On the Normalizer of the Congruence Subgroup $H_5^5(I)$ of the Hecke Group $H^5$

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Abstract

Let $\lambda = 2 \cos \frac{\pi}{5}$ and let $H^5$ be the Hecke group associated to $\lambda$. In this paper, the normalizers of the congruence subgroups $H_5^5(I)$ in $PSL(2, \mathbb{Z}[\lambda])$ are studied in the case where $I = (2)^\alpha I'$, $(2, I') = 1$ and $I'$ is a prime ideal.

Key Words: Normalizer, Congruence subgroup, Hecke group.

1. Introduction

The congruence subgroups of the Hecke group $H^q (q = 3, 4, 6)$ and the normalizers of these groups in $H^q$, in $PSL(2, \mathbb{Z}[\lambda])$ and in $PSL(2, \mathbb{R})$ were studied by various authors (see [1], [2], [4], [5], [8], [9], [16]). The normalizers of the congruence subgroups of the Hecke group $H^5$ in $PSL(2, \mathbb{R})$ were given for prime ideals (see [11],[12]). In this paper, we investigate the normalizer of the congruence subgroup $H_5^5(I)$ of the Hecke group $H^5$ in $PSL(2, \mathbb{Z}[\lambda])$. Furthermore, in [8], it is conjectured that the normalizer of $H_5^5(I)$ in $H^5$ is $H_5^5((2)^\alpha I')$, where $I = (2)^\alpha I'$ is an ideal of $\mathbb{Z}[\lambda]$, $(2, I') = 1$ and $\alpha' = \alpha - \min (2, \lceil \frac{\alpha}{2} \rceil)$. We give a proof to the conjecture in the case where $I'$ is a prime ideal.

We start by recalling definitions, notations, and some preliminary results of these concepts. By a Hecke group we mean a discrete subgroup of $PSL(2, \mathbb{R})$ generated by $T$ and $U_q$, where $T$ and $U_q$ are the Möbius transformations given by $T(z) = -\frac{1}{z}$, and $U_q(z) = z + \lambda_q$. Hecke [6] showed that these groups are discrete if and only if $\lambda_q = 2 \cos \frac{\pi}{q}$.

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or $\lambda_q > 2$. This group is denoted by $H^q$. It is known that a presentation for $H^q$ is

$$\langle T, U_q \rangle = \langle T, S_q | T^2 = S_q^q = I \rangle,$$

where $S_q = TU_q$, and so $H^q$ is isomorphic to the free product $C_2 * C_q$.

We have the following table of the values of $\lambda_q$ for small $q$:

<table>
<thead>
<tr>
<th>$q$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_q$</td>
<td>1</td>
<td>$\sqrt{2}$</td>
<td>$1+\sqrt{5}/2$</td>
<td>$\sqrt{3}$</td>
</tr>
</tbody>
</table>

The best known example is when $q = 3$, and $H^3$ is the modular group $\Gamma = \text{PSL}(2, \mathbb{Z})$ so the above can be thought of as a natural generalization of $\Gamma$. Furthermore, we have the following geometric interpretation: the modular group $\Gamma$ is the triangle group $(2, 3, \infty)$ and the Hecke group $H^q$ is the triangle group $(2, q, \infty)$.

Let $\mathbb{H} := \{ z \in \mathbb{C} | \text{Im}(z) > 0 \}$ and $\hat{\mathbb{H}} := \mathbb{H} \cup \mathbb{Q}(\lambda_q) \cup \{ \infty \}$. Then the Hecke group $H^q$, namely a subgroup of $SL_2(\mathbb{Z}[\lambda_q]) / \{ \pm I \}$, acts on $\hat{\mathbb{H}}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

As usual, we denote an element of $H^q$ as a $2 \times 2$ matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ remembering to identify any such matrix with its negative.

