On Cartan Spaces with \((\alpha, \beta)\)-metric

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Abstract

É. Cartan [2] has originally introduced a Cartan space, which is considered as dual of Finsler space. H. Rund [10], F. Brickell [1] and others studied the relation between these two spaces. The theory of Hamilton spaces was introduced and studied by R. Miron ([8], [9]). He proved that Cartan space is a particular case of Hamilton space. T. Igrashi ([5], [6]) introduced the notion of the \((\alpha, \beta)\)-metric in Cartan spaces and obtained the metric tensor and the invariants \(\rho\) and \(r\) which characterize the special classes of Cartan spaces with \((\alpha, \beta)\)-metric. This paper presents a study of Cartan spaces with \((\alpha, \beta)\)-metric admitting h-metrical d-connection. We prove the conditions for these spaces to be locally Minkowski and conformally flat.

Key Words: Cartan space, \((\alpha, \beta)\)-metric, h-metrical d-connection, Conformally flat.

1. Introduction

Let \(M\) be a real smooth manifold and \((T^*M, \pi, M)\) its cotangent bundle. Let \(C^n = (M, K(x, p))\), where \(K : T^*M \to R\) is a scalar function which is differentiable on \(T^*\tilde{M} = T^*M - \{0\}\), and is homogeneous on the fibres of \(T^*M\). The hessian of \(K^2\), i.e. \(g^{ij}(x, p) = \frac{1}{2}\partial^i \partial^j K^2\), where \(\partial^i = \frac{\partial}{\partial p^i}\), is positively defined on \(T^*\tilde{M}\). Here \(C^n\) is called the Cartan space and the functions \(K(x, p)\) and \(g^{ij}(x, p)\) are called, respectively, the fundamental function and the metric tensor of the Cartan space \(C^n\).

The reciprocal \(g_{ij}(x, p)\) of \(g^{ij}(x, p)\) is given by \(g_{ij}(x, p)g^{ij}(x, p) = \delta^k_j\), where \(g_{ij}(x, p)\)

2000 Mathematics Subject Classification: 53C60, 53B40
and $g^{ij}(x, p)$ are both symmetric and homogeneous of order 0 in $p_j$.

A Cartan space $C^n = (M, K)$ is said to be with $(\alpha, \beta)$-metric if $K(x, p)$ is a function of the variables $\alpha(x, p) = (a^{ij} p_i p_j)^{1/2}$, $\beta(x, p) = p_i b^i(x)$, where $a^{ij}(x)$ is a Riemannian metric and $b^i(x)$ is a vector field depending only on $x$. Clearly $K$ must satisfy the conditions imposed to the fundamental function of a Cartan space.

In this paper, we consider the Cartan spaces with $(\alpha, \beta)$-metric admitting h-metrical d-connection in section 2 and their conformal change in section 3. The fundamental tensor $g^{ij}(x, p)$ and its reciprocal $g_{ij}(x, p)$ of the Cartan space $C^n = (M, K(\alpha, \beta))$ are given by [6] the relation

$$g^{ij} = \rho a^{ij} + \rho b^i b^j + \rho_{-1}(b^i p^j + b^j p^i) + \rho_{-2} p^i p^j, \quad (1.1)$$

where $\rho, \rho_0, \rho_{-1}, \rho_{-2}$ are the invariants given by

$$\rho = \frac{1}{2} \alpha^{-1} K_\alpha, \quad \rho_{-1} = \frac{1}{2} \alpha^{-1} K_{\alpha\beta}, \quad \rho_{-2} = \frac{1}{2} \alpha^{-2} (K_{\alpha\alpha} - \alpha^{-1} K_\alpha) \quad (1.2)$$

$$\rho_0 = \frac{1}{2} K_{\beta\beta},$$

and

$$g_{ij} = \sigma a_{ij} - \sigma_0 b_i b_j + \sigma_{-1}(b_i p_j + b_j p_i) + \sigma_{-2} p_i p_j \quad (1.3)$$

where

$$\sigma = \frac{1}{\rho}, \quad \sigma_0 = \frac{\rho_0}{\rho \tau}, \quad \tau = \sigma + \sigma_0 B^2 + \rho_{-1} \beta, \quad \sigma_{-1} = \frac{\rho_{-1}}{\rho \tau}, \quad \sigma_{-2} = \frac{\rho_{-2}}{\rho \tau}, \quad (1.4)$$

and where $B^2 = b^i b_i$.

