Abstract

Let $G$ be locally compact Abelian group with Haar measure. First is discussed some properties of $L^1(G) \cap L(p, q)(G)$ spaces. Then is mentioned the multipliers space on $L^1(G) \cap L(p, q)(G)$ spaces.

1. Introduction and Preliminaries

Let $G$ be a locally compact abelian group with Haar measure $\mu$. The spaces $B^p(G) = L^1(G) \cap L^p(G)$, $1 \leq p < \infty$ have been studied in [11], [13] and the others. The space $B^p(G)$ is a Banach algebra with respect to the norm $\|f\|_{B^p}$ defined by $\|f\|_{B^p} = \|f\|_1 + \|f\|_p$ and the usual convolution product. The purpose of this paper is to extend some of the results on $B^p(G)$ to spaces

$$B(p, q)(G) = L^1(G) \cap L(p, q)(G),$$

and to discuss the properties of multipliers spaces of $B(p, q)(G)$, where $L(p, q)(G)$ is the usual Lorentz spaces. Many authors are discussed the space of multipliers of Segal algebras, multipliers from $L^1(G)$ into Segal algebras and multipliers from $L^1(G)$ into Banach spaces of functions in literature. Some of them are multipliers from $L^1(G)$ into Lorentz spaces in [3], multipliers of Banach spaces of functions in [5] and multipliers on $L^p(G, A)$ in [8]. The techniques mentioned in this papers will be used frequently. For convenience of the reader, we now review briefly what we need from the theory of $L(p, q)(G)$ spaces.

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Let \((G, \Sigma, \mu)\) be a measure space and let \(f\) be a measurable function on \(G\). For each \(y > 0\) let
\[
\lambda_f(y) = \mu\{x \in G : |f(x)| > y\}.
\]

The function \(\lambda_f\) is called the distribution function of \(f\). The rearrangement of \(f\) is defined by
\[
f^*(t) = \inf\{y > 0 : \lambda_f(y) \leq t\} = \sup\{y > 0 : \lambda_f(y) > t\}, \quad t > 0,
\]
where \(\inf \phi = \infty\). Also, the average function of \(f\) is defined by
\[
f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) \, ds, \quad t > 0.
\]

Note that \(\lambda_f(\cdot), f^*(\cdot)\) and \(f^{**}(\cdot)\) are non-increasing and right continuous on \((0, \infty)\) ([2]). For \(p, q \in (0, \infty)\) we define
\[
\|f\|_{p,q}^* = \|f\|_{p,q,\mu}^* = \left(\frac{q}{p} \int_0^\infty [f^*(t)]^q t^{\frac{p-1}{q}} dt\right)^{\frac{1}{q}}
\]
and
\[
\|f\|_{p,q} = \|f\|_{p,q,\mu} = \left(\frac{q}{p} \int_0^\infty [f^{**}(t)]^q t^{\frac{p-1}{q}} dt\right)^{\frac{1}{q}}.
\]

Also, if \(0 < p, q = \infty\) we define
\[
\|f\|_{p,\infty}^* = \sup_{t>0} t^p f^*(t) \quad \text{and} \quad \|f\|_{p,\infty} = \sup_{t>0} t^p f^{**}(t).
\]

For \(0 < p < \infty\) and \(0 < q \leq \infty\), the Lorentz spaces are denoted by \(L(p,q)(G,\mu)\) (or in short, \(L(p,q)(G)\)) is defined to be the vector space of all (equivalence classes of) measurable functions \(f\) on \(G\) such that \(\|f\|_{p,q}^* < \infty\). We know that \(\|f\|_{p,p}^* = \|f\|_{p,p}\) and so \(L^p(\mu) = L(p,p)(G)\) where \(L^p(\mu)\) is the usual Lebesgue space. Also, \(L(p,q_1)(G) \subset L(p,q_2)(G)\) for \(q_1 < q_2\). In particular,
\[
L(p,q_1)(G) \subset L(p,p)(G) = L^p(G) \subset L(p,q_2)(G) \subset L(p,\infty)(G)
\]
for $0 < q_1 \leq p \leq q_2 \leq \infty$ ([2, 6]). It is also known that if $1 < p < \infty$ and $1 \leq q \leq \infty$, then

$$\|f\|_{p,q}^* \leq \|f\|_{p,q} \leq \frac{p}{p-1} \|f\|_{p,q}^*$$

for each $f \in L(p,q)(G)$ and \(L(p,q)(G), \|\cdot\|_{p,q}\) is a Banach space ([6, 7]).

