefore, he considered warped product CR-submanifolds in the form \( M = M_\tau \times_f M_\perp \), which is called CR-warped product, where

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MT is an invariant submanifold and M⊥ is an anti-invariant submanifold of Kaehlerian manifold \( \bar{M} \).[5]

We note that CR-warped products in Kaehlerian manifold correspond to semi-invariant warped products in Riemannian product manifolds. Recently, we showed that there exist no warped product semi-invariant submanifolds in the form \( M = MT \times f M⊥ \) in contrast to Kaehlerian manifolds [1, 5]. So, in the remainder of this paper we consider warped product semi-invariant submanifolds in the form \( M = M⊥ \times f MT \), where \( M⊥ \) is an anti-invariant submanifold and \( MT \) is an invariant submanifold of Riemannian product manifold \( M \) and it is called warped product semi-invariant submanifold in the rest of this paper.

2. Preliminaries

Let \( M \) be a Riemannian manifold and \( M \) be an isometrically immersed submanifold in \( \bar{M} \). Then the formulas of Gauss and Weingarten for submanifold \( M \) in \( \bar{M} \) are given, respectively, by

\[
\bar{\nabla}_X Y = \nabla_X Y + h(X, Y)
\]

and

\[
\bar{\nabla}_X V = -A_V X + \nabla_X^\perp V,
\]

for any vector fields \( X, Y \) tangent to \( M \) and \( V \) normal to \( M \), where \( \bar{\nabla}, \nabla \) denote the Levi-Civita connections on \( \bar{M}, M \), respectively, \( \nabla^\perp \) is the normal connection in \( TM^\perp \), \( A_V \) is the shape operator of \( M \) with respect to \( V \) and \( h \) is also the second fundamental form of \( M \). Moreover, the second fundamental form \( h \) and shape operator \( A \) are related by

\[
g(A_V X, Y) = g(h(X, Y), V),
\]

where \( g \) denotes the Riemannian metric on \( M \) as well as \( \bar{M} \).

Moreover, the equations of Gauss and Codazzi are given, respectively, by

\[
\bar{R}(X, Y)Z = R(X, Y)Z + A_{h(X, Z)}Y - A_{h(Y, Z)}X + (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z)
\]
and
\[
\{\tilde{R}(X,Y)Z\}^\perp = (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z),
\]
for any \(X, Y, Z\) and \(W\) tangent to \(M\), where \(\tilde{R}\) and \(R\) denote the Riemannian curvature tensors of \(\tilde{M}\) and \(M\), respectively, and \(\{\tilde{R}(X,Y)Z\}^\perp\) is the normal component of \(\tilde{R}(X,Y)Z\).

The covariant derivative of \(h\) is defined by
\[
(\bar{\nabla}_X h)(Y,Z) = \nabla^\perp_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z),
\]
(6)

Now, let \(M_1\) and \(M_2\) be two Riemannian manifolds with Riemannian metrics \(g_1\) and \(g_2\), respectively, and \(f > 0\) be a smooth function on \(M_1\). We consider the product manifold \(M_1 \times f M_2\) with its projections \(\pi : M_1 \times M_2 \rightarrow M_1\) and \(\eta : M_1 \times M_2 \rightarrow M_2\). The warped product \(M = M_1 \times f M_2\) is a manifold \(M_1 \times M_2\) equipped with the Riemannian metric tensor such that
\[
g(X,Y) = g_1(\pi_* X, \pi_* Y) + (f \circ \pi)^2 g_2(\eta_* X, \eta_* Y)
\]
(7)
for any \(X, Y \in \Gamma(TM)\), where \(\Gamma(TM)\) means the set of all differentiable vector fields on \(M\) and \(\ast\) the symbol stands for differential. Thus we have \(g = g_1 + f^2 g_2\).

The function \(f\) is called warping function of the warped product manifold \(M = M_1 \times f M_2\). If warping function is constant, then warped product is called Riemannian product. Furthermore, if we denote the Levi-Civita connection on \(M\) by \(\nabla\), then we have the following proposition for the warped product manifold.

