Some Remarks on the $L^p - L^q$ Boundedness of $uC_\varphi$

M. R. Jabbarzadeh

Abstract

In this paper we will consider the weighted composition operators between two different $L^p$-spaces and then we characterize the functions $u$ and transformations $\varphi$ that induce weighted composition operator $uC_\varphi$ between $L^p(X, \Sigma, \mu)$-spaces by using some properties of conditional expectation operator, pair $(u, \varphi)$ and the measure space $(X, \Sigma, \mu)$.

Key Words: Weighted composition operator, conditional expectation, multiplication operator.

1. Preliminaries And Notations

Let $(X, \Sigma_X, \mu)$ be a sigma finite measure space. By $L(X)$, we denote the linear space of all $\Sigma_X$-measurable functions on $X$. When we consider any sub-sigma algebra $\mathcal{A}$ of $\Sigma_X$, we assume they are completed. For any sigma finite algebra $\mathcal{A} \subseteq \Sigma_X$ and $1 \leq p \leq \infty$ we abbreviate the $L^p$-space $L^p(X, \mathcal{A}, \mu|\mathcal{A})$ to $L^p(\mathcal{A})$, and denote its norm by $\|\cdot\|_p$. We understand $L^p(\mathcal{A})$ as a subspace of $L^p(\Sigma_X)$ and as a Banach space. We define the support of a function $f \in L(X)$ as $\sigma(f) = \{x \in X; f(x) \neq 0\}$. All comparisons between two functions or two sets are to be interpreted as holding up to a $\mu$-null set.

Next, let $(Y, \Sigma_Y, \nu)$ be another sigma finite measure space. Similarly, we use the symbols $L(Y)$ and $L^p(\Sigma_Y)$ to denote the linear space of all $\Sigma_Y$-measurable functions on $Y$ and the $L^p$-space $L^p(Y, \Sigma_Y, \nu)$, respectively. Take a function $u \in L(Y)$ and let $\varphi : Y \to X$ be a non-singular measurable function; i.e. $\varphi^{-1}(\Sigma_X) \subseteq \Sigma_Y$.

2000 AMS Mathematics Subject Classification: Primary 47B20; Secondary 47B38.
\( \nu \circ \varphi^{-1}(A) = \nu(\varphi^{-1}(A)) = 0 \) for all \( A \in \Sigma_X \) such that \( \mu(A) = 0 \). Then the non-singularity of \( \varphi \) means that \( \nu \circ \varphi^{-1} \) is absolutely continuous with respect to \( \mu \) (we write \( \nu \circ \varphi^{-1} \ll \mu \), as usual). Let \( h_\varphi \in L(X) \) be the Radon-Nikodym derivative \( h_\varphi = d\nu \circ \varphi^{-1}/d\mu \).

Associated with each sigma algebra \( A \subseteq \Sigma_Y \), there exists an operator \( E^A_\varphi = E \), which is called conditional expectation operator, on the set of all non-negative measurable functions \( f \) or for each \( f \in L^q(\Sigma_Y) \) for any \( q, 1 \leq q \leq \infty \), and is uniquely determined by the conditions

(i) \( E(f) \) is \( A \)- measurable, and

(ii) if \( A \) is any \( A \)- measurable set for which \( \int_A f d\mu \) exists, we have \( \int_A f d\mu = \int_A E(f) d\mu \).

This operator is at the central idea of our work, and we list here some of its useful properties:

E1. If \( g \) is \( A \)-measurable then \( E(fg) = E(f)g \).
E2. \( E(1) = 1 \).
E3. If \( f > 0 \) then \( E(f) > 0 \).
E4. If \( f \geq 0 \) then \( E(f) \geq 0 \) and \( \sigma(f) \subseteq \sigma(E(f)) \).

Properties E1 and E2 imply that \( E \) is an idempotent; and as operator on \( L^q(\Sigma_Y) \) we have \( E(L^q(\Sigma_Y)) = L^q(A) \). Hence \( E \) is the identity operator \( I \) on \( L^q(\Sigma_Y) \), if and only if \( A = \Sigma_Y \). If we put \( A = \varphi^{-1}(\Sigma_X) \), it is easy to show that, for each non-negative \( \Sigma_Y \)-measurable function \( f \) or for each \( f \in L^q(\Sigma_Y) \), there exists a \( \Sigma_X \)-measurable function \( g \) such that \( E(f) = g \circ \varphi \). We can assume that \( \sigma(g) \subseteq \sigma(h_\varphi) \), and there exists only one \( g \) with this property. We then write \( g = E(f) \circ \varphi^{-1} \), though we make no assumptions regarding the invertibility of \( \varphi \) (see [1]). For a deeper study of the properties of \( E \) see [5]. For any non-singular measurable function \( \varphi \) from \( Y \) into \( X \) and \( u \in L(Y) \), the pair \( (u, \varphi) \) induce a linear operator \( uC_\varphi \) from \( L^p(\Sigma_X) \) into \( L(Y) \) defined by

\[
uC_\varphi(f) = u.f \circ \varphi \quad (f \in L^p(\Sigma_X)).
\]