In [14], it was found that $B(p,q)(G)$ is a normed space with the norm $\|\cdot\|_B$ defined by $\|\cdot\|_B = \|\cdot\|_1 + \|\cdot\|_{p,q}$ and is a Segal Algebra; namely, it satisfies the properties:

1. $(B(p,q), \|\cdot\|_B)$ is a Homogeneous Banach space
2. $B(p,q)(G)$ is a Banach algebra with its norm $\|\cdot\|_B \geq \|\cdot\|_1$
3. $B(p,q)(G)$ is a dense subspace of $L^1(G)$ according to $\|\cdot\|_1$ norm.

Before beginning the next part of the paper, let’s give some basic propositions about $B(p,q)(G)$ which are easy to prove using the properties of $L(p,q)(G)$ mentioned in [1],[2],[3],[6],[7] and [14]. On the other hand, we will say a few words about proofs of some propositions.

**Proposition 1** $(B(p,q), \|\cdot\|_B)$ is strongly character invariant and the map $f \to M_t f$ and the function $t \to M_t f$ are continuous where $M_t f(x) = \langle x, t \rangle f(x)$ for all $f \in B(p,q), x \in G$ and $t \in \hat{G}$ ([3],[12]).

**Proof.** $L^1(G)$ and $L(p,q)(G)$ are strongly character invariant and the functions $t \to M_t f$ and $f \to M_t f$ are continuous in both spaces. \(\square\)

**Proposition 2** For $0 < q_1 \leq p \leq q_2 \leq \infty$, we have the following inclusions as similar to Lorentz spaces [2],[6] and [12]

\[ B(p,q_1)(G) \subset B(p,p)(G) = B^p(G) \subset B(p,q_2)(G) \subset B(p,\infty)(G). \]

**Proposition 3** $(B(p,q), \|\cdot\|_B)$ has a minimal approximate identity in $L^1(G)$ for $1 < p < \infty$ and $1 \leq q < \infty$ ([3]).

**Proposition 4** $(B(p,q), \|\cdot\|_B)$ is an essential Banach $L^1(G)$-module.
Proof. Let \( f \in L^1(G) \) and \( g \in B(p,q) \). Since \( L(p,q) \) is an essential Banach \( L^1(G) \)-module for \( 1 < p < \infty, 0 \le q < \infty, ([1]) \) we have
\[
\|f \ast g\|_B = \|f \ast g\|_1 + \|f \ast g\|_{pq} \le \|f\|_1\|g\|_B.
\]
Also, by using the approximate identity of \( L^1(G) \), say \((e_a)\); we have \( \|e_a \ast g - g\|_B \to 0 \). Therefore we get that \( (B(p,q), \|\cdot\|_B) \) is an essential Banach \( L^1(G) \)-module. \( \square \)

2. Multipliers space on \( B(p,q)(G) \)

Let us denote the space of all bounded linear operators on \( B(p,q) \) as \( M_{pq} \), which is a Banach algebra under the usual operator norm. Besides this, let \( \text{Hom}_{L^1(G)}(B(p,q)(G), B(p,q)(G)) \) be the space of all module homomorphisms of \( L^1(G) \)-module \( B(p,q)(G) \), that is, an operator \( T \in M_{pq} \) satisfies \( T(f \ast g) = f \ast T(g) \) for all \( f \in L^1(G) \) and \( g \in B(p,q)(G) \). The module homomorphisms space, called the multipliers space
\[
\text{Hom}_{L^1(G)}(B(p,q)(G), B(p,q)(G)) = \text{Hom}_{L^1(G)}(B(p,q)(G))
\]
is a Banach \( L^1(G) \)-module by \( (f \circ T)(g) = f \ast T(g) = T(f \ast g) \) for all \( g \in B(p,q)(G) \).

Now, let us fix \( f \in L^1(G) \) and define \( W_f : B(p,q) \to B(p,q) \) as \( W_f(g) = f \ast g \) for all \( f \in L^1(G) \) and \( g \in B(p,q) \). It is easy to see that \( W_f \) is linear and bounded.