**Proposition 2.1** Let \(M = M_1 \times f M_2\) be a warped product manifold. If \(X, Y \in \Gamma(TM_1)\) and \(V, W \in \Gamma(TM_2)\), then we have
1. \(\nabla_X Y \in \Gamma(TM_1)\) is the lift of \(\nabla_X Y\) on \(M_1\); and
2. \(\nabla_X V = \nabla_Y X = X(\ln f) V\);
3. \(\nabla_W = \nabla^{M_2} W - 2W \frac{\Delta W}{f} \text{grad} f\)
where, \(\nabla^{M_2}\) is the Levi-Civita connection on \(M_2[7]\).

Let \(\tilde{M}\) be a \(m\)-dimensional Riemannian manifold with Riemannian metric \(g\) and \(\{e_1, e_2, ..., e_m\}\) be an orthonormal frame on \(\tilde{M}\). For a smooth function \(\phi\) on \(\tilde{M}\), the Gradient \(\text{grad} \phi\), the Hessian \(H^\phi\) and the Laplacian \(\Delta \phi\) of \(\phi\) are defined, respectively, by
\[
g(\text{grad} \phi, X) = X \phi
\]
(8)
\[ H^\phi(X,Y) = XY\phi - (\nabla_XY)\phi = g(\nabla_X\text{grad}\phi, Y) \] (9)

and

\[ \Delta \phi = \sum_{k=1}^{m} \left\{ (\nabla_{e_k} e_k)\phi - e_k e_k \phi \right\} \] (10)

for any vector fields \( X, Y \) tangent to \( \bar{M} \). From (9) and (10), it is easily seen that the Laplacian is negative of the Hessian as a specific case.

Let \( \bar{M} \) be a compact orientable Riemannian manifold without boundary. Then we have

\[ \int_{\bar{M}} \Delta \phi dV = 0, \] (11)

where \( dV \) is the volume element of \( \bar{M} \)[10].

A real space form is a connected Riemannian manifold of constant sectional curvature \( c \), denoted by \( \bar{M}(c) \). Then the Riemannian curvature tensor of \( \bar{M}(c) \) is given by

\[ R(X,Y)Z = c\{g(X,Z)Y - g(Y,Z)X\} \] (12)

for any vector fields \( X, Y, Z \) tangent to \( \bar{M} \).

Let \( \bar{M} \) be an m-dimensional manifold with a tensor \( F \) of type (1,1) such that \( F^2 = I \), \( F \neq \mp I \) then \( \bar{M} \) is said to be an almost product manifold with almost product structure \( F \). If an almost product manifold \( \bar{M} \) has a Riemannian metric \( g \) such that \( g(FX,Y) = g(X,FY) \), for any \( X,Y \in \Gamma(T\bar{M}) \), then \( \bar{M} \) is called an almost Riemannian product manifold. We denote the Levi-Civita connection on \( \bar{M} \) by \( \nabla \) with respect to \( g \). If \( (\bar{\nabla}_X F)Y = 0 \), for any \( X,Y \in \Gamma(TM) \), then \( \bar{M} \) is called a locally Riemannian product manifold[10].

Let \( \bar{M} \) be a Riemannian manifold with almost Riemannian product structure \( F \) and \( \bar{M} \) be an isometrically immersed submanifold in \( \bar{M} \). For each \( x \in M \), we denote by \( D_x \) the maximal invariant subspace of the tangent space \( T_xM \) of \( M \). If the dimension of \( D_x \) is the same for all \( x \) in \( M \), then \( D_x \) gives an invariant distribution on \( M \).

**Definition 2.1** \( M \) is called a semi-invariant submanifold of a locally Riemannian product manifold \( \bar{M} \) if there exist two orthogonal distributions \( D_1 \) and \( D_2 \) on \( M \) such that
1) $TM$ has the orthogonal direct sum $TM = D_1 \oplus D_2$;
2) The distribution $D_1$ is invariant, i.e., $F(D_1) = D_1$;
3) The distribution $D_2$ is anti-invariant, i.e., $F(D_2) \subset TM^\perp$.