Let $I$ be an ideal of $\mathbb{Z}[\lambda_q]$. The principal congruence subgroup of level $I$ is

$$H^q(I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H^q | a-1, b, c, d-1 \equiv 0 (\text{mod} I) \right\}$$

and any subgroup $\Lambda^q$ of $H^q$ containing $H^q(I)$ is called a congruence subgroup of level $I$. The two most important of these are

$$H^q_0(I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H^q | c \equiv 0 (\text{mod} I) \right\}$$

and

$$H^q_1(I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H^q | a-1, c, d-1 \equiv 0 (\text{mod} I) \right\}.$$
It is easy to see that we have the inclusions $H^q(I) \leq H^q_0(I) \leq H^q(I) \leq H^q_0(I)$ and we can also see that $H^q(I)$ is normal in $H^q$ and $H^q_0(I)$ is normal in $H^q_0(I)$.

Again, $H^q_0(I)$ is a natural generalization of the congruence subgroups $\Gamma_0(n)$ of $\Gamma$. It works because the elements of $H^q$ sit naturally in the ring $\mathbb{Z}[\lambda]$.

Recall that $H^q$ is commensurable with $\text{PSL}(2, \mathbb{Z})$ if and only if $q = 4$ and 6. The elements of such groups are completely known (see [17]).

Suppose $H^q$ is not commensurable with $\text{PSL}(2, \mathbb{Z})$. By the result of A. Leutbecher ([14],[15]), $\mathbb{Q}(\lambda) \cup \{ \infty \}$ is the set of cusps of $H^q$ if and only if $q = 5$. Also, 5 is the only $q$ other than 4, 6 for which $\mathbb{Q}(\lambda)$ is a quadratic field. For all other $q$'s, the degree is > 2. As a consequence, $q = 5$ is the next most workable and interesting $q$. Some of the classical results on the modular group can be generalized to $H^5$ (see [3], [9], [10], [11]).

From now on, $q$ will be 5, so $\lambda := \lambda_5$, then $\mathbb{Z}[\lambda] = \mathbb{Z}[\lambda_5]$ and $U := U_5$, or $U = \left( \begin{array}{cc} 1 & \lambda \\ 0 & 1 \end{array} \right)$.

The main facts used in our proofs:

(a) $\mathbb{Z}[\lambda]$ is a principal ideal domain. The norm of any element $u + v\lambda$ of $\mathbb{Z}[\lambda]$ is defined by $\text{Nor}(u + v\lambda) = u^2 - v^2 + uv$. Let $I$ be a non-zero ideal of $\mathbb{Z}[\lambda]$, we say that $a$ and $I$ are relatively prime if there exist elements $x \in \mathbb{Z}[\lambda]$ and $b \in I$ such that $ax + b = 1$, and this is denoted by $(a, I) = 1$.

Let $a, b \in \mathbb{Z}[\lambda]$. The element $a$ is said to be congruent to $b$ modulo $I$ (denoted by $a \equiv b \text{ (mod } I)$) if $a - b \in I$.

(b) The set of cusps of $H^5$ is $\mathbb{Q}(\lambda) \cup \{ \infty \}$ ([14],[15]). Furthermore, if $x \in \mathbb{Q}(\lambda)$ is a cusp, $x$ has a unique reduced form $x = \frac{a}{b}$ ([13]). By definition, this means that $a, c \in \mathbb{Z}[\lambda]$ with $c \geq 0$ and there exists $b, d \in \mathbb{Z}[\lambda]$ such that $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in H^5$. Clearly, $(a, c) = 1$ so that if $x = \frac{a}{b}$ with $(a', c') = 1$, then $a = \mu a'$, $c = \mu c'$ where $\mu$ is a unit in $\mathbb{Z}[\lambda]$.

(c) (Corollary 5 of [13]) $\left( \begin{array}{cc} 1 & b \\ c & 1 \end{array} \right) \in H^5$ if and only if $b = m\lambda$, $m \in \mathbb{Z}$.

