A Note on Kaehlerian Manifolds

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Abstract

The main purpose of the present paper is to study nearly Kaehlerian manifolds. We give the condition for an almost Hermitian manifold to be nearly Kaehlerian.

Key Words: Hybrid tensor, Hermitian manifold, Kaehlerian manifold, Tachibana operator.

1. Introduction

Let $M$ be an almost Hermitian manifold with almost complex structure $\varphi$ and hybrid Riemannian metric tensor field $g$. Then

$$\varphi^2 = -I, \quad g(\varphi X, \varphi Y) = g(X, Y) \quad (1)$$

for any vector field $X$ and $Y$ on $M$. We denote by $\nabla$ the operator of covariant differentiation with respect to $g$ in $M$. If the almost complex structure $\varphi$ of $M$ satisfies

$$(\nabla_X \varphi)Y + (\nabla_Y \varphi)X = 0$$

for any vector field $X$ and $Y$ on $M$, then the manifold $M$ is called a nearly Kaehlerian manifold (Tachibana spaces). The condition above reduces to

$$(\nabla_X \varphi)X = 0.$$
Let $N$ be the Nijenhuis tensor field of $\varphi$ defined by

$$N(X, Y) = [\varphi X, \varphi Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y] - [X, Y]$$

any vector field $X$ and $Y$ on $M$. By a simple computation we have

$$N(X, Y) = -4\varphi(\nabla_X \varphi)Y.$$

**Proposition:** *If the Nijenhuis torsion $N$ of a nearly Kaehlerian manifold vanishes, then $M$ is a Kaehlerian manifold.*

We define a Tachibana operator [3] (see also [2, 4]) $\Phi_\varphi \xi$ associated with an almost complex structure $\varphi$ and an arbitrary $X \in \mathfrak{X}(M)$ and applied to a tensor $\xi \in \mathfrak{T}^0_2(M)$ as

$$\Phi_\varphi \xi(X, Z_1, Z_2) = (L_\varphi X \xi)(Z_1, Z_2) - L_X(\xi \circ \varphi)(Z_1, Z_2)$$

$$+ \xi(Z_1, \varphi(L_X Z_2)) - \xi(\varphi Z_1, L_X Z_2),$$

where $L_X$ denotes the operator of Lie derivation with respect to $X$ and $(\xi \circ \varphi)(Z_1, Z_2) = \xi(\varphi Z_1, Z_2)$. Expression (2) defines a tensor field $\Phi_\varphi \xi \in \mathfrak{T}^0_2(M)$ if and only if $\xi$ as a pure tensor [4]. When

$$\Phi_\varphi \xi(X, Z_1, Z_2) = (L_\varphi X \xi)(Z_1, Z_2) - L_X(\xi \circ \varphi)(Z_1, Z_2) = 0$$

for a pure tensor $\xi$ and for any $X, Z_1, Z_2 \in \mathfrak{X}(M)$, $M$ being a manifold with almost complex structure $\varphi$, $\xi$ is said to be almost analytic [3].

2. **Operator $\Phi$ Applied to a Hybrid Tensor**

Let $g$ be a hybrid Riemannian metric tensor. The following formulas are known (see [1]):

$$(L_X g)(Y_1, Y_2) = X(g(Y_1, Y_2)) - g([X, Y_1], Y_2) - g(Y_1, [X, Y_2]).$$
\[ L_X Y = [X, Y] = \nabla_X Y - \nabla_Y X - T(X, Y) = \nabla_X Y - \nabla_Y X, \]  
(5)

\[ (\nabla K)(X_1, X_2, \ldots, X_s, X) = (\nabla_X K)(X_1, X_2, \ldots, X_s) = \nabla_X (K(X_1, X_2, \ldots, X_s)) \]  
(6)

\[ - \sum_{i=1}^{s} K(X, \ldots, \nabla_X X_i, \ldots, X_s), \quad K \in \mathfrak{X}_1(M), \]

where \( \nabla \) denotes the operator of the Riemannian covariant derivation. By virtue of (1), (4) and (5), from (2) we get

\[ (\Phi_\varphi g)(X, Z_1, Z_2) = \varphi(X)(g(Z_1, Z_2)) - g(\nabla_\varphi X Z_1 - \nabla Z_1 \varphi(X), Z_2) \]  
(7)

\[ -g(\nabla_\varphi Z_1, Z_2) - g(\varphi Z_1 - \nabla_\varphi X Z_2, \varphi(X), Z_2) \]  
(8)

and making use of (6), we have

\[ g(\nabla Z_1 \varphi(X), Z_2) - g(\varphi(\nabla Z_1 X), Z_2) + g(Z_1, \nabla Z_2 \varphi(X)) - g(Z_1, \varphi(\nabla Z_2 X)) \]  
(8)

Substitution (8) into (7) may be written as
(Φφg)(X, Z_1, Z_2) = ϕ(X)(g(Z_1, Z_2)) − X(g(ϕZ_1, Z_2)) + g((∇ϕ)(X, Z_1), Z_2) + g(Z_1, (ϕX)(X, Z_2))
− g(Z_1, (∇ϕX)(X, Z_2)) = 0 \tag{9}

On the other hand, with respect to the Riemannian connection, we have

ϕ(X)(g(Z_1, Z_2)) − g(ϕXZ_1, Z_2) = (ϕXg)(Z_1, Z_2) = 0 \tag{10}

and

X(g(ϕZ_1, Z_2)) − g(ϕZ_1, XZ_2) = (ϕXg)(ϕZ_1, Z_2) = 0 \tag{11}

⇒ −X(g(ϕZ_1, Z_2)) + g(ϕZ_1, XZ_2) = −g(ϕXϕ(Z_1), Z_2).