Furthermore, let $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2$ be a locally Riemannian product manifold. If the factor manifolds $\tilde{M}_2$ and $\tilde{M}_2$ have constant sectional curvatures $c_1$ and $c_2$, denoted $\tilde{M}_1(c_1)$ and $\tilde{M}_2(c_2)$, respectively, then the Riemannian curvature tensor of locally Riemannian product manifold $\tilde{M} = \tilde{M}_1(c_1) \times \tilde{M}_2(c_2)$ is given by

$$R(X, Y)Z = \frac{1}{4}(c_1 + c_2)\{g(Y, Z)X - g(X, Z)Y + g(FY, Z)FX - g(FX, Z)FY\}$$
$$+ \frac{1}{4}(c_1 - c_2)\{g(Y, Z)FX - g(X, Z)FY + g(FY, Z)X - g(FX, Z)Y\}$$

(13)

for any vector fields $X, Y, Z$ tangent to $\tilde{M}$.

Let $M$ be a semi-invariant submanifold of a locally Riemannian product manifold $\tilde{M}$. From the definition we have

$$TM = D \oplus D^\perp.$$  \hspace{2cm} (14)

We denote the orthogonal complementary subbundle of $F(D^\perp)$ in $TM^\perp$ by $\nu$, then we have direct sum

$$TM^\perp = F(D^\perp) \oplus \nu.$$ \hspace{2cm} (15)

It is easily to see that $\nu$ is an invariant subbundle of $\Gamma(TM)$ with respect to $F$. Moreover, for any vector field $X$ tangent to $M$ we put

$$FX = TX + \omega X,$$ \hspace{2cm} (16)

where $TX$ and $\omega X$ are the tangent part and normal part of $FX$, respectively. Also, for any vector field $V$ normal to $M$ we put

$$FV = BV + CV,$$ \hspace{2cm} (17)

where $BV$ and $CV$ are the tangent part and normal part of $FV$, respectively. According to the decomposition (15), we consider the projections $P_1$ and $P_2$ of $TM^\perp$ on $F(D^\perp)$ and $\nu$, respectively, then we can write the second fundamental form $h$ of $M$ as

$$h(X, Y) = h_1(X, Y) + h_2(X, Y)$$ \hspace{2cm} (18)
for any $X, Y \in \Gamma(TM)$, where we have put

$$h_1(X, Y) = P_1 h(X, Y) \quad \text{and} \quad h_2(X, Y) = P_2 h(X, Y).$$

3. Warped Product Submanifolds In Locally Riemannian Product Manifolds

Next we will give the following lemma for later use.

**Lemma 3.1** Let $M = M_\perp \times_f M_T$ be a warped product submanifold of a locally Riemannian product manifold $\tilde{M} = \tilde{M}_1(c_1) \times \tilde{M}_2(c_2)$. Then we have

$$\frac{1}{4}(c_1 + c_2)\|X\|^2\|Y\|^2 = \|h(X, Y)\|^2 - H \ln f(Y, Y)\|X\|^2 - \|X\|^2(Y \ln(f))^2 - g(h(X, h(Y, FX)), FY)$$

for any $X \in \Gamma(TM_T)$ and $Y \in \Gamma(TM_\perp)$.

**Proof.** From (5), Proposition 2.1(3) and Levi-Civita connection $\nabla$ connection, we have

$$g(\tilde{R}(X, Y)FX, FY) = g(\tilde{\nabla}_X h(Y, FX) - (\tilde{\nabla}_Y h)(X, FX), FY)$$

$$= g(\tilde{\nabla}_X h(Y, FX) - h(\nabla X Y, FX) - h(\nabla X FX, Y), FY)$$

$$= g(\tilde{\nabla}_X h(Y, FX), FY) - g(h(\nabla X Y, FX), FY)$$

$$= g(h(\nabla X FX, Y), FY) - Yg(h(X, FX), FY)$$

$$= g(h(X, FX), FY) + g(h(Y, FX), FY)$$

$$= Xg(h(Y, FX), FY) - g(h(Y, FX), FX)$$

$$= g(h(\nabla X FX, Y), FY) - Yg(h(X, FX), FY)$$

$$= g(h(X, FX), \nabla Y FY) + g(h(Y, FX), X)$$

for any $X \in \Gamma(TM_T)$ and $Y \in \Gamma(TM_\perp)$.

