A Generalization of Ankeny and Rivlin’s Result on the Maximum Modulus of Polynomials not Vanishing in the Interior of the Unit Circle

V. K. Jain

Abstract

For an arbitrary entire function $f(z)$, let

$$M(f, r) = \max_{|z|=r} |f(z)|.$$  

For a polynomial $p(z)$ of degree $n$, it is known that

$$M(p, R) \leq R^n M(p, 1), \quad R > 1.$$  

By considering the polynomial $p(z)$ with no zeros in $|z| < 1$, Ankeny and Rivlin obtained the refinement

$$M(p, R) \leq \{(R^n + 1)/2\} M(p, 1), \quad R > 1.$$  

By considering the polynomial $p(z)$ with no zeros in $|z| < k$, ($k \geq 1$) and simultaneously thinking of $s^{th}$ derivative ($0 \leq s < n$) of the polynomial, we have obtained the generalization

$$M(p^{(s)}(z), R) \leq \left\{ \begin{array}{ll} (1/2)\{(R^n + k^n)(1/(1 + k))\} M(p, 1), & R \geq k, \\
(1/(R^n + k^n))(\{(d^s/dz^s)(1 + x^n)\}_{x=1})\{(R + k)/(1 + k)\}^n M(p, 1), & 1 \leq R \leq k, \end{array} \right.$$  

of Ankeny and Rivlin’s result.

Key words and phrases: Polynomial, maximum modulus principle, not vanishing in the interior of unit circle, generalization, $s^{th}$ derivative.

AMS Mathematics Subject Classification: Primary 30 C10, Secondary 30 A10
1. Introduction and statement of results

For an arbitrary entire function $f(z)$, let

$$M(f, r) = \max_{|z|=r} |f(z)|.$$

As a consequence of maximum modulus principle, we have the following result.

**Theorem A** \hspace{1em} If $p(z)$ is a polynomial of degree $n$, then

$$M(p, R) \leq R^n M(p, 1), \quad R > 1,$$

with equality only for $p(z) = \lambda z^n$.

Ankeny and Rivlin [1] considered polynomials not vanishing in the interior of the unit circle and obtained the following refinement of Theorem A.

**Theorem B** \hspace{1em} If $p(z)$ is a polynomial of degree $n$, having no zeros in $|z| < 1$, then

$$M(p, R) \leq \{(1 + R^n)/2\} M(p, 1), \quad R > 1,$$

with equality only for $p(z) = \lambda + \mu z^n$, with $|\lambda| = |\mu|$.

In this paper, we have obtained a generalization of Theorem B, by considering polynomials with no zeros in $|z| < k$, $k \geq 1$ and simultaneously thinking of the $s^{th}$ derivative, $0 \leq s < n$, of the polynomial, instead of the polynomial itself. More precisely we have proved the following theorem.

**Theorem** \hspace{1em} If $p(z)$ is a polynomial of degree $n$ having no zeros in $|z| < k$, $(k \geq 1)$ then for $0 \leq s < n$

$$M(p^{(s)}, R) \leq \begin{cases} 
(1/2) \left( \frac{d^s}{dx^s} (R^n + k^n) \right) \left( \frac{2}{1 + k} \right)^n M(p, 1), & R \geq k, \\
\left( \frac{1}{(1 + k)^n} \right) \left( \frac{d^s}{dx^s} (1 + x^n) \right)_{x=1} \left( (R + k)/(1 + k) \right)^n M(p, 1), & 1 \leq R < k.
\end{cases}$$ \hspace{1em} (1.1)

Equality holds in (1.1) (with $k = 1$ & $s = 0$) for $p(z) = z^n + 1$ and equality holds in (1.2) (with $s = 1$) for $p(z) = (z + k)^n$.

2. Lemmas

For the proof of the theorem we require following lemmas.
Lemma 1  Let $P(z)$ be a polynomial of degree $n$ having all its zeros in $|z| \leq 1$. If $p(z)$ is a polynomial of degree at most $n$ such that

$$|p(z)| \leq |P(z)|, \quad |z| = 1, \quad (2.1)$$

then for $0 \leq s < n$,

$$|p^{(s)}(z)| \leq |P^{(s)}(z)|, \quad |z| \geq 1. \quad (2.2)$$

Proof of Lemma 1  By using (2.1), we can say that an application of maximum modulus principle to the function $p(z)/P(z)$ will yield

$$|p(z)| \leq |P(z)|, \quad |z| \geq 1. \quad (2.3)$$

Therefore the polynomial

$$p(z) - \lambda P(z)$$

will not vanish in $|z| > 1$ for every $\lambda$ with $|\lambda| > 1$. Gauss-Lucas theorem will then imply that polynomial

$$p^{(s)}(z) - \lambda P^{(s)}(z), \quad 1 \leq s < n$$

will not vanish in $|z| > 1$ for every $\lambda$ with $|\lambda| > 1$ and therefore

$$|p^{(s)}(z)| \leq |P^{(s)}(z)|, \quad |z| > 1,$$

leading to

$$|p^{(s)}(z)| \leq |P^{(s)}(z)|, \quad |z| \geq 1, \quad 1 \leq s < n,$$

which, on being combined with (2.3), completes the proof of Lemma 1.