The Cartan tensor $C^{ijk}$ is given by

$$C^{ijk} = -\frac{1}{2} \left[ r_{-1} b^i b^j b^k + \{ \rho_{-1} \alpha^{ij} b^k + \rho_{-2} \alpha^{ij} p^k + \rho_{-3} b^i p^j p^k + \rho_{-4} p^i p^j p^k \} + r_{-2} b^i b^j b^k \right], \quad (1.5)$$

where

$$r_{-1} = \frac{1}{2} K_{\beta\beta\beta}, \quad r_{-2} = \frac{1}{2} \alpha^{-1} K_{\alpha\beta\beta}, \quad r_{-3} = \frac{1}{2} \alpha^{-2} (K_{\alpha\alpha\beta} - \alpha^{-1} K_{\alpha\beta}), \quad r_{-4} = \frac{1}{2} \alpha^{-3} (K_{\alpha\alpha\alpha} - 3 \alpha^{-1} K_{\alpha\alpha} + 3 \alpha^{-2} K_\alpha). \quad (1.6)$$

364
NAGARAJA

Let ‘.’ denote the covariant differentiation with respect to Christoffel symbols $\gamma_{jk}^i$ constructed from $a_{ij}$. Since $a_{ij,.k} = 0$ and $p_{i,.k} = 0$, if $b_{i,.k} = 0$, then $g_{ij,.k} = 0$. Using the Christoffel symbols $\Gamma_{ijk}^j(p) = \frac{1}{2}g^{ir}(\partial_j g_{rk} + \partial_k g_{jr} - \partial_r g_{jk})$ constructed from $g_{ij}(x,p)$, we can define canonical $N$-connection

$$N_{ij} = \Gamma_{ijk}^k p_k - \frac{1}{2} \Gamma_{ijk}^h p_k p^r \partial^h g_{ij}.$$ (1.7)

We consider the canonical d-connection

$$D\Gamma = (N_{jk}, H_{jk}^i, C_{ij}^k)$$ (1.8)

where

$$H_{jk}^i = \frac{1}{2} g^{ir}(\partial_j g_{rk} + \partial_k g_{jr} - \partial_r g_{jk}).$$ (1.9)

The d-tensor field of type $(2,1)$ $C_{ij}^k$ is given by

$$C_{ij}^k = -\frac{1}{2} g_{ir} \partial^r g^{jk} = g_{ir} C_{rj}^k,$$ (1.10)

Let $'\mid k'$ denote the h-covariant differentiation with respect to $D\Gamma$.

**Definition 1.1** A d-connection $D\Gamma$ of a Cartan space $C^n$ with $(\alpha, \beta)$-metric is called the h-metrical d-connection if it satisfies the conditions

- h-deflection tensor $D_{ij} (= p_{ij}) = 0$;
- $\alpha_{ij,.k} = 0$;
- $g_{ij,.k} = 0$.

2. **Cartan Spaces with $(\alpha, \beta)$-metric admitting h-metrical d-connection**

If the connection $D\Gamma$ is h-metrical, then $\alpha_{,h} = 0$, from which we get that

$$0 = K_{,h} = \alpha_{,h} K_\alpha + \beta_{,h} K_\beta = \beta_{,h} K_\beta$$

and

$$\beta_{,h} = b_{,h}^i p_i = 0$$ (2.1)
From (1.1), we have
\[ g^{ij}|_{k} = b^{i}|_{k}(\rho_{0}b^{j} + \rho_{-1}p^{j}) + b^{j}|_{k}(\rho_{0}b^{i} + \rho_{-1}p^{i}) = 0 \]
Transvecting the above with \( p_{i} \), and by virtue of (2.1) we get
\[ b^{j}|_{k}(\rho_{0}\beta + \rho_{-1}\alpha) = 0, \]
which gives \( b^{j}|_{k} = 0 \).

Now from \( a^{ij}|_{k} = 0 \), we can get \( H^{i}_{jk} = \gamma^{i}_{jk} \). Hence we have
\[ b^{i}|_{k} = 0, \quad (2.2) \]
and also the curvature tensor \( D^{i}_{jk} \) of \( D\Gamma \) coincides with the curvature tensor \( R^{i}_{jk} \) of Riemannian connection \( R\Gamma = (\gamma^{i}_{jk}, \gamma^{i}_{jk}y^{i}, 0) \).

If \( R^{i}_{jk} = 0 \) then \( D^{i}_{jk} = 0 \). Thus we have the following proposition.

**Proposition 2.1** A Cartan space \( C^{n} \) with \( (\alpha, \beta) \) metric admitting a h-metrical d-connection is locally flat if and only if the associated Riemannian space is locally flat.

If the connection \( D\Gamma \) is h-metrical, then \( g^{ij}|_{h} = 0, \alpha|_{h} = 0, a^{ij}|_{h} = 0, b^{k}|_{h} = 0, p^{k}|_{h} = 0 \), from which we get \( r_{-1}|_{h} = 0, r_{-2}|_{h} = 0, r_{-3}|_{h} = 0 \) and \( r_{-4}|_{h} = 0 \).