Slightly, $\left( \begin{array}{cc} 1 & 0 \\ c & 1 \end{array} \right) \in H^5$ if and only if $c = n\lambda$, $n \in \mathbb{Z}$.

(d) (Proposition 6 of [13]) Suppose $x_i$, $x_j$ are $H^5$-rationals with reduced form $\frac{a_i}{b_i}$ and $\frac{b_j}{d_j}$, respectively, and suppose that $x_i < x_j$. Then the following statements are equivalent:
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(i) \( \begin{pmatrix} a_i & b_j \\ c_i & d_j \end{pmatrix} \in H^5 \);

(ii) \((x_i, x_j)\) is an even line, that is, it is the image of the complete hyperbolic geodesic with ends at 0 and \(\infty\) under the action of some \(A \in H^5\);

(iii) \(a_id_j - bjc_i = 1\).

(e) \(A \in H\) if and only if it is a finite word in the generators \(T\) and \(U\). The word can be written as

\[
A = U^{r_0}TU^{r_1}T \ldots TU^{r_{n+1}} \tag{1.1}
\]

where \(r_i\) are non-zero integers except \(r_0\) and \(r_{n+1}\) which may be 0. The word in turn gives rise to the matrix \(A\). By judicious applications of the generators the word can be made unique ([18]).

(f) (Lemma 1 of [3]) If \(I\) is a non-zero ideal of \(Z[\lambda]\), then

\[
[H^5 : H^0_I] = N(I) \prod_{P \mid I} \left( 1 + \frac{1}{N(P)} \right),
\]

where the product is over the set of all prime ideals \(P\) which divide \(I\). Here, for a non-zero ideal \(I\) of \(Z[\lambda]\), \(N(I)\) denotes the absolute norm of \(I\).

(g) (Theorem 4.5 of [7]) If \(K, H, G\) are groups with \(K < H < G\), then \([G : K] = [G : H][H : K]\). If any two of these indices are finite, then so is the third.

(h) (Corollary 2 of [10]) The indices of the congruence subgroups of \(H^5\) of level \(I = (2)\) are \([H^5 : H^5_I] = 10\), and \([H^5 : H^5_{I'}] = [H^5 : H^5_{I'}] = 5\).

The rest of this paper is organized as follows. In the next section, we give some results concerning congruence subgroup \(H^0_I\), where \(I = (2)^\alpha\), \((\alpha = 1, 2)\) or \(I\) is a prime ideal, which will be needed later. In section 3, we find the normalizer of the congruence subgroup \(H^0_I\) in \(PSL(2, Z[\lambda])\), where \(I = (2)^\alpha I', (2, I') = 1\), and the proof of the conjecture in [8] for this case is given in Corollary 17.

2. Congruence subgroup \(H^0_I\)

Lemma 1. Let the ideal \(I = (2) = 2Z[\lambda]\) and let \(A \in H^5\). Then

\[
A \in H^0_I if and only if A \equiv \pm \begin{pmatrix} 1 & r\lambda \\ 0 & 1 \end{pmatrix} \pmod{I},
\]

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where $r = 0, 1$.

**Proof.** By using (g) and (h), we have that

$$[H^5_0(I): H^5(I)] = 2.$$ 

Then, since $U = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \notin H^5(I)$, the partition of $H^5_0(I)$ associated to the subgroup $H^5(I)$ is

$$H^5_0(I) = H^5(I) \cup UH^5(I). \quad (2.2)$$

Thus, for every matrix $A$ in $H^5_0(I)$, from (2.2), there are two cases as follows.

**Case 1.** If $A \in H^5(I)$, then we get

$$A \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{I}.$$ 

**Case 2.** If $A \in UH^5(I)$, then we have

$$A \equiv \pm \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \pmod{I}.$$ 

This completes the proof of the lemma. \hfill \Box

**Corollary 2.** Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $H^5_0(I)$. In this case,

(i) if $I = (2)$, then $a - d \equiv 0 \pmod{(2)}$

(ii) if $I = (4) = (2)^2$, then $a - d \equiv 0 \pmod{(4)}$.