Proposition 5 The set
\[
\Lambda = \overline{\text{span} \{W_f \mid f \in L^1(G)\}} = \{W_f \mid f \in L^1(G)\}
\]
is a complete subalgebra of \( M_{pq} \) and it possesses a minimal approximate identity.

Proof. By the definition of \( \Lambda \), it is easy to see that \( \Lambda \) is a complete subalgebra of \( M_{pq} \) under the operator norm with usual composition. For each \( f \in L^1(G) \) and \( h \in B(p,q) \), if we define \( W_f(h) = f \ast h \), then we have
\[
\|W_f\| = \sup_{\|h\|_p \le 1} \|W_f(h)\|_B = \sup_{\|h\|_p \le 1} \|f \ast h\|_B \le \|f\|_1 , \tag{1}
\]
and for all \( f, g \in L^1(G), h \in B(p,q) \), one can write
\[
(W_f - W_g)(h) = f \ast h - g \ast h = (f - g) \ast h = W_{f-g}(h) \tag{2}
\]
\[
(W_f \circ W_g)(h) = W_f(g \ast h) = f \ast (g \ast h) = W_{f+g}(h).
\]
Let \( f \in L^1(G) \). Using (1),(2) and the minimal approximate identity of \( L^1(G) \) say \( (e_\alpha) \), we get

\[
\lim_{\alpha} \| W_{e_\alpha} \circ W_f - W_f \| = \lim_{\alpha} \| W_{e_\alpha} \ast f - W_f \|
\]

\[
= \lim_{\alpha} \| W_{e_\alpha} \ast f - W_f \|
\]

\[
\leq \lim_{\alpha} \| e_\alpha \ast f - f \|_1 = 0.
\]

Consequently, we have \( \lim_{\alpha} \| W_{e_\alpha} \circ T - T \| = 0 \) for all \( T \in \Lambda \). \( \square \)

**Proposition 6** The space \( \Lambda \) is a complete subalgebra of \( \text{Hom}_{L^1(G)} (B(p,q)(G)) \).

**Proof.** Let \( f \in L^1(G) \), then \( W_f \in M_{pq} \). Since \( B(p,q) \) is an essential Banach \( L^1(G) \)-module, we have

\[
W_f (g \ast h) = f \ast g \ast h = g \ast W_f (h)
\]

for all \( g \in L^1(G) \) and \( h \in B(p,q) \). Thus \( W_f \) belongs to \( \text{Hom}_{L^1(G)} (B(p,q)(G)) \). Since \( \text{Hom}_{L^1(G)} (B(p,q)(G)) \) is a Banach space under the usual operator norm, \( \Lambda \) is a complete subalgebra of \( \text{Hom}_{L^1(G)} (B(p,q)(G)) \). \( \square \)

**Proposition 7** The space \( \Lambda \) is an essential Banach \( L^1(G) \)-module.

**Proof.** Let \( g \in L^1(G) \) and \( W_f \in \Lambda \). Define \( g \circ W_f : B(p,q) \rightarrow B(p,q) \) by letting \( (g \circ W_f)(h) = W_f (h \ast g) = W_f (g \ast h) \) for each \( h \in B(p,q) \). In this case, we find

\[
\| g \circ W_f \| = \sup_{\| h \|_B \leq 1} \| (g \circ W_f)(h) \|_B = \sup_{\| h \|_B \leq 1} \| W_f (g \ast h) \|_B
\]

\[
\leq \| W_f \| \sup_{\| h \|_B \leq 1} \| g \ast h \|_B \leq \| W_f \| \| g \|_1.
\]

