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Lyapunov-type Inequalities for Certain Nonlinear Systems on Time Scales

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Abstract

In this study, we prove Lyapunov-type inequalities for certain nonlinear systems on an arbitrary time scale $\mathbb{T}$ by using elementary time scale calculus. These inequalities enable us to obtain a criterion of disconjugacy for such systems. Special cases of our results contain the classical Lyapunov inequality for both differential and difference equations.

Key Words: Nonlinear systems; Hamiltonian systems; Lyapunov - type inequality; Discongugacy; Generalized zero; Time Scale.

1. Introduction

In this study, we present Lyapunov-type inequalities for the nonlinear system

$$\begin{aligned} x^{\Delta}(t) &= \alpha_1(t)x(\sigma(t)) + \beta_1(t)\,|u(t)|^{\gamma-2}\,u(t) \\ u^{\Delta}(t) &= -\beta_2(t)\,|x(\sigma(t))|^{\alpha-2}\,x(\sigma(t)) - \alpha_1(t)u(t) \end{aligned} \tag{1}$$

on a time scale $\mathbb{T}$ (an arbitrary nonempty closed subset of real numbers $\mathbb{R}$) where α_1, β_1 and β_2 are real rd–continuous functions on $\mathbb{T}$ with $1-\mu(t)\alpha_1(t) \neq 0$ and $\beta_1(t) > 0$, $\alpha > 1$ constant and α is the conjugate number of γ, *i.e.*, $\frac{1}{\alpha} + \frac{1}{\gamma} = 1$.

Notice that the second order half–linear dynamic equation

$$[r(t)\left|x^{\Delta}(t)\right|^{\alpha-2} x^{\Delta}(t)]^{\Delta} + q(t)\,|x(\sigma(t))|^{\alpha-2}\,x(\sigma(t)) = 0, \tag{2}$$

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where r and q are real rd–continuous functions with $r(t) > 0$ for all $t \in \mathbb{T}$ and $\alpha > 1$, can be written as an equivalent nonlinear system (1) on $\mathbb{T}$. Indeed, let $x(t)$ be a solution of (2) and set $u(t) = r(t)\left|x^{\Delta}(t)\right|^{\alpha-2} x^{\Delta}(t)$. Then we have

$$x^{\Delta}(t) = r^{1-\gamma}(t)\,|u(t)|^{\gamma-2}\,u(t), \qquad u^{\Delta}(t) = -q(t)\,|x(\sigma(t))|^{\alpha-2}\,x(\sigma(t)). \tag{3}$$

Hence (2) is equivalent to (1) with

$$\alpha_1(t) \equiv 0, \qquad \beta_1(t) = r^{1-\gamma}(t), \qquad \beta_2(t) = q(t).$$

We also remark that the nonlinear system (1) with $\alpha = \gamma = 2$ cover not only the recent paper [18] by Jiang and Zhou

$$\begin{aligned} x^{\Delta}(t) &= \alpha_1(t)x(\sigma(t)) + \beta_1(t)u(t) \\ u^{\Delta}(t) &= -\beta_2(t)x(\sigma(t)) - \alpha_1(t)u(t) \end{aligned}$$

but also the linear Hamiltonian system (when $\mathbb{T} = \mathbb{R}$, see [14] and [23])

$$\begin{aligned} x'(t) &= \alpha_1(t)x(t) + \beta_1(t)u(t) \\ u'(t) &= -\beta_2(t)x(t) - \alpha_1(t)u(t) \end{aligned}$$

and the discrete Hamiltonian system (when $\mathbb{T} = \mathbb{Z}$, see [3] and [14])

$$\begin{aligned} \Delta x(t) &= \alpha_1(t)x(t+1) + \beta_1(t)u(t) \\ \Delta u(t) &= -\beta_2(t)x(t+1) - \alpha_1(t)u(t) \end{aligned}.$$

For completeness, we now recall the classical Lyapunov inequality [22] which states that if the nontrivial solution $x(t)$ of

$$x''(t) + q(t)x(t) = 0$$

has two zeros at a and b, $a < b$, then

$$\int_a^b |q(s)|\,ds > \frac{4}{b-a}. \tag{4}$$

This result and many of its generalizations have proved to be useful tools in oscillation theory, disconjugacy, eigenvalue problems, and numerous other applications for the theories of differential and difference equations. A thorough literature of Lyapunov inequalities and their applications can be found in the survey paper [7] by Cheng and the references quoted therein. For authors who contributed the above results, we refer to Reid [27], [28], Hartman [15], Hochstadt [17], Eliason [11], Singh [29], Kwong [19] and Cheng [6]. We should also mention here that inequality (4) has been generalized to second order nonlinear differential equations by Eliason [12] and Pachpatte [25], to delay differential equations of the second order by Eliason [13], by Dahiya and Singh [8], and to certain higher order differential equations by Pachpatte [24]. Lyapunov - type inequalities for the Emden - Fowler type equations can be found in Pachpatte's paper [25]. Lyapunov - type inequalities for the half - linear equation were obtained for the first time by Elbert [10], but the proof of its extension can be found in Došlý and Řehák's recent book ([9], p. 190). Lyapunov - type inequalities for the half–linear equation have been rediscovered by Lee et al. [20] and Pinasco [26].

The paper is organized as follows. In section 2, we recall some basic definitions, concepts and theorems of an arbitrary time scale $\mathbb{T}$. Much of the material in this section is contained in an introductory book by Bohner and Peterson [4]. In section 3, being motivated by the recent papers of Tiryaki *et al.* [30], Ünal *et al.* [31], Guseinov and Kaymakçalan [14], and Jiang and Zhou [18], we set up and prove our main theorems for the nonlinear system (1) on an arbitrary time scale $\mathbb{T}$. In section 4, we obtain a disconjugacy criterion to show that the inequalities proposed in section 3 can be used as a handy tool in the study of the qualitative nature of solutions.