**Proof.** (i) Since $a^2 - 1 = (a - 1)(a + 1)$, by Lemma 1, we have

$$a^2 - 1 \equiv 0 \pmod{(2)^2}. \quad (2.3)$$

Since $A \in H^5_0(2)$ and $ad - bc = 1$, we have

$$ad \equiv 1 \pmod{(2)}. \quad (2.4)$$

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Suppose that $a - d \equiv u \pmod{2}$ for some $u \in \mathbb{Z}[\lambda]$. Multiplying by $a$, one has $a^2 - ad \equiv au \pmod{2})$. In this case, since $a \not\equiv 0 \pmod{2}$, by (2.3) and (2.4), we have $u \equiv 0 \pmod{2})$. This implies that $a - d \equiv 0 \pmod{2})$.

(ii) Since $A \in H_0^5(4)$ and $ad - bc = 1$, it is clear that $ad \equiv 1 \pmod{4})$. Here, using a similar argument as in the proof of (i), we have $a - d \equiv 0 \pmod{2})$.

Remark 3. Let \[
\begin{pmatrix} a & b \\ c & d \end{pmatrix}
\] be an element of $H_0^5(2)$. Then $a - d \equiv 0 \pmod{2^2})$ is not necessarily true.

Example 4. From (1.1),

\[A = U T U^2 T = \begin{pmatrix} 1 + 2\lambda & -\lambda \\ 2\lambda & -1 \end{pmatrix} \in H_0^5(2).\]

Then, for the matrix $A$, we have $a - d = 2(1 + \lambda) \not\equiv 0 \pmod{2^2})$.

Remark 5. Let \[
\begin{pmatrix} a & b \\ c & d \end{pmatrix}
\] be an element of $H_0^5(4)$. Then $a - d \equiv 0 \pmod{2^3})$ is not necessarily true.

Example 6. If we take $A = \begin{pmatrix} 1 + 2\lambda & -\lambda \\ 2\lambda & -1 \end{pmatrix}$, then, for matrices

\[A^2 = \begin{pmatrix} 3 + 6\lambda & -2 - 2\lambda \\ 4(1 + \lambda) & -1 - 2\lambda \end{pmatrix} \in H_0^5((2)^2) \text{ and } A^4 = \begin{pmatrix} 29 + 48\lambda & -4(3 + 5\lambda) \\ 8(3 + 5\lambda) & -11 - 16\lambda \end{pmatrix} \]

$\in H_0^5((2)^3)$, we have $a - d = 4(1 + \lambda) \not\equiv 0 \pmod{2^3})$ and $a - d = 4(10 + 16\lambda) \not\equiv 0 \pmod{2^3})$.

Remark 7. If the ideal $I \neq (2)$, then Corollary 2 (i) and (ii) are not true.

Example 8. Let $I = (3)$. From (1.1),

\[B = U T U^4 T U^2 T U T U^{-1} T = \begin{pmatrix} 25 + 40\lambda & 10 + 17\lambda \\ 9(2 + 3\lambda) & 8 + 11\lambda \end{pmatrix} \in H_0^5(3).\]
For the matrix $B$, $a = 25 + 40\lambda$. Then it is easily seen that $a^2 - 1 \not\equiv 0 \pmod{3}$. It follows that $a - d \not\equiv 0 \pmod{3}$.

**Example 9.** Let $I = (2 + \lambda)$. From (1.1),

$$C = TU^2TU^3TU^2TU^2T = \begin{pmatrix}
12 + 25\lambda & 5 + 6\lambda \\
-(2 + \lambda)^2(4 + 5\lambda) & -4(3 + 5\lambda)
\end{pmatrix} \in H_0^5(2 + \lambda).$$

For the matrix $C$, $a = 12 + 25\lambda$ and $a^2 - 1 = (11 + 25\lambda)(13 + 25\lambda)$. From (a), we have Nor$(a^2 - 1) = 131.229$ and Nor$(2 + \lambda) = 5$. In this case, since Nor$(a^2 - 1) \equiv 4 \pmod{5}$, we obtain $a^2 - 1 \not\equiv 0 \pmod{(2 + \lambda)}$. This implies that $a - d \not\equiv 0 \pmod{(2 + \lambda)}$.