As a result, \( \Lambda \) is a Banach \( L^1(G) \)-module. On the other hand, since \( L^1(G) \) has a bounded approximate identity \( (e_\alpha) \), \( \| e_\alpha \| \geq 0 \) which is in \( C_c(G) \), the set of all continuous functions with a compact support, such that it is also an approximate identity in \( B(p,q) \).
by proposition 3. Then, for any \( W_f \in \Lambda \), we have
\[
\| e_\alpha \circ W_f - W_f \| = \sup_{\|u\|_B \leq 1} \| (e_\alpha \circ W_f - W_f)(u) \|_B
\]
\[
= \sup_{\|u\|_B \leq 1} \| f * u * e_\alpha - f * u \|_B
\]
\[
\leq \sup_{\|u\|_B \leq 1} \| f * e_\alpha - f \|_1 \| u \|_B
\]
by proposition 4. Therefore \( \Lambda \) is an essential Banach \( L^1(G) \) -module. Also for any \( f \in L^1(G) \) and \( W_{e_\alpha} \in \Lambda \), we have
\[
\lim_{\alpha} \| f - f \circ W_{e_\alpha} \| = \lim_{\alpha} \left( \sup_{\|u\|_B \leq 1} \| (f - f \circ W_{e_\alpha})(u) \|_B \right)
\]
\[
\leq \lim_{\alpha} \left( \sup_{\|u\|_B \leq 1} \| f - e_\alpha \ast f \|_1 \| u \|_B \right)
\]
\[
\leq \lim_{\alpha} \| f - e_\alpha \ast f \|_1 = 0.
\]
So \( f \in L^1(G) \cup \Lambda \), namely \( f \in \Lambda \). That is to say \( L^1(G) \subset \Lambda \). \( \square \)

**Proposition 8** Let \( T \in Hom_{L^1(G)}(B(p,q)(G)) \). Therefore \( T \circ W \in \Lambda \) for each \( W \in \Lambda \).

**Proof.** Since \( B(p,q)(G) \) is a Segal algebra, it is easy to see that
\[
\Lambda = \text{span} \left\{ W_f \mid f \in L^1(G) \right\} = \text{span} \left\{ W_g \mid g \in B(p,q)(G) \right\}.
\]
Let us take any \( W_g \in \Lambda \). Then for all \( h \in B(p,q)(G) \), we get
\[
(T \circ W_g)(h) = T(g \ast h) = T(g) \ast h = W_{T(g)}(h)
\]
and \( T \circ W_g \in \Lambda \), since \( T(g) \in B(p,q)(G) \). Now take any \( W \in \Lambda \). By the definition of \( \Lambda \), for all \( \varepsilon > 0 \) we can find \( g \in B(p,q)(G) \) such that \( \| W - W_g \| < \frac{\varepsilon}{\| T \|} \). Since \( T \circ W_g \in \Lambda \)
and $T$ is bounded on $B(p,q)(G)$, we have

\[
\|T \circ W - T \circ W_g\| = \sup_{\|h\|_p \leq 1} \| (T \circ W)(h) - (T \circ W_g)(h) \|_B
\]

\[
= \sup_{\|h\|_p \leq 1} \| T(W(h)) - T(g * h) \|_B
\]

\[
\leq \|T\| \sup_{\|h\|_p \leq 1} \| W(h) - g * h \|_B
\]

\[
= \|T\| \sup_{\|h\|_p \leq 1} \| W(h) - W_g(h) \|_B
\]

\[
= \|T\| \| W - W_g \| < \varepsilon.
\]

Therefore we say that $T \circ W \in \text{span} \{W_g \mid g \in B(p,q)(G)\} = \Lambda$. 

\[\blacksquare\]

**Theorem 9** Let $G$ be a locally compact abelian group. Then $M(\Lambda)$, the space of multipliers on Banach algebra $\Lambda$, is isometrically isomorphic to the space $\text{Hom}_{L^1(G)}(B(p,q)(G))$.

**Proof.** Define a mapping $\Psi : \text{Hom}_{L^1(G)}(B(p,q)(G)) \to M(\Lambda)$ by letting $\Psi(T) = \rho_T$ for each $T \in \text{Hom}_{L^1(G)}(B(p,q)(G))$, where $\rho_T(S) = T \circ S$ for all $S \in \Lambda$. Note that $\Psi$ is well-defined by Proposition 8; and moreover, if $\rho_T(S \circ K) = T \circ S \circ K = \rho_T(S) \circ K$ for all $S, K \in \Lambda$, then we see that $\Psi(T) = \rho_T \in M(\Lambda)$. It is obvious that the mapping $\Psi$ is linear and injective. Also, for $T \in \text{Hom}_{L^1(G)}(B(p,q)(G))$ and any $S \in \Lambda$, we have