Lemma 2  If $p(z)$ is a polynomial of degree at most $n$ then for $0 \leq s < n$,

$$|p^{(s)}(z)| + |q^{(s)}(z)| \leq \left\{ \left| \frac{d^s}{dz^s}(1) \right| + \left| \frac{d^s}{dz^s}(z^n) \right| \right\} M(p, 1), \quad |z| \geq 1, \quad (2.4)$$

where

$$q(z) = z^n p(1/z). \quad (2.5)$$
Proof of Lemma 2  We consider the polynomial
\[ t(z) = p(z) - \lambda M(p, 1), \quad |\lambda| > 1, \]
of degree at most \(n\). Then the polynomial
\[ T(z) = z^n t(1/z) = q(z) - \overline{\lambda} M(p, 1) z^n, \quad \text{(by (2.5))}, \]
of degree \(n\), possesses the characteristic
\[ |t(z)| \leq |T(z)|, \quad |z| = 1 \]
and has all its zeros in \(|z| \leq 1\). Therefore on applying Lemma 1 to polynomials \(t(z)\) and \(T(z)\) we get for \(0 \leq s < n\) and \(|\lambda| > 1\)
\[ |p^{(s)}(z) - \lambda M(p, 1) \frac{d^s}{dz^s}(1)| \leq |q^{(s)}(z) - \overline{\lambda} M(p, 1) \frac{d^s}{dz^s}(z^n)|, \quad |z| \geq 1, \]
which, by choosing \(\arg\lambda\) suitably, can be rewritten as
\[ |p^{(s)}(z)| - |\lambda| M(p, 1) \frac{d^s}{dz^s}(1)| \leq ||\lambda| M(p, 1) \frac{d^s}{dz^s}(z^n)| - |q^{(s)}(z)||, \quad |z| \geq 1. \quad (2.6) \]
We can apply Lemma 1 to polynomials \(q(z)\) and
\[ z^n M(p, 1) \]
also, and obtain for \(0 \leq s < n\)
\[ |q^{(s)}(z)| \leq M(p, 1) \frac{d^s}{dz^s}(z^n)|, \quad |z| \geq 1, \]
which helps us to rewrite(2.6) as
\[ |p^{(s)}(z)| - |\lambda| M(p, 1) \frac{d^s}{dz^s}(1)| \leq |\lambda| M(p, 1) \frac{d^s}{dz^s}(z^n)| - |q^{(s)}(z)|, \quad |z| \geq 1, |\lambda| > 1, 0 \leq s < n. \]
Now on letting
\[ |\lambda| \to 1, \]
(2.4) follows. \(\square\)
Lemma 3 If $P(z)$ is a polynomial of degree $n$, having no zeros in $|z| < k, (k \geq 1)$, with $M(P,1) = 1$

then for $1 \leq R \leq k^2$

$$M(P,R) \leq ((R + k)/(1 + k))^n.$$  

This lemma is due to Aziz and Mohammad [2].

Lemma 4 Let $P(z)$ be a polynomial of degree $n$, having no zeros in $|z| < k, (k \geq 1)$. Then

$$|P(z)| \leq 1 \quad \text{for} \quad |z| \leq 1$$

implies

$$|P^{(s)}(z)| \leq n(n-1)\ldots(n-s+1)/(1+k^s) \quad \text{for} \quad |z| \leq 1 \quad \text{and} \quad s \geq 1.$$  

This lemma is due to Govil and Rahman [3].

From Lemma 4 we easily get

Lemma 5 If $P(z)$ is a polynomial of degree $n$, having no zeros in $|z| < k, (k \geq 1)$ then for $0 \leq s < n$

$$M(P^{(s)},1) \leq (1/(1+k^s))M(P,1)[\{d^s/dx^s(1+x^n)\}_{x=1}].$$

3. Proof of Theorem 1

We consider the polynomial

$$P(z) = p(kz). \quad (3.1)$$

Then the polynomial

$$Q(z) = z^n P(1/z)$$

possesses the characteristic

$$|P(z)| \leq |Q(z)|, \quad |z| = 1$$
and has all its zeros in |z| ≤ 1. Therefore on applying Lemma 1 to the polynomials P(z) and Q(z) we get for 0 ≤ s < n and t ≥ 1

\[ |P^{(s)}(te^{i\theta})| \leq |Q^{(s)}(te^{i\theta})|, \quad 0 \leq \theta \leq 2\pi. \]  

(3.2)

Further, by Lemma 2 we have for t ≥ 1 and 0 ≤ s < n

\[ |P^{(s)}(te^{i\theta})| + |Q^{(s)}(te^{i\theta})| \leq \left\{ \frac{ds}{dt}s(t^n + 1) \right\} M(P, 1), \quad 0 \leq \theta \leq 2\pi, \]

which, by (3.2), implies that

\[ |P^{(s)}(te^{i\theta})| \leq \frac{1}{2}\left\{ \frac{ds}{dt}s(1 + t^n) \right\} M(p, 1), \]

i.e.

\[ |p^{(s)}(kte^{i\theta})| \leq \frac{1}{2}(2k^n)\left\{ \frac{ds}{dt}(1 + t^n) \right\} M(p, k), \quad \text{(by (3.1))}, \]

\[ \leq \frac{1}{2}(2k^n)(2k/(1 + k))^n M(p, 1)\left\{ \frac{ds}{dt}(1 + t^n) \right\}, \quad \text{(by Lemma 3)}, \]

thereby leading to inequality (1.1).

Now by applying Lemma 5 to the polynomial p(Rz), (1 ≤ R ≤ k), having no zeros in |z| < k/R, we have for 0 ≤ s < n

\[ M(p^{(s)}, R) \leq \frac{1}{(R^s + k^s)}M(p, R)\left\{ \frac{ds}{dx}(1 + x^n) \right\}_{x=1}, \]

and the inequality (1.2) follows by using Lemma 3. This completes the proof of Theorem 1.

\[ \square \]

References