**Example 10.** Let $I = (4 - \lambda)$. From (1.1),

$$D = TU^7TU^{-5}T = \begin{pmatrix}
5\lambda \\
-(4 - \lambda)^3(2 + 3\lambda) & 1
\end{pmatrix} \in H_0^5(4 - \lambda).$$

For the matrix $D$, $a = 5\lambda$. Using a similar argument as in Example (9), we have $a^2 - 1 \not\equiv 0 \pmod{(4 - \lambda)}$. This implies that $a - d \not\equiv 0 \pmod{(4 - \lambda)}$.

**Corollary 11.** $H_0^5(2) = H_1^5(2)$.

**Lemma 12.** Let $I = (\tau)$ be a prime ideal of $\mathbb{Z}[\lambda]$. Let $p$ be the positive rational prime which lies below $\tau$. Then

(i) $(p) = (\tau)$ if and only if $H_0^5(p) = H_0^5(\tau)$.

(ii) $(p) \neq (\tau)$ if and only if $H_0^5(p) \not\subseteq H_0^5(\tau)$.

**Proof.** (i) If $(p) = (\tau)$, then it is easily seen that $H_0^5(p) = H_0^5(\tau)$. Suppose that $H_0^5(p) = H_0^5(\tau)$. Let $x = \frac{1}{\tau} \in \mathbb{Q}(\lambda)$. By Leutbecher’s Theorem ([14], [15]), $x$ is a cusp of $H^5$. By (b), the reduced form for $x$ is of the form $\frac{c}{d}$, where $c$ is a unit in $\mathbb{Z}[\lambda]$. Thus, by (d), $H_0^5(\tau)$ contains an element of the form

$$A_c = \begin{pmatrix}
c & b \\
c & d
\end{pmatrix}.$$  \hspace{1cm} (2.5)

In this case, since $H_0^5(p) = H_0^5(\tau)$, it follows that $\tau = c^{-1}pu$, where $u \in \mathbb{Z}[\lambda]$. Thus we have $(p) = (\tau)$. 

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(ii) Let \((p) \neq (\tau)\). Since \((p) \subset (\tau)\), it is clear that \(H_0^5(p) < H_0^5(\tau)\). By (2.5), \(A_c \not\in H_0^5(p)\). This implies that \(H_0^5(p) \not\subset H_0^5(\tau)\).

Conversely, from (2.5), we have \((p) \neq (\tau)\).

3. Upper bound for \(N(H_0^5(2^\alpha \tau))\)

Let \(I'\) be an ideal of \(\mathbb{Z}[\lambda]\). Since \(\mathbb{Z}[\lambda]\) is a principal ideal domain, \(I' = (\tau)\) for some \(\tau\). Note that we may assume that \(\tau\) is positive.

From now on, we take the ideal
\[
I = (2)^\alpha I' = (2)^\alpha \cap I' = (2^\alpha \tau),
\]
(3.6)
where \((2, I') = 1\) and \(I' = (\tau)\) is a prime ideal. Denote by \(N(H_0^5(2^\alpha \tau))\) the normalizer of \(H_0^5(2^\alpha \tau)\) in \(PSL(2, \mathbb{Z}[\lambda])\). Let \(X = \begin{pmatrix} x & z \\ y & t \end{pmatrix} \in N(H_0^5(2^\alpha \tau))\), and \(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H_0^5(2^\alpha \tau)\). Then, we have that
\[
XAX^{-1} = \begin{pmatrix} atx - bxy - dyz + ctz & -(a - d)xz + bx^2 - cz^2 \\ (a - d)ty - by^2 + ct^2 & -ayz + byx + dxt - ctz \end{pmatrix} \in H_0^5(2^\alpha \tau)
\]
(3.7)
\[
X^{-1}AX = \begin{pmatrix} * & * \\ -(a - d)xz - by^2 + cx^2 & * \end{pmatrix} \in H_0^5(2^\alpha \tau).
\]
(3.8)