On the other hand, taking account of $\tilde{M}$ being a locally Riemannian product manifold and $M = M_\perp \times_f M_T$ is a warped product, then we have

$$g(h(Y, FX), FY) = g(\tilde{\nabla}_Y FX, FY) = g(\tilde{\nabla}_Y X, Y)$$

$$= Y \ln f g(X, Y) = 0$$

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\[ g(h(X, FX), FY) = g(\bar{\nabla}_Y FX, FY) = g(\nabla_X X, Y) = -g(\nabla_X Y, X) \]
\[ = -Y \ln f g(X, X). \tag{22} \]

Moreover, considering the ambient space is locally Riemannian product, with Proposition 2.1(3), and using \( M_\perp \), is totally geodesic in \( M \) we get

\begin{align*}
g(h(X, FX), F \bar{\nabla}_Y Y) &= g(h(X, FX), Fh(Y, Y) + F \bar{\nabla}_Y Y) \\
&= g(Fh(X, FX), h(Y, Y)) + g(Fh(X, FX), \nabla_Y Y) \\
&= g(F \bar{\nabla}_X FX - F \nabla_X FX, h(Y, Y)) \\
&= g(F \bar{\nabla}_X FX, h(Y, Y)) + g(FX, \nabla_Y Y) \\
&= g(F \bar{\nabla}_X X, h(Y, Y)) + g(X, FX) g(F \frac{\text{grad} f}{f}, h(Y, Y)) \\
&= g(F \bar{\nabla}_X X, \nabla_Y Y) - g(F \nabla_X FX, \nabla_Y Y) \\
&= g(h(Y, Y), h(X, X)) + \frac{1}{f} g(X, FX) g(h(Y, Y), F \text{grad} f) \\
&= g(h(X, X), h(Y, Y)) + \frac{1}{f} g(X, FX) g(h(Y, Y), F \text{grad} f) \\
&= g(h(X, X), h(Y, Y)) + \frac{1}{f} g(X, FX) g(h(Y, Y), F \text{grad} f) \\
&= -\frac{1}{f} g(X, X) g(\nabla_Y Y, \text{grad} f). \tag{23} \end{align*}

Furthermore, using (1), (2), taking account of \( A \) and \( h \) being symmetric, we infer

\begin{align*}
g(h(Y, Y), \text{grad} f) &= g(\bar{\nabla}_Y Y, F \text{grad} f) = g(\bar{\nabla}_Y FY, \text{grad} f) = -g(A_{FY} Y, \text{grad} f) \\
&= Y(Y, F \text{grad} f) - g(Y, \nabla_Y F \text{grad} f) = -g(\nabla_{FY} \text{grad} f, FY) \\
&= -g(h(Y, \text{grad} f), FY) = -g(\bar{\nabla}_{\text{grad} f} FY, Y) \\
&= g(A_{FY} \text{grad} f, Y). \tag{24} \end{align*}

which implies that

\[ g(h(Y, Y), F \text{grad} f) = 0. \tag{25} \]
Thus equation (23) becomes

\[ g(h(X, FX), F\nabla_Y Y) = g(h(X, X), h(Y, Y)) - g(X, X)g(\nabla_Y Y, \text{grad}_f) \]

\[ = g(h(X, X), h(Y, Y)) - g(X, X)(\nabla_Y Y) \ln f. \quad (26) \]

In the (20), our assertion becomes

\[ g(h(Y, FX), \nabla_X FY) = g(h(Y, FX), Fh(X, Y) + F\nabla_X Y) \]

\[ = g(h(Y, FX), Fh(X, Y)) + g(h(Y, FX), Y \ln f(FX)) \]

\[ = g(Fh(Y, FX), h(X, Y)) \]

\[ = g(F\nabla_Y FX - F\nabla_Y FX, h(X, Y)) \]

\[ = g(\nabla_Y X, h(X, Y)) - Y \ln f g(X, h(X, Y)) \]

\[ = \|h(X, Y)\|^2. \quad (27) \]