Here, the non-singularity of \( \varphi \) guarantees that \( uC_\varphi \) is well defined as a mapping of equivalence classes of functions on \( \sigma(u) \). If \( uC_\varphi \) takes \( L^p(\Sigma_X) \) into \( L^q(\Sigma_Y) \), then \( uC_\varphi \) is bounded, by the closed graph theorem. In this case we call \( uC_\varphi \) a weighted composition operator \( L^p(\Sigma_X) \) into \( L^q(\Sigma_Y) \). If \( X = Y \) and \( \varphi \) is the identity, we write \( uC_\varphi \) as \( M_u \) and call it the multiplication operator induced by \( u \). In case that \( u \equiv 1 \) we write \( uC_\varphi \) as \( M_uC_\varphi \) as \( C_\varphi \) and call it the composition operator induced by \( \varphi \).
Boundedness of $uC_\varphi$

Boundedness of composition operators in $L^p$-spaces ($1 \leq p < \infty$) for finite measures appeared already in the Dunford-Schwarz book [2, Lemma 7, pp.664–665] and for $\sigma$-finite measures in [6] and [7]. In this section we turn attention to the follow-up problem.

Which function $u \in L(Y)$ and measurable function $\varphi : Y \to X$ induce a weighted composition operator from $L^p(\Sigma_X)$ into $L^q(\Sigma_Y)$ in the case $1 \leq q \leq p < \infty$?

The next lemma will be crucial in what follows. In fact, it is a slight generalization of proposition 2.1 in [3].

**Lemma 1** Suppose $1 \leq p, q < \infty$, $u \in L(Y)$ and let the pair $(u, \varphi)$ induce a weighted composition operator from $L^p(\Sigma_X)$ into $L^q(\Sigma_Y)$. Then for any $f \in L^p(\Sigma_X)$ we have

$$\|uC_\varphi f\|_{L^q(\Sigma_Y)} = \|M_J f\|_{L^q(\Sigma_X)},$$

where $J = (h_{\varphi} E(|u|^{q}) \circ \varphi^{-1})^{\frac{1}{q}}$.

**Proof.** Let $f \in L^p(\Sigma_X)$. As an application of the properties of the conditional expectation and using the change of variable formula we have

$$\|uC_\varphi f\|_{L^q(\Sigma_Y)}^q = \int_Y |u.f \circ \varphi|^q d\nu = \int_Y E(|u|^q) |f|^q \circ \varphi d\nu$$

$$= \int_X E(|u|^q) \circ \varphi^{-1} |f|^q d\nu \circ \varphi^{-1} = \int_X (h_{\varphi} E(|u|^{q}) \circ \varphi^{-1}) |f|^q d\mu$$

$$= \int_X |Jf|^q d\mu = \|M_J f\|_{L^q(\Sigma_X)}^q.$$

So we proved that the pair $(u, \varphi)$ induce a weighted composition operator $uC_\varphi : L^p(\Sigma_X) \to L^q(\Sigma_Y)$ if and only if $J$ induces a multiplication operator $M_J : L^p(\Sigma_X) \to L^q(\Sigma_X)$ and $\|uC_\varphi\| = \|M_J\|$.

The proof of the following proposition can be obtained by adapting the proof of theorem 2.3 in [4].
Suppose \( 1 \leq q < p < \infty \) and \( \frac{1}{p} + \frac{1}{q} = \frac{1}{r} \). Let \( u \in L(Y) \) and \( \varphi : Y \to X \) be a non-singular measurable function. Then the pair \((u, \varphi)\) induce a weighted composition operator \(uC_\varphi\) from \(L^p(\Sigma_X)\) into \(L^q(\Sigma_Y)\) if and only if \( J \in L^r(\Sigma_X) \) and its norm given by \( \|uC_\varphi\| = \|J\|_{L^r(\Sigma_X)} \).

**Corollary 3** Under the same assumptions as in proposition 2.2, \( \varphi \) induces a composition operator \(C_\varphi : L^p(\Sigma_X) \to L^q(\Sigma_Y)\) if and only if \( h_\varphi \in L^{\frac{r}{r}}(\Sigma_X) \). Also, if \( X = Y \), \( u \) induces a multiplication operator \(M_u : L^p(\Sigma_X) \to L^q(\Sigma_X)\) if and only if \( u \in L^r(\Sigma_X) \). In these cases we have \( \|uC_\varphi\| = \|h_\varphi\|_{L^r(\Sigma_X)} \) and \( \|M_u\| = \|u\|_{L^r(\Sigma_X)} \).