If we take \(A = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\), then,
\[
XAX^{-1} = \begin{pmatrix} 1 - xy\lambda & x^2\lambda \\ -y^2\lambda & 1 + xy\lambda \end{pmatrix} \in H_0^5(2^\alpha \tau)
\]
(3.9)
\[
X^{-1}AX = \begin{pmatrix} 1 + ty\lambda & t^2\lambda \\ -y^2\lambda & 1 - ty\lambda \end{pmatrix} \in H_0^5(2^\alpha \tau).
\]
(3.10)

Lemma 13. Let \(X = \begin{pmatrix} x & z \\ y & t \end{pmatrix} \in N(H_0^5(2^\alpha \tau))\). Then
\[y \equiv 0 (mod \ (2^\alpha \tau)),\]
where \( \alpha' = \alpha - \min \{2, \left\lfloor \frac{\alpha}{2} \right\rfloor \} \).

**Proof.** Since \( \lambda \) is a unit in \( \mathbb{Z}[\lambda] \), by (3.9) and (3.10), then
\[
y^2 \equiv 0 \pmod{(2^\alpha \tau)}. \tag{3.11}
\]
Since \( ad - bc = 1 \) and \( xt - yz = 1 \), from (3.8), we have
\[
(a^2 - 1)y \equiv 0 \pmod{(2^\alpha \tau)}. \tag{3.12}
\]
Here, since \( I' = (\tau) \) and \( (2) \) are prime ideals, by (3.6), (3.11) and (3.12), we obtain
\[
y \equiv 0 \pmod{(2)} \tag{3.13}
\]
\[
(a^2 - 1)y \equiv 0 \pmod{(2)}. \tag{3.14}
\]
and
\[
y \equiv 0 \pmod{(\tau)} \tag{3.15}
\]
\[
(a^2 - 1)y \equiv 0 \pmod{(\tau)}. \tag{3.16}
\]
By (3.13), there exists \( \alpha' \in \mathbb{Z}_+ \) such that
\[
y \equiv 0 \pmod{(2^{\alpha'})} \text{ and } y \not\equiv 0 \pmod{(2^{\alpha' + 1})}. \tag{3.17}
\]
This implies that
\[
y^2 \equiv 0 \pmod{(2^{2\alpha'})} \tag{3.18}
\]
For \( \alpha \) and \( \alpha' \), there are two cases:

**Case 1.** Let \( \alpha < \alpha' \). Then (3.11) and (3.12) are always true.

**Case 2.** Let \( \alpha \geq \alpha' \). From (3.11) and (3.18), we obtain
\[
\alpha \leq 2\alpha' \Rightarrow \frac{\alpha}{2} \leq \alpha' \leq \alpha. \tag{3.19}
\]
From (3) and (3.17), we get
\[
(a^2 - 1)y \equiv 0 \pmod{(2^{\alpha' + 2})}. \tag{3.20}
\]
By using (3.12) and (3.20), we have that
\[ \alpha \leq \alpha' + 2 \Rightarrow 0 \leq \alpha - \alpha' \leq 2. \] (3.21)

Thus, the smallest element \( \alpha' \in \mathbb{Z}_+ \) which satisfies (3.19) and (3.21) must be found.

(i) For \( \alpha = 1, 2 \) and 3, by (3.19) and (3.21), we have that \( \alpha' = 1, 1 \) and 2, respectively.