Also, considering (21) and Proposition 2.1(3), we arrive at

\[ g(h(\nabla_X FX, Y), FY) = g(h(\nabla_X^M FX - \frac{\text{grad}_f}{f} g(X, FX), Y), FY) \]

\[ = g(h(\nabla_X^M FX, Y), FY) \]

\[ - g(X, FX) \frac{1}{f} g(\text{grad}_f, Y), FY) \]

\[ = -g(X, FX) \frac{1}{f} g(h(\text{grad}_f, Y), FY), \quad (28) \]

and here, taking account of (25) we get

\[ g(h(\text{grad}_f, Y), FY) = g(\nabla_Y \text{grad}_f, FY) = -g(\nabla_Y FY, \text{grad}_f) \]

\[ = -g(\nabla_Y Y, F\text{grad}_f) = -g(h(Y, Y), F\text{grad}_f) = 0. \quad (29) \]
Substituting these equations into (20), consider (22) again and using (8) and (9), we find

\[
g(\bar{R}(X,Y)FX, FY) = -\|h(X,Y)\|^2 + Y((Y\ln f)g(X, X)) \\
+ g(h(X, X), h(Y, Y)) - g(X, X)(\nabla_Y Y)\ln f \\
+ Y\ln fg(h(X, FX), FY) \\
= -\|h(X,Y)\|^2 + Y(Y\ln f)g(X, X) \\
+ g(h(X, X), h(Y, Y)) - g(X, X)(\nabla_Y Y)\ln f \\
- (Y\ln f)^2g(X, X) \\
= -\|h(X,Y)\|^2 + Y(Y\ln f)g(X, X) + 2Y\ln fg(\nabla_Y X, X) \\
+ g(h(X, X), h(Y, Y)) - (\nabla_Y Y)\ln fg(X, X) \\
- (Y\ln f)^2g(X, X) \\
= -\|h(X,Y)\|^2 + Y(Y\ln f)g(X, X) + (Y\ln f)^2g(X, X) \\
+ g(h(X, X), h(Y, Y)) - (\nabla_Y Y)\ln fg(X, X) \\
= -\|h(X,Y)\|^2 + Y\ln f(Y, Y)g(X, X) + (Y\ln f)^2g(X, X) \\
+ g(h(X, X), h(Y, Y)). \tag{30}
\]

Taking account (13) and (30), we conclude that

\[
- \frac{1}{4}(c_1 + c_2)\|X\|^2\|Y\|^2 - \frac{1}{4}(c_1 - c_2)\|Y\|^2g(FX, X) = -\|h(X,Y)\|^2 \\
+ \frac{1}{4}\ln f(Y, Y)g(X, X) + (Y\ln f)^2g(X, X) + g(h(X, X), h(Y, Y)). \tag{31}
\]

Here \(g(FX, X) = 0\) can be choosen. Because, in the locally Riemannian product
manifolds, choosing the vector fields $\pi_*X$ and $\eta_*X$ which have the same length, then from (7), $X$ and $FX$ be orthogonal vector which gives our assertion and (31) becomes

$$
\|h(X, Y)\|^2 = \frac{1}{4}(c_1 + c_2)\|X\|^2\|Y\|^2 + H^{ln \ f}(Y, Y)g(X, X)
+ (\ln f)^2 g(X, X) + g(h(X, X), h(Y, Y)).
$$

(32)

This completes proof of the lemma.

\[\Box\]

**Theorem 3.1** Let $M = M_\perp \times_f M_T$ be a compact orientable warped product semi-invariant submanifold in a locally Riemannian product manifold $\bar{M} = \bar{M}_1(c_1) \times \bar{M}_2(c_2)$. Then $M$ is a Riemannian product submanifold if and only if

$$
\sum_{j=1}^{q} \sum_{i=1}^{p} \sum_{k=1}^{s} (h_{ij}^{k})^2 = \sum_{j=1}^{q} \sum_{i=1}^{p} \sum_{k=1}^{s} h_{ii}^{k} h_{jj}^{k} + \frac{pq}{4}(c_1 + c_2),
$$

(33)

where $h_{ij}^{k} = g(h(e^j, e^i), N^k)$ and $\{e_1, e_2, ..., e_p\}$, $\{e^1, e^2, ..., e^q\}$ and $\{N_1, N_2, ..., N_s\}$ are orthonormal bases for $\Gamma(TM_\perp)$, $\Gamma(TM_T)$ and $\Gamma(\nu)$, respectively.