Hence from (1.5), (1.6) and (1.10), we have
\[ C^{ij}_{k|h} = 0. \quad (2.3) \]

**Definition 2.1** A Cartan space \( C^{n} \) is a Berwald space if and only if \( C^{ij}_{k|h} = 0 \).

Hence from (2.3) we have the following proposition.

**Proposition 2.2** A Cartan space with \( (\alpha, \beta) \) metric admitting a h-metrical d-connection is a Berwald space.

As it is well known [11] a locally Minkowski space is a Berwald space in which the curvature tensor vanishes.
Hence from Proposition (2.1) and Proposition (2.2), we have the following proposition.

**Theorem 2.1** A Cartan space with \( (\alpha, \beta) \) metric admitting a h-metrical d-connection is locally Minkowski if and only if the associated Riemannian space is locally flat.
3. Conformal Change of a Cartan Space

Let $C^n = (M, K)$ be an $n$-dimensional Cartan space with $(\alpha, \beta)$-metric $K = K(\alpha, \beta)$. By a conformal change $\sigma : K \rightarrow \overline{K}$ : $\overline{K}(\overline{\alpha}, \overline{\beta}) = e^\sigma K(\alpha, \beta)$, we have another Cartan space $C^n = (M, \overline{K}(\overline{\alpha}, \overline{\beta}))$, where $\overline{\alpha} = e^\sigma \alpha$ and $\overline{\beta} = e^\sigma \beta$.

Putting $\alpha = (a^{ij}(x)p_i p_j)^{\frac{1}{2}}$ and $\beta = b^i(x)p_i$, we get $\overline{\alpha}^j = e^{2\sigma} a^{ij}$ and $\overline{\beta}^i = e^\sigma b^i$. Then the Christoffel symbols $\overline{\gamma}^i_{jk}$ constructed from $\overline{\alpha}^j$ are written as

$$\overline{\gamma}^i_{jk} = \gamma^i_{jk} + B^i_{jk}$$

(3.1)

where $B^i_{jk} = \delta^i_j \sigma_k + \delta^i_k \sigma_j - \sigma^i a_{jk}$, $\sigma^i = \sigma_j a^{ij}$.

Taking covariant derivative of $\overline{\beta}^i$ with respect to $\overline{\gamma}^i_{jk}$, we get

$$\overline{\beta}^{i'}_{,k} = e^\sigma (b^{i'}_{,k} + 2 \sigma_j b^j + b^r \sigma r \delta^i_j - \sigma^i b^r a_{rk}).$$

Transvecting the above by $\overline{b}^r$, and putting

$$M^i = \frac{1}{B^2} \left( B^k b^i_{,k} - \frac{b^r b^i}{n+4} \right),$$

(3.2)

we have $\sigma^i = M^i - M_i$, from which we get $\sigma_i = \overline{M}_i - M_i$. Substituting this in (3.1) and putting

$$D^i_{hj} = \gamma^i_{hj} + \delta^i_h M_j + \delta^i_j M_h - M^i a_{hj},$$

we have

$$\overline{D}^i_{hj} = D^i_{hj}.$$ (3.3)

$D^i_{hj}$ is a symmetric conformally invariant linear connection on $M$.

Thus we have the following proposition.

**Proposition 3.1** In a Cartan space with $(\alpha, \beta)$-metric, there exists a conformally invariant symmetric linear connection $D^i_{hj}$.

If we denote by $D^i_{hjk}$ the curvature tensor of $D^i_{hj}$, we have from (3.3)

$$\overline{D}^i_{hjk} = D^i_{hjk}.$$ (3.4)

Since $b^i_{,k} = 0$, we have $M^i = 0$. Hence $D^i_{hjk} = \gamma^i_{hjk}$ and $R^i_{hjk} = R^i_{hjk}$. Thus we have the next proposition.
Proposition 3.2 In a Cartan space \( C^n \) with \((\alpha, \beta)\)-metric admitting \( h \)-metrical \( d \)-connection, \( M^i = 0 \) and there exists a conformally invariant symmetric linear connection \( D^i_{jk} \) such that \( D^i_{jk} = \gamma^i_{jk} \) and its curvature tensor \( D_{hjk}^i = R_{hjk}^i \).

If the associated Riemannian space \((M, \alpha)\) is locally flat \( (R_{hjk}^i = 0) \), then from (3.4) and Proposition (3.2), we have \( D_{hjk}^i = 0 \), i.e. the space \( C^n \) is conformally flat.

Thus we conclude that.

Theorem 3.1 A Cartan space \( C^n = (M, K(\alpha, \beta)) \) with \((\alpha, \beta)\) metric admitting \( h \)-metrical \( d \)-connection is conformally flat if and only if associated Riemannian space is locally flat.

References


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Received 04.05.2006