2. Preliminaries on Time Scales

In 1988, Stefan Hilger [16] in his Ph.D. thesis (supervised by Bernd Auibach) added a new wrinkle to the calculus by introducing the calculus on time scale, which is a unification and extension of the theories of continuous and discrete analyses. A time scale is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$, and we usually denote it by the symbol $\mathbb{T}$. The two most popular examples are $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. Some other interesting time scales exist, and they give rise to plenty of applications such as the study of population dynamics model (see [4], page 15, 71). We define the forward and backward jump operators σ, $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \text{ and } \rho(t) := \sup\{s \in \mathbb{T} : s < t\}$$

(supplemented by $\inf \varnothing = \sup \mathbb{T}$ and $\sup \varnothing = \inf \mathbb{T}$). A point $t \in \mathbb{T}$ with $t > \inf \mathbb{T}$ is called *right-scattered, right-dense, left-scattered* and *left-dense,* if $\sigma(t) > t$, $\sigma(t) = t$, $\rho(t) < t$ and $\rho(t) = t$ holds, respectively. Points are left-dense and right-dense at the same time are called *dense.* The set $\mathbb{T}^\kappa$ is derived from $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. *The graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

Hence the graininess function is 0 if $\mathbb{T} = \mathbb{R}$ while it is 1 if $\mathbb{T} = \mathbb{Z}$. Let f be a function defined on $\mathbb{T}$, then we define the delta derivative of f at $t \in \mathbb{T}^\kappa$, denoted by $f^\Delta(t)$, to be the number (provided it exists) with the property such that for every $\epsilon > 0$, there exists a neighborhood $\mathbb{U}$ of t with

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)\,[\sigma(t) - s] \right| \leq \epsilon\,|\sigma(t) - s| \qquad \text{for all } s \in \mathbb{U}.$$

Some elementary facts concerning the delta derivative are contained in the following lemma.

Lemma 1 *Let f, $g : \mathbb{T} \to \mathbb{R}$ be two function and $t \in \mathbb{T}^\kappa$. Then we have the following:*

i) If f is differentiable at t, then f is continuous at t.

ii) If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

iii) If f is differentiable and t is right-dense, then

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

iv) If f is differentiable at t, then

$$f^\sigma(t) = f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t).$$

v) If f and g are differentiable at t, then fg is differentiable at t with

$$(fg)^\Delta(t) = f^\sigma(t) g^\Delta(t) + f^\Delta(t) g(t) = f(t) g^\Delta(t) + f^\Delta(t) g^\sigma(t).$$

vi) If f and g are differentiable at t and $g(t)g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is differentiable at t with

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

We say $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous provided f is continuous at right-dense points in $\mathbb{T}$ and its left-sided limit exists (finite) at left-dense points in $\mathbb{T}$. One of the important property of rd-continuous fuctions is that *every rd-continuous function possesses an antiderivative.* A function $F : \mathbb{T}^{\kappa} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^{\kappa}$. In this case we define the integral of f by

$$\int_a^t f(s)\Delta s = F(t) - F(a)$$

for all $t \in \mathbb{T}$.

Other useful formulas are as follows

$$\int_t^{\sigma(t)} f(s)\Delta s = \mu(t)f(t), \tag{5}$$

$$\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t, \tag{6}$$

$$\int_a^b f(t)g^{\Delta}(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^{\Delta}(t)g^{\sigma}(t)\Delta t. \tag{7}$$

Let $f,\, g : \mathbb{T} \to \mathbb{R}$ be rd-continuous and $a,\, b \in \mathbb{T}$. If $|f(t)| \leq g(t)$ on $[a, b)$, then

$$\left|\int_a^b f(t)\Delta t\right| \leq \int_a^b g(t)\Delta t. \tag{8}$$

We will also need the following version of Hölder's inequality on time scales in the proof of our main theorems and its proof can be found in [1] and [2].

Lemma 2 *Let $a, b \in \mathbb{T}$. For rd-continuous $f, g : [a, b] \to \mathbb{R}$ we have*

$$\int_a^b |f(t)g(t)|\,\Delta t \le \left(\int_a^b |f(t)|^\gamma\,\Delta t\right)^{1/\gamma} \left(\int_a^b |g(t)|^\alpha\,\Delta t\right)^{1/\alpha}$$

where $\frac{1}{\gamma} + \frac{1}{\alpha} = 1$.

A comprehensive treatment of calculus on time scales can be found, for instance, in [4], [5], and [21].

3. Main Results

Since our attention is restricted to the Lyapunov-type inequalities for the nonlinear system (1) on an arbitrary time scale $\mathbb{T}$, we shall assume the existence of nontrivial real solution (x, u) of the nonlinear system (1).

We recall that a nontrivial real solution (x, u) of system (1) has a *generalized zero* at $\sigma(t)$ if $t \in \mathbb{T}$ is either dense and $x(t) = 0$ or right–scattered, and $x(t)x(\sigma(t)) < 0$ or $x(\sigma(t)) = 0$. We note that under the condition $\beta_1(t) > 0$ for $t \in \mathbb{T}$, the definition of generalized zero, a special case of that given in [4], is consistent with what is used for the generalized zero in the discrete case (see [18]).