If \( p = q \), then \( r \) must be \( \infty \). So \( uC_\varphi(L^p(\Sigma_X)) \subseteq L^p(\Sigma_Y) \) if and only if \( J \in L^\infty(\Sigma_X) \). In this case \( \|uC_\varphi\| = \|J\|_{L^\infty(\Sigma_X)} \). This fact is well known. For direct proof see [6].

**Examples.** (i) Suppose \( X = [0, a^4] \) and \( Y = [-a^2, a^2] \) for some \( a > 0 \). Let \( \varphi : (Y, \Sigma_Y, \nu) \rightarrow (X, \Sigma_X, \mu) \) be defined on Lebesgue measure spaces by \( \varphi(x) = a^4 - x^2 \).

If we consider \(uC_\varphi : L^2(\Sigma_X) \to L^2(\Sigma_Y)\) as \(uC_\varphi f(x) = xf(a^4 - x^2)\), then a simple computation gives \( h_\varphi = 1/2\sqrt{a^4 - x} \notin L^\infty(\Sigma_X) \). Then \( C_\varphi \) does not define a bounded composition operator. However it is easy to see that

\[
J(x) = \left( \frac{1}{2\sqrt{a^4 - x}} \right) = \sqrt{a^4 - x} \in L^\infty(\Sigma_X).
\]

So \( uC_\varphi \) is bounded and \( \|uC_\varphi\| = a \).

(ii) Let \((X, \Sigma_X, \mu)\) be the unit circle in complex plane and Lebesgue measurable sets equipped with normalized Lebesgue measure, and \( \varphi(z) = z^2 \). If we consider \(uC_\varphi \) from \( L^2(X, \Sigma_X, \mu) \) into \( L^2(X, \Sigma_X, \mu \circ \varphi^{-1}) \), then we have

\[
\|uC_\varphi\|^2_{L^2(X, \Sigma_X, \mu \circ \varphi^{-1})} = \int_X h_\varphi |f| |f| \circ \varphi^{-1} d\mu \\
= \int_X h_\varphi E(h_\varphi |u|^2) \circ \varphi^{-1} d\mu = \int_X G |f|^2 d\mu,
\]

where \( G = h_\varphi E(h_\varphi |u|^2) \circ \varphi^{-1} \). Hence \( uC_\varphi \) is bounded if and only if \( G \in L^\infty(X, \Sigma_X, \mu) \). We note that by a simple computation we have

\[
G(z) = \frac{1}{2} \sum_{\zeta^2 = z} |u(\zeta)|^2 h(\zeta), \quad (z \in X).
\]

260
Now, we try to give another characterization of boundedness for $uC_\varphi$ from $L^p(\Sigma_X)$ into $L^q(\Sigma_Y)$. Let $u \in L(Y)$ and $\varphi : Y \to X$ be a non-singular measurable function. Define the measure $\mu_{u,\varphi}$ by

$$\mu_{u,\varphi}(A) = \int_{\varphi^{-1}(A)} |u|^qd\nu, \quad (A \in \Sigma_X).$$

Since $\nu \circ \varphi^{-1} \ll \mu$, then for each $A \in \Sigma_X$ with $\mu(A) = 0$, we have $\nu(\varphi^{-1}(A)) = 0$; so $\mu_{u,\varphi}(A) = 0$. Then $\mu_{u,\varphi} \ll \mu$. Put $\theta = (d\mu_{u,\varphi}/d\mu)^{1/q}$ which, of course, is a non-negative $\Sigma_X$-measurable function.

**Lemma 4** Fixing $1 \leq q < \infty$ and given $u \in L(Y)$. Then, for any non-negative $\Sigma_X$-measurable function $f$,

$$\int_X f d\mu_{u,\varphi} = \int_Y |u|^q f \circ \varphi d\nu$$

in the sense that, if one of the Integrals exists, then so does the other, and they are equal.