(ii) For \( \alpha \geq 4 \), there exists an element \( \beta \in \mathbb{N} \) such that \( \alpha = \beta + 4 \). In this case, by (3.19) and (3.21),
\[ 2 \leq \alpha' \text{ and } \beta + 2 \leq \alpha'. \]

Since \( \alpha' \) is smallest, it follows that \( \alpha' = \beta + 2 \). Thus, for every \( \alpha \in \mathbb{Z}_+ \) such that \( \alpha \geq 4 \), we obtain \( \alpha' = \alpha - 2 \). Consequently, from (i) and (ii), we have \( y \equiv 0 \pmod{(2^\alpha') \tau} \), where \( \alpha' = \alpha - \min \left(2, \left\lfloor \frac{\alpha}{2} \right\rfloor \right) \).

This completes the proof of the lemma. \( \square \)

**Lemma 14.** (Lemma 1 of [12]) If \( I \) is a prime ideal of \( \mathbb{Z}[\lambda] \), then
\[ N(H_0^5(I)) = H_0^5(I). \]

**Remark 15.** If \( I \) is not a prime ideal of \( \mathbb{Z}[\lambda] \), then Lemma 14 is not necessarily true as in the following theorem.

**Theorem 16.** Let the ideal \( I = (2^\alpha \tau) \) as in (6). Then
\[ N(H_0^5(2^\alpha \tau)) = H_0^5(2^\alpha \tau) \]
where \( \alpha' = \alpha - \min \left(2, \left\lfloor \frac{\alpha}{2} \right\rfloor \right) \).

**Proof.** By Corollary 2 (i) and (ii), it is clear that
\[ H_0^5(2^\alpha \tau) \leq N(H_0^5(2^\alpha \tau)) \] (3.22)
where \( \alpha' = \alpha - \min \left(2, \left\lfloor \frac{\alpha}{2} \right\rfloor \right) \). Now we prove the converse inclusion, that is,
\[ H_0^5(2^\alpha \tau) \geq N(H_0^5(2^\alpha \tau)) \]
where \( \alpha' = \alpha - \min(2, \left\lfloor \frac{\alpha}{2} \right\rfloor) \). Let \( X = \begin{pmatrix} x & z \\ y & t \end{pmatrix} \in N(H_0^5(2^\alpha \tau)) \). Then, by Lemma 13 and (3.10), it is clear that

\[
y \equiv 0 \pmod{2^\alpha' \tau}.
\]

This implies that \( y = c2^\alpha' \tau \) for some \( c \in \mathbb{Z}[\lambda] \). Suppose \( c \neq 0 \). Recall that \( a = \frac{x}{c^2} \in \mathbb{Q}(\lambda) \) is a cusp of \( H^5 \) as in (b). Let \( a = \frac{x'}{y'} \) be the reduced form for \( a \). Then \( H^5 \) contains an element of the form

\[
Y = \begin{pmatrix} x' & z' \\ y' & t' \end{pmatrix}.
\]

Since \( (x, c2^\alpha' \tau) = 1 \), \( y' = \mu c2^\alpha' \tau \) where \( \mu \) is a unit of \( \mathbb{Z}[\lambda] \). Hence \( y' \) is a multiple of \( 2^\alpha' \tau \). This implies that \( Y \in H_0^5(2^\alpha' \tau) \leq N(H_0^5(2^\alpha \tau)) \). Since \( X \infty = Y \infty \), it follows that

\[
Y^{-1}X = \begin{pmatrix} u & v \\ 0 & u^{-1} \end{pmatrix} \in N(H_0^5(2^\alpha \tau)),
\]

where \( u, v \in \mathbb{Z}[\lambda] \). Applying (9) and (10) to \( Y^{-1}X \), we have that

\[
\begin{pmatrix} 1 & u^2 \lambda \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & u^{-2} \lambda \\ 0 & 1 \end{pmatrix}
\]

are elements of \( H_0^5(2^\alpha \tau) \). By (c), \( u = \pm 1 \). Multiplying \( Y^{-1}X \) by \(-I\) if necessary, we may assume that \( u = 1 \) and

\[
Y^{-1}X = \begin{pmatrix} 1 & x + y \lambda \\ 0 & 1 \end{pmatrix},
\]

where \( x, y \in \mathbb{Z} \). Note that

\[
\begin{pmatrix} 1 & y \lambda \\ 0 & 1 \end{pmatrix} \in N(H_0^5(2^\alpha \tau)).
\]