Theorem 1 *Suppose $\beta_1(t) > 0$ on $[a, \sigma(b)]$. Let a, $b \in \mathbb{T}$ with $\sigma(a) < b$. Assume that (1) has a real solution (x, u) such that $x(\sigma(a)) = 0 = x(\sigma(b))$ and x is not identically zero on $[\sigma(a), b]$. Then the inequality*

$$2 \le \int_{\sigma(a)}^b |\alpha_1(t)|\,\Delta t + \left(\int_{\sigma(a)}^{\sigma(b)} \beta_1(t)\Delta t\right)^{1/\gamma} \left(\int_{\sigma(a)}^b \beta_2^+(t)\Delta t\right)^{1/\alpha} \tag{9}$$

holds, where $\beta_2^+(t) = \max\{0, \beta_2(t)\}$ and $\frac{1}{\alpha} + \frac{1}{\gamma} = 1$.

Proof. Let $(x(t), u(t))$ be nontrivial real solution of system (1) such that $x(\sigma(a)) = 0 = x(\sigma(b))$ and x is not identically zero on $[\sigma(a), b]$. Then multiplying the first equation of (1) by $u(t)$ and the second one by $x(\sigma(t))$, then adding them up yields

$$(xu)^\Delta(t) = \beta_1(t)\,|u(t)|^\gamma - \beta_2(t)\,|x(\sigma(t))|^\alpha\,. \tag{10}$$

Integrating (10) from $\sigma(a)$ to $\sigma(b)$ and taking into account that $x(\sigma(a)) = 0 = x(\sigma(b))$, we have

$$0 = \int_{\sigma(a)}^{\sigma(b)} \beta_1(t) |u(t)|^{\gamma} \Delta t - \int_{\sigma(a)}^{\sigma(b)} \beta_2(t) |x(\sigma(t))|^{\alpha} \Delta t. \tag{11}$$

Since $x(\sigma(b)) = 0$, we have

$$\begin{aligned}
\int_{\sigma(a)}^{\sigma(b)} \beta_1(t) |u(t)|^{\gamma} \Delta t &= \int_{\sigma(a)}^{\sigma(b)} \beta_2(t) |x(\sigma(t))|^{\alpha} \Delta t \\
&= \int_{\sigma(a)}^{b} \beta_2(t) |x(\sigma(t))|^{\alpha} \Delta t + \int_{b}^{\sigma(b)} \beta_2(t) |x(\sigma(t))|^{\alpha} \Delta t \\
&= \int_{\sigma(a)}^{b} \beta_2(t) |x(\sigma(t))|^{\alpha} \Delta t + \mu(b)\beta_2(b) |x(\sigma(b))|^{\alpha} \\
&= \int_{\sigma(a)}^{b} \beta_2(t) |x(\sigma(t))|^{\alpha} \Delta t. \qquad (12)
\end{aligned}$$

Choose $\tau \in (\sigma(a), \sigma(b))$ such that $|x(\tau)| = \max\limits_{\sigma(a) \leq t \leq \sigma(b)} |x(t)|$. Since x is not identically zero on $[\sigma(a), b]$, we have $|x(\tau)| > 0$. Integrating the first equation of (1) from $\sigma(a)$ to τ and using $x(\sigma(a)) = 0$, we obtain

$$x(\tau) = \int_{\sigma(a)}^{\tau} \alpha_1(t) x(\sigma(t)) \Delta t + \int_{\sigma(a)}^{\tau} \beta_1(t) |u(t)|^{\gamma-2} u(t) \Delta t, \tag{13}$$

and hence

$$|x(\tau)| \leq \int_{\sigma(a)}^{\tau} |\alpha_1(t)| |x(\sigma(t))| \Delta t + \int_{\sigma(a)}^{\tau} \beta_1(t) |u(t)|^{\gamma-1} \Delta t. \tag{14}$$

Similarly, since $x(\sigma(b)) = 0$, we have

$$\begin{aligned}
-x(\tau) &= \int_{\tau}^{\sigma(b)} \alpha_1(t)x(\sigma(t))\Delta t + \int_{\tau}^{\sigma(b)} \beta_1(t)\,|u(t)|^{\gamma-2}\,u(t)\Delta t \\
&= \int_{\tau}^{b} \alpha_1(t)x(\sigma(t))\Delta t + \int_{b}^{\sigma(b)} \alpha_1(t)x(\sigma(t))\Delta t + \int_{\tau}^{\sigma(b)} \beta_1(t)\,|u(t)|^{\gamma-2}\,u(t)\Delta t \\
&= \int_{\tau}^{b} \alpha_1(t)x(\sigma(t))\Delta t + \mu(b)\alpha_1(b)x(\sigma(b)) + \int_{\tau}^{\sigma(b)} \beta_1(t)\,|u(t)|^{\gamma-2}\,u(t)\Delta t \\
&= \int_{\tau}^{b} \alpha_1(t)x(\sigma(t))\Delta t + \int_{\tau}^{\sigma(b)} \beta_1(t)\,|u(t)|^{\gamma-2}\,u(t)\Delta t, \qquad (15)
\end{aligned}$$

and hence

$$|-x(\tau)| = |x(\tau)| \leq \int_{\tau}^{b} |\alpha_1(t)|\,|x(\sigma(t))|\,\Delta t + \int_{\tau}^{\sigma(b)} \beta_1(t)\,|u(t)|^{\gamma-1}\,\Delta t. \qquad (16)$$

Summing up (14) and (16) yields

$$2\,|x(\tau)| \leq \int_{\sigma(a)}^{b} |\alpha_1(t)|\,|x(\sigma(t))|\,\Delta t + \int_{\sigma(a)}^{\sigma(b)} \beta_1(t)\,|u(t)|^{\gamma-1}\,\Delta t. \qquad (17)$$

