On $P$-Sasakian Manifolds Satisfying Certain Conditions on the Concircular Curvature Tensor

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Abstract

We classify $P$-Sasakian manifolds, which satisfy the conditions $Z(\xi, X) \cdot Z = 0$, $Z(\xi, X) \cdot R = 0$, $R(\xi, X) \cdot Z = 0$, $Z(\xi, X) \cdot S = 0$ and $Z(\xi, X) \cdot C = 0$.

Key Words: $P$-Sasakian manifold, concircular curvature tensor, Weyl conformal curvature tensor.

1. Introduction

A Riemannian manifold $M$ is locally symmetric if its curvature tensor $R$ satisfies $\nabla R = 0$, where $\nabla$ is Levi-Civita connection of the Riemannian metric. As a generalization of locally symmetric spaces, many geometers have considered semi-symmetric spaces and in turn their generalizations. A Riemannian manifold $M$ is said to be semi-symmetric if its curvature tensor $R$ satisfies

$$R(X, Y) \cdot R = 0, \quad X, Y \in TM,$$

where $R(X, Y)$ acts on $R$ as a derivation.

Locally symmetric and semisymmetric $P$-Sasakian manifolds are studied in [2] and [5]. After the curvature tensor, the Weyl conformal curvature tensor $C$ and the concircular curvature tensor $Z$ are the next most important tensors. In this paper, we study several derivation conditions on $P$-Sasakian manifolds. The paper is organized as follows. In

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section 2, we give a brief account of $P$-Sasakian manifolds, the Weyl conformal curvature tensor and the concircular curvature tensor. In section 3, we find necessary and sufficient conditions for $P$-Sasakian manifolds satisfying the conditions like $Z(\xi, X) \cdot Z = 0$, $Z(\xi, X) \cdot R = 0$, $R(\xi, X) \cdot Z = 0$, $Z(\xi, X) \cdot S = 0$ and $Z(\xi, X) \cdot C = 0$. In Section 4, we prove that for an $n$-dimensional $P$-Sasakian manifold $M$ the following three statements are equivalent: (a) $M$ is locally symmetric, (b) $M$ is concircularly symmetric and (c) $M$ is locally isometric to the Hyperbolic space $H^n(-1)$.

2. $P$-Sasakian Manifolds

An $n$-dimensional differentiable manifold $M$ is called an \textit{almost paracontact manifold} if it admits an almost paracontact structure $(\varphi, \xi, \eta)$ consisting of a $(1, 1)$ tensor field $\varphi$, a vector field $\xi$, and a 1-form $\eta$ satisfying

$$\varphi^2 = \text{Id} - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi \xi = 0, \quad \eta \circ \varphi = 0. \quad (2.1)$$

The first and one of the remaining three relations in (2.1) imply the other two relations in (2.1). Let $g$ be a compatible Riemannian metric with $(\varphi, \xi, \eta)$, that is,

$$g(X, Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y) \quad (2.2)$$

or equivalently,

$$g(X, \varphi Y) = g(\varphi X, Y) \quad \text{and} \quad g(X, \xi) = \eta(X) \quad (2.3)$$

for all $X, Y \in TM$. Then, $M$ becomes an \textit{almost paracontact Riemannian manifold} equipped with an almost paracontact Riemannian structure $(\varphi, \xi, \eta, g)$.

An almost paracontact Riemannian manifold is called a $P$-\textit{Sasakian manifold} if it satisfies

$$(\nabla X \varphi) Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \quad X, Y \in TM, \quad (2.4)$$

where $\nabla$ is Levi-Civita connection of the Riemannian metric. From the above equation it follows that

$$\nabla \xi = \varphi, \quad (2.5)$$

$$(\nabla_X \eta) Y = g(X, \varphi Y) = (\nabla_Y \eta) X, \quad X \in TM. \quad (2.6)$$
In an $n$-dimensional $P$-Sasakian manifold $M$, the curvature tensor $R$, the Ricci tensor $S$, and the Ricci operator $Q$ satisfy

\begin{align*}
R(X, Y)\xi &= \eta(X)Y - \eta(Y)X, \\
R(\xi, X)Y &= \eta(Y)X - g(X, Y)\xi, \\
R(\xi, X)\xi &= X - \eta(X)\xi, \\
S(X, \xi) &= -(n-1)\eta(X), \\
Q\xi &= -(n-1)\xi,
\end{align*}

\begin{align*}
\eta(R(X, Y)U) &= g(X, U)\eta(Y) - g(Y, U)\eta(X), \\
\eta(R(\xi, X)Y) &= \eta(X)\eta(Y) - g(X, Y).
\end{align*}

An almost paracontact Riemannian manifold $M$ is said to be $\eta$-Einstein [2] if the Ricci operator $Q$ satisfies

\begin{equation}
Q = a \text{Id} + b \eta \otimes \xi,
\end{equation}

where $a$ and $b$ are smooth functions on the manifold. In particular, if $b = 0$, then $M$ is an Einstein manifold. For more details about almost paracontact Riemannian manifolds we refer to [2], [6] and [7].