**Proof.** Let $f = \sum_{i=1}^n \alpha_i \chi_{A_i}$, where $A_i \in \Sigma_X$ and $0 < \mu(A_i) < \infty$. We have that

$$\int_X f d\mu_{u,\varphi} = \sum_{i=1}^n \alpha_i \mu_{u,\varphi}(A_i)$$

$$= \sum_{i=1}^n \alpha_i \int_{\varphi^{-1}(A_i)} |u|^qd\nu = \int_Y |u|^q \left( \sum_{i=1}^n \alpha_i \chi_{\varphi^{-1}(A_i)} \right) d\nu$$

$$= \int_Y |u|^q \left( \sum_{i=1}^n \alpha_i \chi_{A_i} \right) \circ \varphi d\nu = \int_Y |u|^q f \circ \varphi d\nu.$$

Now, if $f$ is a non-negative function in $L(X)$, we take an increasing sequence $\{f_n\}_{n=1}^\infty$ of non-negative simple functions such that $f_n \to f$ a.e. Then we have $\int_X f_n d\mu_{u,\varphi} \to \int_X f d\mu_{u,\varphi}$. On the other hand $\{|u|^q f_n \circ \varphi\}_{n=1}^\infty$ is an increasing sequence such that $|u|^q f_n \circ \varphi \to |u|^q f \circ \varphi$ a.e., so $\int_X f_n d\mu_{u,\varphi} = \int_Y |u|^q f_n \circ \varphi d\nu \to \int_Y |u|^q f \circ \varphi d\nu.$

$\square$

Now, we present the main result of this paper.
Theorem 5 Suppose \(1 \leq q < p < \infty \) and \( \frac{1}{p} + \frac{1}{r} = \frac{1}{q} \). Let \( u \in L(Y) \) and \( \varphi : Y \to X \) be a non-singular measurable function. Then the following assertions are equivalent:

(i) The pair \((u, \varphi)\) induce a weighted composition operator \(uC\varphi\) from \(L^p(\Sigma_X)\) into \(L^q(\Sigma_Y)\).

(ii) \(\theta\) belongs to \(L^r(\Sigma_X)\).

(iii) There is a partition \(\{F_n\}_{n=1}^\infty\) of \(X\) such that \(\sum_{n=1}^\infty \|\theta|_{F_n}\|_{\infty} \mu(F_n) < \infty\), where \(\|\theta|_{F_n}\|_{\infty} = \text{ess} \sup_{x \in F_n} \theta(x)\).

Proof. Suppose that (i) holds, and \(f \in L^p(\Sigma_X)\). By using lemma 2.4 we have

\[
\|uC\varphi\|_{L^q(\Sigma_Y)}^q = \int_Y |u|^q |f|^q \circ \varphi \ d\nu = \int_X |f|^q d\mu_u,\varphi
\]

\[
= \int_X |\theta f|^q d\mu = \|M_\theta f\|_{L^q(\Sigma_X)}^q.
\]

Hence by corollary 2.3, \(uC\varphi\) is bounded if and only if \(\theta \in L^r(\Sigma_X)\). Thus we obtain the equivalence of (i) and (ii).

Assume that (iii) does not hold. Choose a number \(a > 1\) arbitrarily, and set \(F_0 = \{x \in X : \theta(x) = 0\}\), \(F_{2n} = \{x \in X : a^{n-1} < \theta(x)^r \leq a^n\}\) and \(F_{2n-1} = \{x \in X : a^{-n} \leq \theta(x)^r < a^{-n+1}\}\). Then \(\{F_n\}_{n=0}^\infty\) clearly becomes a partition of \(X\). So we have

\[
\int_X \theta^r d\mu = \sum_{i=1}^\infty \int_{F_{2n}} \theta^r d\mu + \sum_{i=1}^\infty \int_{F_{2n-1}} \theta^r d\mu
\]

\[
\geq \sum_{i=1}^\infty a^{n-1} \mu(F_{2n}) + \sum_{i=1}^\infty a^{-n} \mu(F_{2n-1})
\]

\[
\geq \frac{1}{a} \left[ \sum_{n=1}^\infty \|\theta|_{F_{2n}}\|_{\infty} \mu(F_{2n}) + \sum_{n=1}^\infty \|\theta|_{F_{2n-1}}\|_{\infty} \mu(F_{2n-1}) \right]
\]

\[
\geq \frac{1}{a} \sum_{n=1}^\infty \|\theta|_{F_n}\|_{\infty} \mu(F_n) = +\infty.
\]

This means that \(\theta \notin L^r(\Sigma_X)\). Hence we proved the implication (ii) \(\Rightarrow\) (iii).

Finally, let \(\{F_n\}_{n=0}^\infty\) be a partition of \(X\) such that \(\sum_{n=1}^\infty \|\theta|_{F_n}\|_{\infty} \mu(F_n) < \infty\), we have

\[
\int_X \theta^r d\mu = \sum_{i=1}^\infty \int_{F_n} \theta^r d\mu \leq \sum_{n=1}^\infty \|\theta|_{F_n}\|_{\infty} \mu(F_n) < \infty.
\]
Thus we proved the implication (iii) ⇒ (i) (⇔ (ii)).

\[ \square \]

Acknowledgments

The author would like to thank the referees for their useful comments.

References