By applying Hölder's inequality to the second integral of the right hand side of (17) with the indices α and γ, we obtain

$$\begin{aligned}\int_{\sigma(a)}^{\sigma(b)} \beta_1(t)\,|u(t)|^{\gamma-1}\,\Delta t &= \int_{\sigma(a)}^{\sigma(b)} \beta_1^{\frac{1}{\gamma}}(t)\beta_1^{\frac{1}{\alpha}}(t)\,|u(t)|^{\gamma-1}\,\Delta t \\ &\le \left(\int_{\sigma(a)}^{\sigma(b)} \beta_1(t)\Delta t\right)^{1/\gamma}\left(\int_{\sigma(a)}^{\sigma(b)} \beta_1(t)\,|u(t)|^{(\gamma-1)\alpha}\,\Delta t\right)^{1/\alpha} \\ &= \left(\int_{\sigma(a)}^{\sigma(b)} \beta_1(t)\Delta t\right)^{1/\gamma}\left(\int_{\sigma(a)}^{\sigma(b)} \beta_1(t)\,|u(t)|^{\gamma}\,\Delta t\right)^{1/\alpha}, \qquad (18)\end{aligned}$$

where $\frac{1}{\alpha}+\frac{1}{\gamma}=1$. Substituting the inequality (18) into (17) yields

$$2\,|x(\tau)| \le \int_{\sigma(a)}^{b} |\alpha_1(t)|\,|x(\sigma(t))|\,\Delta t + \left(\int_{\sigma(a)}^{\sigma(b)} \beta_1(t)\Delta t\right)^{1/\gamma}\left(\int_{\sigma(a)}^{\sigma(b)} \beta_1(t)\,|u(t)|^{\gamma}\,\Delta t\right)^{1/\alpha}. \qquad (19)$$

Substituting equality (12) into (19), we get

$$\begin{aligned}2\,|x(\tau)| &\le \int_{\sigma(a)}^{b} |\alpha_1(t)|\,|x(\sigma(t))|\,\Delta t + \left(\int_{\sigma(a)}^{\sigma(b)} \beta_1(t)\Delta t\right)^{1/\gamma}\left(\int_{\sigma(a)}^{b} \beta_2(t)\,|x(\sigma(t))|^{\alpha}\,\Delta t\right)^{1/\alpha} \\ &\le |x(\tau)|\int_{\sigma(a)}^{b} |\alpha_1(t)|\,\Delta t + |x(\tau)|\left(\int_{\sigma(a)}^{\sigma(b)} \beta_1(t)\Delta t\right)^{1/\gamma}\left(\int_{\sigma(a)}^{b} \beta_2^{+}(t)\Delta t\right)^{1/\alpha}. \qquad (20)\end{aligned}$$

Dividing the latter inequality by $|x(\tau)|$, we obtain inequality (9). □

Remark 1 *We should note here that Theorem 1 reduces to Corollary 2 in [30] when* $\mathbb{T}=\mathbb{R}$ *and to Theorem 1 with* $\beta=\alpha$ *in [31] when* $\mathbb{T}=\mathbb{Z}$.

Theorem 2 *Suppose* $1-\mu(t)\alpha_1(t)>0$ *and* $\beta_1(t)>0$ *on* $[a,\sigma(b)]$. *Let* $a,\ b\in\mathbb{T}$ *with* $\sigma(a)<b$. *Assume that (1) has a real solution* (x,u) *such that* $x(\sigma(a))=0$ *and*

$x(b)x(\sigma(b)) < 0$. *Then the inequality*

$$1 < \int_{\sigma(a)}^{b} |\alpha_1(t)| \Delta t + \left(\int_{\sigma(a)}^{b} \beta_1(t) \Delta t \right)^{1/\gamma} \left(\int_{\sigma(a)}^{b} \beta_2^+(t) \Delta t \right)^{1/\alpha} \tag{21}$$

holds, where $\beta_2^+(t) = \max\{0, \beta_2(t)\}$ *and* $\frac{1}{\alpha} + \frac{1}{\gamma} = 1$.

Proof. Integrating (10) from $\sigma(a)$ to b and observing that $x(\sigma(a)) = 0$ we obtain

$$u(b)x(b) = \int_{\sigma(a)}^{b} \beta_1(t) |u(t)|^{\gamma} \Delta t - \int_{\sigma(a)}^{b} \beta_2(t) |x(\sigma(t))|^{\alpha} \Delta t. \tag{22}$$

Rewriting the first equation of (1) by using the formula $f^{\sigma}(t) = f(t) + \mu(t) f^{\Delta}(t)$, we get

$$(1 - \mu(t)\alpha_1(t)) x(\sigma(t)) = x(t) + \mu(t)\beta_1(t) |u(t)|^{\gamma - 2} u(t). \tag{23}$$

Taking $t = b$ in (23) and multiplying this result by $x(b)$ yields

$$(1 - \mu(b)\alpha_1(b)) x(b) x(\sigma(b)) = x^2(b) + \mu(b)\beta_1(b) |u(b)|^{\gamma - 2} u(b) x(b). \tag{24}$$

Since $x(b)x(\sigma(b)) < 0$, it is easy to see that $\mu(b) > 0$. Also since $1 - \mu(t)\alpha_1(t) > 0$ and $\beta_1(t) > 0$ for all $t \in \mathbb{T}$ by the hypothesis of theorem, (24) gives rise to $u(b)x(b) < 0$. Therefore, it follows from (22) that the inequality

$$\int_{\sigma(a)}^{b} \beta_1(t) |u(t)|^{\gamma} \Delta t < \int_{\sigma(a)}^{b} \beta_2(t) |x(\sigma(t))|^{\alpha} \Delta t \leq \int_{\sigma(a)}^{b} \beta_2^+(t) |x(\sigma(t))|^{\alpha} \Delta t \tag{25}$$

holds. Choose $\tau \in [\sigma(a), b)$ such that $|x(\tau)| = \max_{\sigma(a) \leq t \leq b} |x(t)|$. Integrating the first equation of (1) from $\sigma(a)$ to τ and noticing that $x(\sigma(a)) = 0$, we obtain

$$x(\tau) = \int_{\sigma(a)}^{\tau} \alpha_1(t) x(\sigma(t)) \Delta t + \int_{\sigma(a)}^{\tau} \beta_1(t) |u(t)|^{\gamma - 2} u(t) \Delta t, \tag{26}$$

and hence

$$|x(\tau)| \leq \int_{\sigma(a)}^{b} |\alpha_1(t)| |x(\sigma(t))| \Delta t + \int_{\sigma(a)}^{b} \beta_1(t) |u(t)|^{\gamma - 1} \Delta t. \tag{27}$$