**Proof.** From (10) and (9), respectively, we have

$$
\Delta \ln f = -\sum_{i=1}^{p} g(\nabla_{e_i} \ln f, e_i) - \sum_{j=1}^{q} g(\nabla_{e^j} \ln f, e^j).
$$

(34)

Making use of $\nabla$ being Levi-Civita connection and (9) we have

$$
H^{ln \ f}(e_i, e_i) = e_i(e_i \ln f) - (\nabla_{e_i} e_i) \ln f
= e_i g(e_i, \ln f) - g(\nabla_{e_i} e_i, \ln f)
= g(\nabla_{e_i} e_i, \ln f) + g(\nabla_{e_i} \ln f, e_i) - g(\nabla_{e_i} e_i, \ln f)
= g(\nabla_{e_i} \ln f, e_i).
$$

(35)

Moreover, taking account of $\nabla f \in \Gamma(TM_\perp)$, Proposition 2.1(3) and consider (34), (35) we arrive at
\[ \Delta \ln f = - \sum_{i=1}^{p} H^{nf}(e_i, e_i) - \sum_{j=1}^{q} g(\nabla_{e_j} \text{grad} \ln f, e^j) \]

\[ = - \sum_{i=1}^{p} H^{nf}(e_i, e_i) - \sum_{j=1}^{q} \{e^j g(\text{grad} \ln f, e^j) - g(\text{grad} f, \nabla_{e_j} e^j)\} \]

\[ = - \sum_{i=1}^{p} H^{nf}(e_i, e_i) - \sum_{j=1}^{q} \{0 - \frac{g(\text{grad} f, -g(e^j, e^j) \frac{1}{f} \text{grad} f)}{f}\} \]

\[ = - \sum_{i=1}^{p} H^{nf}(e_i, e_i) - \frac{1}{f^2} \sum_{j=1}^{q} g(e^j, e^j) g(\text{grad} f, \text{grad} f) \]

\[ = - \sum_{i=1}^{p} H^{nf}(e_i, e_i) - \frac{q}{f^2} \|\text{grad} f\|^2. \quad (36) \]

Now, let \( \{Fe_1, Fe_2, ..., Fe_p, N_1, N_2, ..., N_s\} \) be an orthonormal basis for \( TM_{\perp} = F(TM_{\perp}) \oplus \nu \) such that \( \{Fe_1, Fe_2, ..., Fe_p\} \) and \( \{N_1, N_2, ..., N_s\} \) are orthonormal bases for \( F(TM_{\perp}) \) and \( \nu \), respectively. By direct calculations, we have

\[ h(X, X) = \sum_{i=1}^{p} g(h(X, X), Fe_i) Fe_i + \sum_{j=1}^{s} g(h(X, X), N_j) N_j \]

\[ = - \sum_{i=1}^{p} g(X, FX)(e_i \ln f) Fe_i + \sum_{j=1}^{s} g(h(X, X), N_j) N_j \quad (37) \]

\[ h(Y, Y) = \sum_{k=1}^{p} g(h(Y, Y), Fe_k) Fe_k + \sum_{l=1}^{s} g(h(Y, Y), N_l) N_l. \quad (38) \]

Here, since \( A \) is self-adjoint, we have

\[ g(h(Y, Y), Fe_k) = g(\nabla_Y Y, Fe_k) = g(\nabla_Y FY, e_k) = -g(A_{FY} Y, e_k) \]

\[ = -g(h(e_k, Y), FY) = -g(\nabla_{e_k} Y, FY) = -g(\nabla_{e_k} FY, Y) \]

\[ = g(A_{FY} e_k, Y) = 0, \]
that is, \( h(Y, Y) \) has no component in \( F(TM_{\bot}) \). Thus we have

\[
h(Y, Y) = \sum_{i=1}^{s} g(h(Y, Y), N_i)N_i.
\]  