Let $(M, g)$ be an $n$-dimensional Riemannian manifold. Then the concircular curvature tensor $Z$ and the Weyl conformal curvature tensor $C$ are defined by [9]

\begin{align*}
Z(X, Y)U &= R(X, Y)U - \frac{r}{n(n-1)} (g(Y, U)X - g(X, U)Y), \\
C(X, Y)U &= R(X, Y)U - \frac{1}{n-2} \{ S(Y, U)X - S(X, U)Y \\
&+ g(Y, U)QX - g(X, U)QY \} \\
&+ \frac{r}{(n-1)(n-2)} \{ g(Y, U)X - g(X, U)Y \}
\end{align*}

for all $X, Y, U \in TM$, respectively, where $r$ is the scalar curvature of $M$. 

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3. Main Results

In this section, we obtain necessary and sufficient conditions for $P$-Sasakian manifolds satisfying the derivation conditions $Z(\xi, X) \cdot Z = 0$, $Z(\xi, X) \cdot R = 0$, $R(\xi, X) \cdot Z = 0$, $Z(\xi, X) \cdot S = 0$ and $Z(\xi, X) \cdot C = 0$.

**Theorem 3.1** An $n$-dimensional $P$-Sasakian manifold $M$ satisfies

$Z(\xi, X) \cdot Z = 0$

if and only if either the scalar curvature $r$ of $M$ is $r = n(1 - n)$ or $M$ is locally isometric to the Hyperbolic space $H^n(-1)$.

**Proof.** In a $P$-Sasakian manifold $M$, we have

$$Z(X, Y)\xi = \left(1 - \frac{r}{n(n-1)}\right) (\eta(Y)X - \eta(X)Y), \quad (3.18)$$

$$Z(\xi, X)Y = \left(1 - \frac{r}{n(n-1)}\right) (g(X, Y)\xi - \eta(Y)X). \quad (3.19)$$

The condition $Z(\xi, U) \cdot Z = 0$ implies that

$$0 = [Z(\xi, U), Z(X, Y)]\xi - Z(Z(\xi, U)X, Y)\xi - Z(X, Z(\xi, U)Y)\xi,$$

which in view of (3.19) gives

$$0 = \left(1 + \frac{r}{n(n-1)}\right) \left\{ -g(U, Z(X, Y)\xi)\xi + g(U, X)Z(\xi, Y)\xi \right.$$ 

$$-\eta(X)Z(U, Y)\xi + g(U, Y)Z(X, \xi)\xi$$

$$- \eta(Y)Z(X, U)\xi + \eta(U)Z(X, Y)\xi - Z(X, Y)U \right\}.$$

Equation (3.18) then gives

$$\left(1 + \frac{r}{n(n-1)}\right) \left(Z(X, Y)U - \left(1 + \frac{r}{n(n-1)}\right) (g(Y, U)X - g(X, U)Y) \right) = 0.$$ 

Therefore either the scalar curvature $r = n(1 - n)$ or

$$Z(X, Y)U - \left(1 - \frac{r}{n(n-1)}\right) (g(Y, U)X - g(X, U)Y) = 0.$$
which in view of (2.16) gives
\[ R(X,Y)U = g(U,X)Y - g(U,Y)X. \]

The above equation implies that \( M \) is of constant curvature \(-1\) and consequently it is locally isometric to the Hyperbolic space \( H^n(-1) \).

Conversely, if \( M \) has scalar curvature \( r = n(1-n) \) then from (3.19) it follows that \( Z(\xi,X) = 0 \). Similarly, in the second case, since \( M \) is of constant curvature \( r = n(1-n) \), therefore we again get \( Z(\xi,X) = 0 \).

Using the fact that \( Z(\xi,X) \cdot R \) denotes \( Z(\xi,X) \) acting on \( R \) as a derivation, we have the following Theorem as a corollary of Theorem 3.1.

**Theorem 3.2** An \( n \)-dimensional \( P \)-Sasakian manifold \( M \) satisfies
\[ Z(\xi,X) \cdot R = 0 \]
if and only if either \( M \) is locally isometric to the Hyperbolic space \( H^n(-1) \) or \( M \) has constant scalar curvature \( r = n(1-n) \).

**Proposition 3.3** Let \((M,g)\) be an \( n \)-dimensional Riemannian manifold. Then \( R \cdot Z = R \cdot R \).

**Proof.** Let \( X, Y, U, V, W \in TM \). Then
\[
\]
So from (2.16) and the symmetry properties of the curvature tensor \( R \) we have
\[
\]
which proves the proposition.

Now, in view of Theorem 2.1 of [2] and Proposition 3.3 we have the following theorem:
Theorem 3.4 An $n$-dimensional $P$-Sasakian manifold $M$ satisfies

$$R(\xi, X) \cdot Z = 0$$

if and only if $M$ is locally isometric to the Hyperbolic space $H^n(-1)$.

Next, we prove the following

Theorem 3.5 An $n$-dimensional $P$-Sasakian manifold $M$ satisfies

$$Z(\xi, X) \cdot S = 0$$

if and only if either $M$ has scalar curvature $r = n(1-n)$ or $M$ is an Einstein manifold with the scalar curvature $r = n(1-n)$.