By using Hölder's inequality in the second integral of the right hand side of (27) with the indices α and γ, we obtain

$$\begin{aligned}\int_{\sigma(a)}^{b} \beta_1(t)\left|u(t)\right|^{\gamma-1}\Delta t &= \int_{\sigma(a)}^{b} \beta_1^{\frac{1}{\gamma}}(t)\beta_1^{\frac{1}{\alpha}}(t)\left|u(t)\right|^{\gamma-1}\Delta t \\ &\leq \left(\int_{\sigma(a)}^{b} \beta_1(t)\Delta t\right)^{1/\gamma}\left(\int_{\sigma(a)}^{b} \beta_1(t)\left|u(t)\right|^{(\gamma-1)\alpha}\Delta t\right)^{1/\alpha} \\ &= \left(\int_{\sigma(a)}^{b} \beta_1(t)\Delta t\right)^{1/\gamma}\left(\int_{\sigma(a)}^{b} \beta_1(t)\left|u(t)\right|^{\gamma}\Delta t\right)^{1/\alpha}, \qquad (28)\end{aligned}$$

where $\frac{1}{\alpha}+\frac{1}{\gamma}=1$. Substituting the inequality (28) into (27) yields

$$|x(\tau)| \leq \int_{\sigma(a)}^{b} |\alpha_1(t)|\,|x(\sigma(t))|\,\Delta t + \left(\int_{\sigma(a)}^{b} \beta_1(t)\Delta t\right)^{1/\gamma}\left(\int_{\sigma(a)}^{b} \beta_1(t)\left|u(t)\right|^{\gamma}\Delta t\right)^{1/\alpha}. \qquad (29)$$

Using (25) in (29), we have

$$\begin{aligned}|x(\tau)| &< \int_{\sigma(a)}^{b} |\alpha_1(t)|\,|x(\sigma(t))|\,\Delta t + \left(\int_{\sigma(a)}^{b} \beta_1(t)\Delta t\right)^{1/\gamma}\left(\int_{\sigma(a)}^{b} \beta_2^{+}(t)\left|x(\sigma(t))\right|^{\alpha}\Delta t\right)^{1/\alpha} \\ &\leq |x(\tau)|\int_{\sigma(a)}^{b} |\alpha_1(t)|\,\Delta t + |x(\tau)|\left(\int_{\sigma(a)}^{b} \beta_1(t)\Delta t\right)^{1/\gamma}\left(\int_{\sigma(a)}^{b} \beta_2^{+}(t)\Delta t\right)^{1/\alpha}. \qquad (30)\end{aligned}$$

Dividing the latter inequality by $|x(\tau)|$, we get the desired inequality (21). □

Theorem 3 *Suppose* $1-\mu(t)\alpha_1(t) > 0$ *and* $\beta_1(t) > 0$ *on* $[a,\sigma(b)]$. *Let* $a,\, b \in \mathbb{T}$ *with* $\sigma(a) < b$. *Assume that (1) has a real solution* (x,u) *such that* $x(a)x(\sigma(a)) < 0$ *and*

$x(\sigma(b)) = 0$. *Then the inequality*

$$1 < \int_{\sigma(a)}^{b} |\alpha_1(t)|\,\Delta t + \left(\int_{\sigma(a)}^{\sigma(b)} \beta_1(t)\Delta t\right)^{1/\gamma} \left(\int_{a}^{b} \beta_2^{+}(t)\Delta t\right)^{1/\alpha} \tag{31}$$

holds, where $\beta_2^{+}(t) = \max\{0, \beta_2(t)\}$ *and* $\frac{1}{\alpha} + \frac{1}{\gamma} = 1$.

Proof. The proof can be obtained easily by the similar method used in the proof of the Theorem 2 with a slight modification. Hence it is omitted. □

Theorem 4 *Suppose* $1-\mu(t)\alpha_1(t) > 0$, $\beta_1(t) > 0$ *and* $\beta_2(t) > 0$ *on* $[a, \sigma(b)]$. *Let* a, $b \in \mathbb{T}$ *with* $\sigma(a) < b$. *Assume that (1) has a real solution* (x, u) *such that* $x(a)x(\sigma(a)) < 0$ *and* $x(b)x(\sigma(b)) < 0$. *Then the inequality*

$$1 < \int_{a}^{b} |\alpha_1(t)|\,\Delta t + \left(\int_{a}^{\sigma(b)} \beta_1(t)\Delta t\right)^{1/\gamma} \left(\int_{a}^{b} \beta_2(t)\Delta t\right)^{1/\alpha} \tag{32}$$

holds, where $\frac{1}{\alpha} + \frac{1}{\gamma} = 1$.

Proof. We have two cases: Either $x(t) \neq 0$ for all $t \in [a, b]$ or $x(t_0) = 0$ for some $t_0 \in (\sigma(a), b)$. The latter case follows from applying Theorem 2 to the point t_0 and b. Hence, we only prove the first case.