By virtue of (6), (9), (10) and (11) reduces to

(Φφg)(X, Z_1, Z_2) = −g(ϕXϕ(Z_1), Z_2) + g(ϕ(XZ_1), Z_2) + g((∇ϕXϕ)(X), Z_2) + g(Z_1, (ϕXg)(X, Z_2))
− g(Z_1, (∇ϕXg)(X, Z_2)) = 0 \tag{12}

The analogue to (12) is

(Φφg)(Z_2, Z_1, X) = −g(∇ϕZ_2ϕ(Z_1), X) + g((∇ϕZ_1ϕ)(Z_2), X) + g(Z_1, (∇ϕXg)(Z_2)) + g(Z_1, ϕ(∇ϕXg)(Z_2)) + g(Z_1, ϕ(∇ϕXg)(Z_2)). \tag{13}
Lemma: If a Riemannian metric tensor $g$ is hybrid, then we have

$$g((\nabla_Y \varphi)(Z), X) = -g(Z, (\nabla_Y \varphi)(X)),$$

(14)

where $\nabla$ denotes the operator of the Riemannian covariant derivative with respect to $g$.

Proof. By virtue of (1) and

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

we have

$$Yg(\varphi Z, X) = -Yg(Z, \varphi X),$$

$$g(\nabla_Y \varphi(Z), X) + g(\varphi Z, \nabla_Y X) = -g(\nabla_Y Z, \varphi X) - g(Z, \nabla_Y \varphi(X))$$

or

$$-g(\nabla_Y Z, \varphi X) - g(\nabla_Y \varphi(Z), X) = g(\varphi Z, \nabla_Y X) + g(Z, \nabla_Y \varphi(X))$$

$$g(\varphi(\nabla_Y Z) - \nabla_Y \varphi(Z), X) = -g(Z, \varphi(\nabla_Y X) + \nabla_Y \varphi(X))$$

and therefore, by (6), the proof is completed. \(\square\)

We have

$$(\Phi_\varphi g)(X, Z_1, Z_2) - (\Phi_\varphi g)(Z_2, Z_1, X) = g(Z_1, (\nabla_X \varphi)(Z_2)) - g(Z_1, (\nabla_X \varphi)(Z_2))$$

$$-g(Z_1, (\nabla_X \varphi)(Z_2)) - g(X, (\nabla_X \varphi)(Z_2))$$

$$+g(Z_1, (\nabla_X \varphi)(X)) - g(Z_1, (\nabla_X \varphi)(X))$$

$$+g(Z_1, \varphi(\nabla_X Z_2)) + g(Z_1, \varphi(\nabla_X Z_2))$$

$$-g(Z_1, \varphi(\nabla_X Z_2)) - g(Z_1, \varphi(\nabla_X Z_2)).$$
\[-2g(X, (\nabla Z_1, \varphi)(Z_2)) + 2g(Z_1, \varphi(\nabla_X Z_2))
\quad - 2g(Z_1, \varphi(\nabla_Z_2 X))
\quad = -2g(X, (\nabla Z_1, \varphi)(Z_2)) + 2g(Z_1, \varphi(L_X Z_2))\]

or

\[(\psi_{\varphi} g)(X, Z_1, Z_2) - (\psi_{\varphi} g)(Z_2, Z_1, X) = -2g(X, (\nabla Z_1, \varphi)(Z_2)) \quad (15)\]

where

\[(\psi_{\varphi} g)(X, Z_1, Z_2) = (\Phi_{\varphi} g)(X, Z_1, Z_2) - g(Z_1, \varphi(L_X Z_2))
\quad = (L_{\varphi X} g)(Z_1, Z_2) - (L_X (g \circ \varphi))(Z_1, Z_2) - g(\varphi Z_1, L_X Z_2),\]

\[(\psi_{\varphi} g)(Z_2, Z_1, X) = (\Phi_{\varphi} g)(Z_2, Z_1, X) - g(Z_1, \varphi(L_Z_2 X))
\quad = (L_{\varphi Z_2} g)(Z_1, X) - (L_Z_2 (g \circ \varphi))(Z_1, X) - g(\varphi Z_1, L_Z_2 X).\]

From (15) we have

\[(\psi_{\varphi} g)(X, Z_1, Z_2) - (\psi_{\varphi} g)(Z_2, Z_1, X) + (\psi_{\varphi} g)(X, Z_2, Z_1) - (\psi_{\varphi} g)(Z_1, Z_2, X)
\quad = -2g(X, (\nabla Z_1, \varphi)(Z_2)) + (\nabla_{Z_2} \varphi)(Z_1)).\]

Thus we have the following theorem.

**Theorem** A necessary and sufficient condition that an almost Hermitian manifold to be nearly Kahlerian is that

\[\text{Alt}_{X, Z_2} (\psi_{\varphi} g)(X, Z_1, Z_2) + \text{Alt}_{X, Z_1} (\psi_{\varphi} g)(X, Z_2, Z_1) = 0.\]
References


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