\[
\|T \circ S\| = \sup_{\|g\|_p \leq 1} \| (T \circ S)(g) \|_B = \sup_{\|g\|_p \leq 1} \| T(S(g)) \|_B
\]

\[
\leq \|T\| \sup_{\|g\|_p \leq 1} \| S(g) \|_B = \|T\| \|S\|,
\]

and so we can obtain the relation

\[
\|\rho_T\| = \sup_{S \in \Lambda} \frac{\|\rho_T(S)\|}{\|S\|} = \sup_{S \in \Lambda} \frac{\|T \circ S\|}{\|S\|} \leq \|T\|.
\]

On the other hand, since $\{W_{e_\alpha}\}$ is a minimal approximate identity for the space $\Lambda$, we get

\[
\|\rho_T\| = \sup_{S \in \Lambda} \frac{\|T \circ S\|}{\|S\|} \geq \sup_{\alpha} \frac{\|T \circ W_{e_\alpha}\|}{\|W_{e_\alpha}\|} \geq \sup_{\alpha} \|T \circ W_{e_\alpha}\| \geq \|T\|
\]

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and \( \| \rho_T \| = ||T|| \).

Finally we show that the mapping \( \Psi : \text{Hom}_{L^1(G)}(B(p,q)(G)) \rightarrow M(\Lambda) \) is onto. Let \( \rho \) be an element of \( M(\Lambda) \) and \( (e_\alpha) \) an approximate identity for \( L^1(G) \). Since \( \Lambda \subset \text{Hom}_{L^1(G)}(B(p,q)(G)) \) and \( \rho e_\alpha \in \Lambda \), for any \( f \in L^1(G) \) and \( g \in B(p,q) \), we have

\[
\rho e_\alpha (f * g) = (f \circ (\rho e_\alpha))(g).
\]

(3)

Also \( M(\Lambda) \subset \text{Hom}_{L^1(G)}(\Lambda) \) implies that

\[
\rho (f * e_\alpha)(g) = (f \circ (\rho e_\alpha))(g).
\]

(4)

Therefore by (3) and (4), we get

\[
\rho e_\alpha (f * g) = (f \circ (\rho e_\alpha))(g) = \rho (f * e_\alpha)(g).
\]

So for each \( f \in L^1(G) \) and \( g \in B(p,q) \), we obtain

\[
\lim_\alpha \| \rho (f * e_\alpha)(g) - \rho f (g) \|_B = \lim_\alpha \| (\rho (f * e_\alpha) - \rho f)(g) \|_B
\]

\[
= \lim_\alpha \| \rho (f * e_\alpha - f)(g) \|_B
\]

\[
\leq \lim_\alpha \| \rho (f * e_\alpha - f) \| \| g \|_B
\]

\[
\leq \| \rho \| \lim_\alpha \| f * e_\alpha - f \|_1 \| g \|_B = 0
\]

Thus we get

\[
\lim_\alpha (\rho e_\alpha)(f * g) = \lim_\alpha (f \circ (\rho e_\alpha))(g) = \lim_\alpha (\rho (f * e_\alpha))(g) = \rho f (g).
\]

Since the space \( B(p,q) \) is an essential Banach \( L^1(G) \)-module by proposition 4, the limit of \( (\rho e_\alpha)(f * g) = (f \circ (\rho e_\alpha))(g) \) exists and equal to \( f * T(g) \in B(p,q) \) while \( T \) is an operator in \( \text{Hom}_{L^1(G)}(B(p,q)) \). Therefore, since the limits \( \lim_\alpha (\rho e_\alpha)(f * g) = \lim_\alpha (f \circ (\rho e_\alpha))(g) = \rho f (g) \) exist, we can write \( f \circ T = \rho f \) for all \( f \in L^1(G) \). Then \( e_\alpha \circ T \circ W = (\rho e_\alpha) \circ W = \rho (e_\alpha \circ W) \) can be written for all \( W \in \Lambda \). By proposition 7, for all \( W \in \Lambda \), we get \( T \circ W = \rho (W) \) or \( \rho_T(W) = \rho(W) \). Therefore \( \rho_T = \rho. \)

\( \square \)
References


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