Assume that $x(t) \neq 0$ for all $t \in [a, b]$. Let b_0 be the smallest number in $(a, b]$ such that $x(b_0)x(\sigma(b_0)) < 0$, then x does not have any generalized zero in $(\sigma(a), b_0)$. Without loss of generality we may assume $x(t) > 0$ for all $t \in [\sigma(a), b_0]$ and it follows from $x(a)x(\sigma(a)) < 0$ and $x(b_0)x(\sigma(b_0)) < 0$ that $x(a) < 0$ and $x(\sigma(b_0)) < 0$ must hold. Let $s \in [a, \sigma(b_0)]$ be such that $|u(s)| = \max_{a \le t \le \sigma(b_0)} |u(t)|$. Integrating the second equality of (1) from a to s and then from s to b_0, we obtain

$$u(s) - u(a) = -\int_{a}^{s} \beta_2(t)\,|x(\sigma(t))|^{\alpha-2}\,x(\sigma(t))\Delta t - \int_{a}^{s} \alpha_1(t)u(t)\Delta t, \tag{33}$$

and

$$u(b_0) - u(s) = -\int_s^{b_0} \beta_2(t) |x(\sigma(t))|^{\alpha-2} x(\sigma(t))\Delta t - \int_s^{b_0} \alpha_1(t)u(t)\Delta t, \tag{34}$$

respectively. We note that for $s = a$ we write solely (34), and for $s = b_0$ only (33) is written. We claim that $u(a) > 0$ and $u(b_0) < 0$. Indeed, multiplying (23) by $x(t)$, we obtain

$$(1 - \mu(t)\alpha_1(t))x(t)x(\sigma(t)) = x^2(t) + \mu(t)\beta_1(t) |u(t)|^{\gamma-2} u(t)x(t). \tag{35}$$

Setting $t = a$ and $t = b_0$ in (35) yields

$$(1 - \mu(a)\alpha_1(a))x(a)x(\sigma(a)) = x^2(a) + \mu(a)\beta_1(a) |u(a)|^{\gamma-2} u(a)x(a),$$

and

$$(1 - \mu(b_0)\alpha_1(b_0))x(b_0)x(\sigma(b_0)) = x^2(b_0) + \mu(b_0)\beta_1(b_0) |u(b_0)|^{\gamma-2} u(b_0)x(b_0),$$

respectively. It is easy to see that $\mu(a) > 0$ and $\mu(b_0) > 0$ from $x(a)x(\sigma(a)) < 0$ and $x(b_0)x(\sigma(b_0)) < 0$, respectively. Combining $1 - \mu(t)\alpha_1(t) > 0$ and $\beta_1(t) > 0$ with the above inequalities, we get $u(a)x(a) < 0$ and $u(b_0)x(b_0) < 0$. Since $x(a) < 0$ and $x(b_0) > 0$, we must have $u(a) > 0$ and $u(b_0) < 0$ as claimed. Now, if $u(s) < 0$ then we have from (33) and $u(a) > 0$ that

$$\begin{aligned} |u(s)| &\le \int_a^s \beta_2(t) |x(\sigma(t))|^{\alpha-1} \Delta t + \int_a^s |\alpha_1(t)| |u(t)| \Delta t \\ &\le \int_a^{b_0} \beta_2(t) |x(\sigma(t))|^{\alpha-1} \Delta t + \int_a^{b_0} |\alpha_1(t)| |u(t)| \Delta t, \end{aligned}$$

and if $u(s) > 0$ then we have from (34) and $u(b_0) < 0$ that

$$\begin{aligned} |u(s)| &\le \int_s^{b_0} \beta_2(t) |x(\sigma(t))|^{\alpha-1} \Delta t + \int_s^{b_0} |\alpha_1(t)| |u(t)| \Delta t \\ &\le \int_a^{b_0} \beta_2(t) |x(\sigma(t))|^{\alpha-1} \Delta t + \int_a^{b_0} |\alpha_1(t)| |u(t)| \Delta t. \end{aligned}$$

So in either case, we have

$$|u(s)| \leq \int_a^{b_0} \beta_2(t) |x(\sigma(t))|^{\alpha-1} \Delta t + \int_a^{b_0} |\alpha_1(t)| |u(t)| \Delta t. \tag{36}$$

Using Hölder's inequality in the first integral of the right hand side of (36) with the indices α and γ, we obtain

$$\begin{aligned} \int_a^{b_0} \beta_2(t) |x(\sigma(t))|^{\alpha-1} \Delta t &\leq \left(\int_a^{b_0} \beta_2(t) \Delta t \right)^{1/\alpha} \left(\int_a^{b_0} \beta_2(t) |x(\sigma(t))|^{(\alpha-1)\gamma} \Delta t \right)^{1/\gamma} \\ &= \left(\int_a^{b_0} \beta_2(t) \Delta t \right)^{1/\alpha} \left(\int_a^{b_0} \beta_2(t) |x(\sigma(t))|^{\alpha} \Delta t \right)^{1/\gamma}. \end{aligned} \tag{37}$$

Substituting (37) into (36) yields

$$|u(s)| \leq \left(\int_a^{b_0} \beta_2(t) \Delta t \right)^{1/\alpha} \left(\int_a^{b_0} \beta_2(t) |x(\sigma(t))|^{\alpha} \Delta t \right)^{1/\gamma} + \int_a^{b_0} |\alpha_1(t)| |u(t)| \Delta t. \tag{38}$$

Integrating (10) from a to $\sigma(b_0)$, we obtain

$$x(\sigma(b_0))u(\sigma(b_0)) - x(a)u(a) = \int_a^{\sigma(b_0)} \beta_1(t) |u(t)|^{\gamma} \Delta t - \int_a^{\sigma(b_0)} \beta_2(t) |x(\sigma(t))|^{\alpha} \Delta t. \tag{39}$$