Proof. The condition $Z(\xi, X) \cdot S = 0$ implies that

$$S(Z(\xi, X)Y, \xi) + S(Y, Z(\xi, X)\xi) = 0,$$

which in view of (3.19) gives

$$0 = \left(1 + \frac{r}{n(n-1)}\right)(-g(X, Y)S(\xi, \xi) + \eta(Y)S(X, \xi) - \eta(X)S(Y, \xi) + S(X, Y)).$$

So by the use of (2.10) we have

$$\left(1 + \frac{r}{n(n-1)}\right)\left(S - (1-n)g\right) = 0.$$

Therefore either the scalar curvature $r$ of $M$ is $r = n(1-n)$ which is of constant or $S = (1-n)g$ which implies that $M$ is an Einstein manifold with the scalar curvature $r = n(1-n)$. The converse statement is trivial. $\square$

Theorem 3.6 An $n$-dimensional $P$-Sasakian manifold $M$ satisfies

$$Z(\xi, X) \cdot C = 0$$

if and only if either $M$ has scalar curvature $r = n(1-n)$ or $M$ is conformally flat, in which case $M$ is a $SP$-Sasakian manifold.
\textbf{Proof.} \( Z(\xi, U) \cdot C = 0 \) implies that
\[
0 = [Z(\xi, U), C(X, Y)] W - C(Z(\xi, U)X, Y) W - C(X, Z(\xi, U)Y) W,
\]
which in view of (3.19) we have
\[
0 = (1 + \frac{r}{n(n-1)}) [\eta(C(X, Y)W)U - C(X, Y, W, U)\xi - \eta(X)C(U, Y)W
+ g(U, X)C(\xi, Y)W - \eta(Y)C(X, U)W + g(U, Y)C(X, \xi)W
- \eta(W)C(X, Y)U + g(U, W)C(X, Y)\xi].
\]
So either the scalar curvature of \( M \) is \( r = n(1 - n) \) or the equation
\[
0 = \eta(C(X, Y)W)U - C(X, Y, W, U)\xi - \eta(X)C(U, Y)W
+ g(U, X)C(\xi, Y)W - \eta(Y)C(X, U)W + g(U, Y)C(X, \xi)W
- \eta(W)C(X, Y)U + g(U, W)C(X, Y)\xi

holds on \( M \). Taking the inner product of the last equation with \( \xi \) we get
\[
0 = \eta(C(X, Y)W)\eta(U) - C(X, Y, W, U)
- \eta(X)\eta(C(U, Y)W) + g(U, X)\eta(C(\xi, Y)W) - \eta(Y)\eta(C(X, U)W
+ g(U, Y)\eta(C(X, \xi)W) - \eta(W)\eta(C(X, Y)U).
\]
Hence using (2.10), (2.12) and (2.17) the equation (3.20) turns the form
\[
0 = g(U, Y)g(X, W) - g(U, X)g(Y, W)
+ \frac{1 - n}{n - 2} \left\{ -g(Y, W)g(X, U) + g(X, W)g(U, Y)
+ g(X, U)\eta(Y)\eta(W) - g(U, Y)\eta(X)\eta(W) \right\}
+ \frac{1}{n - 2} \left\{ S(Y, U)\eta(X)\eta(W) - S(X, U)\eta(Y)\eta(W)
+ g(Y, W)S(X, U) - g(X, W)S(Y, U) \right\} - R(X, Y, W, U).
\]
Hence by a suitable contraction of (3.21) we have
\[
S(Y, W) = (1 + \frac{r}{n-1})g(Y, W) + (-n + \frac{r}{1-n})\eta(Y)\eta(W),
\]
which implies that \( M \) is an \( \eta \)-Einstein manifold. So using (3.22) in (3.20) we obtain \( C = 0 \) on \( M \). Thus using the fact from [1] that a conformally flat \( P \)-Sasakian manifold is an \( SP \)-Sasakian, \( M \) becomes an \( SP \)-Sasakian manifold. The converse statement is trivial.

\[ \square \]

4. An application

A Riemannian manifold is said to be \textit{concircularly symmetric} if the concircular curvature tensor \( Z \) is parallel, that is, \( \nabla Z = 0 \). Now, we prove the following theorem.

**Theorem 4.1** In a \( P \)-Sasakian manifold \( M \) the following conditions are equivalent:

(a) \( M \) is locally symmetric,

(b) \( M \) is concircularly symmetric,

(c) \( M \) is locally isometric to the Hyperbolic space \( H^n(-1) \).

**Proof.** It is obvious that the condition \( \nabla T = 0, T \in \{R, Z\} \), implies the condition \( R \cdot T = 0 \). From Theorem 2.1 of [2] and Theorem 3.4, it follows that \( M \) satisfies the condition \( R(\xi, X) \cdot T = 0, T \in \{R, Z\} \) if and only if \( M \) is locally isometric to the Hyperbolic space \( H^n(-1) \).

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