As a consequence,

\[
\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in N(H_0^5(2^\alpha \tau)).
\]
Suppose that \( x \neq 0 \). Since \( \lambda \in \mathbb{R} \setminus \mathbb{Q} \), for any \( \epsilon > 0 \), there exist \( k \) and \( l \) such that

\[
\begin{pmatrix}
1 & x \\
0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & \lambda \\
0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & \delta \\
0 & 1 \\
\end{pmatrix} = \sigma \in N(H_0^5(2^\alpha \tau)),
\]

where \( 0 < |\delta| < \epsilon \). As a consequence,

\[
\sigma
\begin{pmatrix}
1 & 0 \\
2^\alpha p \lambda & 1 \\
\end{pmatrix}
\sigma^{-1} = \begin{pmatrix}
1 + 2^\alpha \delta p & 2^\alpha \delta^2 p \\
0 & 1 - 2^\alpha \delta p \\
\end{pmatrix} \in H_0^5(2^\alpha \tau),
\]

where \( p \) is the positive rational prime which lies below \( \tau \). This implies that \( H_0^5(2^\alpha \tau) \) is not discrete, giving a contradiction. Hence \( x = 0 \) and \( Y^{-1}X \in H_0^5(2^{\alpha'} \tau) \). Since \( Y \in H_0^5(2^{\alpha'} \tau) \), then we obtain \( X \in H_0^5(2^{\alpha'} \tau) \).

Suppose \( y = 0 \). From the above argument, we have that \( X \in H_0^5(2^{\alpha'} \tau) \). Consequently,

\[
N(H_0^5(2^\alpha \tau)) \leq H_0^5(2^{\alpha'} \tau),
\]

where \( \alpha' = \alpha - \min(2, [\frac{\alpha}{2}]) \).

This completes the proof of the theorem. \( \square \)

**Corollary 17.** Let \( I = (2)^\alpha I' \) be an ideal of \( \mathbb{Z}[\lambda] \), where \( I' \) is a prime ideal of \( \mathbb{Z}[\lambda] \) and \( (2, I') = 1 \). Then the normalizer of \( H_0^5(I) \) in \( H^5 \) is \( H_0^5((2)^{\alpha'} I') \), where \( \alpha' = \alpha - \min(2, [\frac{\alpha}{2}]) \).

**Proof.** From Theorem 16, it is clear that

\[
N(H_0^5(I)) \cap H^5 = H_0^5((2)^{\alpha'} I'),
\]

where \( \alpha' = \alpha - \min(2, [\frac{\alpha}{2}]) \). \( \square \)

**Theorem 18.** Let \( I = (2)^\alpha I' \) be an ideal of \( \mathbb{Z}[\lambda] \), and \( (2, I') = 1 \). Then

\[
[H_0^5((2)^{\alpha'} I') : H_0^5((2)^\alpha I')] = \begin{cases}
1, & \alpha = 1 \\
4, & \alpha = 2, 3 \\
16, & \alpha \geq 4
\end{cases}
\]

where \( \alpha' = \alpha - \min(2, [\frac{\alpha}{2}]) \).
Proof. By using (f) and (g), we have that

\[
[H_0^5((2)^\alpha'I') : H_0^5((2)^\alpha'I')] = \left\{ \begin{array}{ll}
1, & \alpha = 1 \\
4, & \alpha = 2, 3 \\
16, & \alpha \geq 4
\end{array} \right.
\]

where \( \alpha' = \alpha - \min\left(2, \left\lceil \frac{\alpha}{2} \right\rceil \right) \).

\[\square\]

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References


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