Notice that the second integral of the right hand side of (39), by using (5) and (6), can be written as

$$\begin{aligned} \int_a^{\sigma(b_0)} \beta_2(t) |x(\sigma(t))|^{\alpha} \Delta t &= \int_a^{b_0} \beta_2(t) |x(\sigma(t))|^{\alpha} \Delta t + \int_{b_0}^{\sigma(b_0)} \beta_2(t) |x(\sigma(t))|^{\alpha} \Delta t \\ &= \int_a^{b_0} \beta_2(t) |x(\sigma(t))|^{\alpha} \Delta t + \mu(b_0)\beta_2(b_0) |x(\sigma(b_0))|^{\alpha}, \end{aligned}$$

and substituting the above equality into (39), we get

$$\begin{aligned} x(\sigma(b_0))u(\sigma(b_0)) \quad &+ \quad \mu(b_0)\beta_2(b_0)\left|x(\sigma(b_0))\right|^{\alpha} - x(a)u(a) \\ &= \quad \int_a^{\sigma(b_0)} \beta_1(t)\left|u(t)\right|^{\gamma} \Delta t - \int_a^{b_0} \beta_2(t)\left|x(\sigma(t))\right|^{\alpha} \Delta t. \end{aligned} \tag{40}$$

We claim that $x(\sigma(b_0))u(\sigma(b_0))+\mu(b_0)\beta_2(b_0)\left|x(\sigma(b_0))\right|^{\alpha} > 0$. To this end, from the second equation of (1) by using the formula $f^{\sigma}(t) = f(t) + \mu(t)f^{\Delta}(t)$, we get

$$u(\sigma(t)) - u(t) = -\mu(t)\beta_2(t)\left|x(\sigma(t))\right|^{\alpha-2} x(\sigma(t)) - \mu(t)\alpha_1(t)u(t),$$

which implies

$$\left(1 - \mu(t)\alpha_1(t)\right) u(t) = u(\sigma(t)) + \mu(t)\beta_2(t)\left|x(\sigma(t))\right|^{\alpha-2} x(\sigma(t)). \tag{41}$$

First taking $t = b_0$ in (41) and then multiplying the result by $x(\sigma(b_0))$ yields

$$\left(1 - \mu(b_0)\alpha_1(b_0)\right) u(b_0)x(\sigma(b_0)) = u(\sigma(b_0))x(\sigma(b_0)) + \mu(b_0)\beta_2(b_0)\left|x(\sigma(b_0))\right|^{\alpha}. \tag{42}$$

It follows from the fact $1 - \mu(b_0)\alpha_1(b_0) > 0$, $u(b_0) < 0$ and $x(\sigma(b_0)) < 0$ that the left hand side of (42) is strictly positive, so is the right hand side as claimed. Hence, $x(\sigma(b_0))u(\sigma(b_0)) + \mu(b_0)\beta_2(b_0)\left|x(\sigma(b_0))\right|^{\alpha} - x(a)u(a) > 0$ since $x(a)u(a) < 0$ which yields

$$0 < \int_a^{\sigma(b_0)} \beta_1(t)\left|u(t)\right|^{\gamma} \Delta t - \int_a^{b_0} \beta_2(t)\left|x(\sigma(t))\right|^{\alpha} \Delta t,$$

and hence

$$\int_a^{b_0} \beta_2(t)\left|x(\sigma(t))\right|^{\alpha} \Delta t < \int_a^{\sigma(b_0)} \beta_1(t)\left|u(t)\right|^{\gamma} \Delta t. \tag{43}$$

Substituting (43) into (38), we obtain

$$\begin{aligned} |u(s)| \quad &< \quad \left(\int_a^{b_0} \beta_2(t)\Delta t\right)^{1/\alpha} \left(\int_a^{\sigma(b_0)} \beta_1(t)\left|u(t)\right|^{\gamma} \Delta t\right)^{1/\gamma} + \int_a^{b_0} \left|\alpha_1(t)\right| \left|u(t)\right| \Delta t \\ &\leq \quad |u(s)| \left(\int_a^{b_0} \beta_2(t)\Delta t\right)^{1/\alpha} \left(\int_a^{\sigma(b_0)} \beta_1(t)\Delta t\right)^{1/\gamma} + |u(s)| \int_a^{b_0} \left|\alpha_1(t)\right| \Delta t, \end{aligned}$$

and hence, dividing the above inequality by $|u(s)|$ and since $b_0 \leq b$, we get

$$\begin{aligned} 1 &< \left(\int_a^{b_0} \beta_2(t)\Delta t\right)^{1/\alpha} \left(\int_a^{\sigma(b_0)} \beta_1(t)\Delta t\right)^{1/\gamma} + \int_a^{b_0} |\alpha_1(t)|\,\Delta t \\ &\leq \left(\int_a^{b} \beta_2(t)\Delta t\right)^{1/\alpha} \left(\int_a^{\sigma(b)} \beta_1(t)\Delta t\right)^{1/\gamma} + \int_a^{b} |\alpha_1(t)|\,\Delta t, \end{aligned}$$

which completes the proof. □

Combining Theorems 1–4, we have the following corollary.

Corollary 1 *Suppose $1-\mu(t)\alpha_1(t) > 0$, $\beta_1(t) > 0$ and $\beta_2(t) > 0$ on $[a,\sigma(b)]$. Let a, $b \in \mathbb{T}$ with $\sigma(a) < b$. Assume that (1) has a real solution (x,u) with generalized zeros in $\sigma(a)$ and $\sigma(b)$ and x is not identically zero on $[\sigma(a), b]$. Then the inequality*

$$1 < \int_a^{\sigma(b)} |\alpha_1(t)|\,\Delta t + \left(\int_a^{\sigma(b)} \beta_1(t)\Delta t\right)^{1/\gamma} \left(\int_a^{\sigma(b)} \beta_2(t)\Delta t\right)^{1/\alpha} \tag{44}$$

holds, where $\frac{1}{\alpha}+\frac{1}{\gamma}=1$.