(39)

Thus from (37) and (39), we have

\[
g(h(X, X), h(Y, Y)) = \sum_{k=1}^{s} \sum_{j=1}^{s} g(h(X, X), N_j)g(h(Y, Y), N_k)
\]

\[
= \sum_{k=1}^{s} g(h(X, X), N_k)g(h(Y, Y), N_k)
\]

\[
= \sum_{k=1}^{s} g(A_{N_k}X, X)g(A_{N_k}Y, Y).
\]  

(40)

In the same way, we have

\[
\|h(X, Y)\|^2 = \sum_{i=1}^{p} g(h(X, Y), Fe_i)g(h(X, Y), Fe_i)
\]

\[
+ \sum_{k=1}^{s} g(h(X, Y), N_k)g(h(X, Y), N_k)
\]

\[
= \sum_{k=1}^{s} g(h(X, Y), N_k)g(h(X, Y), N_k).
\]  

(41)

Let \( X = e^1, e^2, ..., e^q \) and \( Y = e_1, e_2, ..., e_p \) be in (32) and taking account of (36), then we infer

\[
\Delta \ln f = -\frac{1}{p} \sum_{k=1}^{s} \sum_{i=1}^{p} \sum_{j=1}^{q} (h_{ij}^k)^2 + \frac{1}{q} \sum_{k=1}^{s} \sum_{i=1}^{p} \sum_{j=1}^{q} h_{ii}^k h_{jj}^k
\]

\[
+ \frac{p}{4}(e_1 + e_2) - \frac{q^2}{f^2}\|\text{grad} f\|^2 + q \sum_{i=1}^{p} (e_i \ln f)^2.
\]  

(42)
On the other hand, we can easily see that

\[
\sum_{i=1}^{p} (e_i \ln f)^2 = \sum_{i=1}^{p} (g(\text{grad} \ln f, e_i))^2 = \frac{1}{f^2} \sum_{i=1}^{p} g(\text{grad} f, e_i)^2
\]

and hence (42) becomes

\[
\Delta \ln f = -\frac{1}{p} \sum_{k=1}^{s} \sum_{i=1}^{p} \sum_{j=1}^{q} (h_{ij}^k)^2 + \frac{1}{q} \sum_{k=1}^{s} \sum_{i=1}^{p} \sum_{j=1}^{q} h_{ii}^k h_{jj}^k + \frac{p}{4} (c_1 + c_2)
\]

\[
+ \frac{(1-q)}{f^2} \|\text{grad} f\|^2.
\]

(43)

From (11) and (43) we conclude

\[
\int_M \left\{ -\frac{1}{p} \sum_{k=1}^{s} \sum_{i=1}^{p} \sum_{j=1}^{q} (h_{ij}^k)^2 + \frac{1}{q} \sum_{k=1}^{s} \sum_{i=1}^{p} \sum_{j=1}^{q} h_{ii}^k h_{jj}^k + \frac{p}{4} (c_1 + c_2)
\]

\[
+ \frac{(1-q)}{f^2} \|\text{grad} f\|^2 \right\} dV = 0.
\]

(44)

Here if (33) is satisfied, (44) implies that \(\text{grad} f = 0\) which is equivalent to \(f\) is a constant function on \(M_\perp\) because \(q \neq 1\). Thus \(M\) is a Riemannian product.

Conversely, if \(M\) is a Riemannian product, then we have

\[
g(\nabla_Z X, Y) = g(\bar{\nabla}_Z FX, FY) = g(h(Z, FX), FY) = 0,
\]

for any \(X \in \Gamma(TM)\), \(Y \in \Gamma(TM_\perp)\) and \(Z \in \Gamma(TM)\), which implies that \(h(Z, FX) \in \Gamma(\nu)\). Then our assertion is valid.

\[\square\]

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References


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