Remark 2 *Taking $\alpha=\gamma=2$ in the nonlinear system (1) yields the following Hamiltonian system on a time scale $\mathbb{T}$*

$$\begin{aligned} x^{\Delta}(t) &= \alpha_1(t)x(\sigma(t)) + \beta_1(t)u(t) \\ u^{\Delta}(t) &= -\beta_2(t)x(\sigma(t)) - \alpha_1(t)u(t) \end{aligned} . \tag{45}$$

Hence, all of above results presented in this section for system (1) is also valid for system (45). Thus, we should remark here that the nonlinear system (1) may be viewed as the natural generalization of the Hamiltonian system (45) on a time scale $\mathbb{T}$. On the other hand, when $\alpha=\gamma=2$ in system (1), it is easy to see that Theorems 1-4 and Corollary 1 reduce to Theorems 1.1-1.4 and Corollary 1.5 of Jiang and Zhou [18], respectively.

4. A Disconjugacy Criterion

Applying the inequalities derived in section 3, we established a disconjugacy criterion for the solution of system (1). Let $a, b \in \mathbb{T}$ with $\sigma(a) < b$. Consider the nonlinear system

$$\begin{array}{r} x^{\Delta}(t) = \alpha_1(t)x(\sigma(t)) + \beta_1(t)\left|u(t)\right|^{\gamma-2}u(t) \\ u^{\Delta}(t) = -\beta_2(t)\left|x(\sigma(t))\right|^{\alpha-2}x(\sigma(t)) - \alpha_1(t)u(t) \end{array}, \qquad t \in [a,b]^{\kappa}. \tag{46}$$

We will assume that the coefficients $\alpha_1(t)$, $\beta_1(t)$ and $\beta_2(t)$ are real rd–continuous functions defined on $[a, \sigma(b)]$, $\gamma > 1$ and $\alpha > 1$ are constants with $\frac{1}{\alpha} + \frac{1}{\gamma} = 1$, and

$$1 - \mu(t)\alpha_1(t) > 0, \quad \beta_1(t) > 0, \quad \beta_2(t) > 0 \text{ for all } t \in [a, \sigma(b)]. \tag{47}$$

Note that each solution (x, u) of system (46) will be a vector valued function defined on $[a, \sigma(b)]$.

We now define the concept of a relatively generalized zero for the component x of a real solution (x, u) of system (46) and also the concept of disconjugacy of this system on $[a, \sigma(b)]$. The definition is relative to the interval $[a, \sigma(b)]$ and the left end–point a is treated separately.

Definition 1 ([18]) *The component x of a real solution (x, u) of (46) has a relatively generalized zero at a if and only if $x(a) = 0$, while we say x has a relatively generalized zero at $\sigma(t_0) > a$ provided (x, u) has a generalized zero at $\sigma(t_0)$. System (46) is called disconjugacy on $[a, \sigma(b)]$ if there is no real solution (x, u) of this system with x nontrivial and having two (or more) relatively generalized zeros in $[a, \sigma(b)]$.*

Notice that when $\mathbb{T} = \mathbb{Z}$ with the condition (47), definitions of a relatively generalized zero and of disconjugacy are equivalent to those given in ([3] p. 354; [14], [18]).

Theorem 5 *Assume condition (47) holds. If*

$$\int_a^{\sigma(b)} \left|\alpha_1(t)\right| \Delta t + \left(\int_a^{\sigma(b)} \beta_1(t)\Delta t \right)^{1/\gamma} \left(\int_a^{\sigma(b)} \beta_2(t)\Delta t \right)^{1/\alpha} \leq 1, \tag{48}$$

then (46) is disconjugate on $[a, \sigma(b)]$.

Proof. Suppose, on the contrary, that system (46) is not disconjugate on $[a, \sigma(b)]$. By Definition 1, there exists a real solution (x, u) of (46) with x nontrivial and such that x has at least two relatively generalized zeros in $[a, \sigma(b)]$. Now, we have two cases to consider.

Case 1: One of the two relatively generalized zeros is at the left end–point a, i.e., $x(a) = 0$, the other is at $\sigma(b_0) \in (a, \sigma(b)]$. Therefore, applying Theorem 1 or Theorem 3, we get

$$\int_a^{\sigma(b_0)} |\alpha_1(t)|\, \Delta t + \left(\int_a^{\sigma(b_0)} \beta_1(t)\Delta t \right)^{1/\gamma} \left(\int_a^{\sigma(b_0)} \beta_2(t)\Delta t \right)^{1/\alpha} > 1,$$

which contradicts to (48).

Case 2: Neither of two relatively generalized zeros is at a. Then x has two generalized zeros at both $\sigma(a_0)$ and $\sigma(b_0)$ with $\sigma(a_0) < \sigma(b_0)$ in $(a, \sigma(b)]$. Therefore, applying Corollary 1, we have

$$\int_{a_0}^{\sigma(b_0)} |\alpha_1(t)|\, \Delta t + \left(\int_{a_0}^{\sigma(b_0)} \beta_1(t)\Delta t \right)^{1/\gamma} \left(\int_{a_0}^{\sigma(b_0)} \beta_2(t)\Delta t \right)^{1/\alpha} > 1,$$

which again contradicts to (48).

By combining above two cases, the proof is now completed. □

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