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Some Random Fixed Point Theorems for Non-Self Nonexpansive Random Operators*

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Abstract

Let (Ω, Σ) be a measurable space, with $\sum$ a sigma-algebra of subsets of Ω, and let E be a nonempty bounded closed convex and separable subset of a Banach space X, whose characteristic of noncompact convexity is less than 1. We prove that a multivalued nonexpansive, non-self operator $T : \Omega \times E \to KC(X)$ satisfying an inwardness condition and itself being a 1-χ-contractive nonexpansive mapping has a random fixed point. We also prove that a multivalued nonexpansive, non-self operator $T : \Omega \times E \to KC(X)$ with a uniformly convex X satisfying an inwardness condition has a random fixed point.

Key Words: Random fixed point, non-self mappings, Nonexpansive random operator, inwardness condition.

1. Introduction

Random fixed point theory has received much attention in recent years; see, Itoh [8] and Shahzad and Latif [15]. Research in this direction was initiated by the Prague School of Probabilists as the originator of random operator theory; see O. Hans [6, 7]. Since then, a lot of efforts have been devoted to random fixed point theory and applications; see Ramírez [11, 12], Tan and Yuan [16], Xu [17, 19, 20], Yuan and Yu [21].

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In 2004, Domínguez Benavides and Ramírez [4] proved a fixed point theorem for a multivalued nonexpansive, non-self mapping and 1-χ-contractive mapping in the framework of Banach spaces whose characteristic of noncompact convexity associated to the separation measure of noncompactness $\varepsilon_\alpha(X)$ is less than 1.

The purpose of the present paper is to prove some random fixed point theorems for nonexpansive non-self random operators. First, we will prove the existence of fixed point for multivalued non-self, nonexpansive random operators in the framework of a Banach spaces with characteristic of noncompact convexity associated to the Kuratowski measure of noncompactness $\varepsilon_\alpha(X)$ being less than 1 and satisfying an inwardness condition, and also being 1-χ-contractive mapping. Moreover, if X is a separable subset of a uniformly convex Banach space, a similar result is proved. Finally we also prove that a multivalued nonexpansive non-self random operator $T : \Omega \times E \to KC(X)$ satisfying an inwardness condition has a random fixed point.

2. Preliminaries and notations

We begin with establishing some preliminaries. By a measurable space we mean a pair (Ω, Σ), where Ω is a nonempty set and Σ is a sigma-algebra of subsets of Ω. Let X be a Banach space and E a nonempty subset of X. We shall denote by 2^E the family of nonempty closed subsets of E, by $CB(E)$ the family of nonempty closed bounded subsets of E, by $K(E)$ the family of nonempty compact subsets of E, and by $KC(E)$ the family of nonempty compact convex subsets of E. Let $H(\cdot,\cdot)$ be the Hausdorff distance on $CB(X)$, i.e.,

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}$$

for $A, B \in CB(X)$, where $d(x,E) = \inf\{d(x,y) | y \in E\}$ is the distance from x to $E \subset X$.

Let E be a nonempty closed subset of a Banach space X. Recall now that a multivalued mapping $T : E \to 2^X$ is said to be upper semicontinuous on E if $\{x \in E : Tx \subset V\}$ is open in E whenever $V \subset X$ is open; T is said to be lower semicontinuous if $T^1(V) := \{x \in E : Tx \cap V \neq \emptyset\}$is open in E whenever $V \subset X$ is open; and T is said to be continuous if it is both upper and lower semicontinuous (cf. [1] and [2] for details). There is another but different kind of continuity for a multivalued operator: $T : E \to CB(X)$ is said to be continuous on E (with respect to the Hausdorff metric H) if $H(Tx_n, Tx) \to 0$ whenever $x_n \to x$. It is not hard to see (see Deimling [2]) that both definitions of continuity are equivalent if Tx is compact for every $x \in E$.

A multivalued operator $T : \Omega \to 2^X$ is called (Σ)-measurable if, for any open subset B of X,

$$T^{-1}(B) := \{\omega \in \Omega : T(\omega) \cap B \neq \emptyset\}$$

belongs to Σ. A mapping $x : \Omega \to X$ is said to be a *measurable selector* of a measurable multivalued operator $T : \Omega \to 2^X$ if $x(\cdot)$ is measurable and $x(\omega) \in T(\omega)$ for all $\omega \in \Omega$. An operator $T : \Omega \times E \to 2^X$ is called a random operator if, for each fixed $x \in E$, the operator $T(\cdot, x) : \Omega \to 2^X$ is measurable. We will denote by $F(\omega)$ the fixed point set of $T(\omega, \cdot)$, i.e.,

$$F(\omega) := \{x \in E : x \in T(\omega, x)\}.$$

Note that, if we do not assume the existence of a fixed point for the deterministic mapping $T(\omega, \cdot) : E \to 2^X, F(\omega)$ may be empty. A measurable operator $x : \Omega \to E$ is said to be a *random fixed point of a operator* $T : \Omega \times E \to 2^X$ if $x(\omega) \in T(\omega, x(\omega))$ for all $\omega \in \Omega$. Recall that $T : \Omega \times E \to 2^X$ is continuous if, for each fixed $\omega \in \Omega$, the operator $T : (\omega, \cdot) \to 2^X$ is continuous.

If E is a closed convex subset of a Banach space X, then a multivalued mapping $T : E \to CB(X)$ is said to be a *contraction* if there exists a constant $k \in [0, 1)$ such that

$$H(Tx, Ty) \leq k\|x - y\|, \quad x, y \in E,$$

and T is said to be *nonexpansive* if

$$H(Tx, Ty) \leq \|x - y\|, \quad x, y \in E,$$

A random operator $T : \Omega \times E \to 2^X$ is said to be *nonexpansive* if, for each fixed $\omega \in \Omega$ the map $T : (\omega, \cdot) \to E$ is nonexpansive.

For later reference, we list the following results related to the concept of measurability.

Lemma 2.1 (cf. Wagner [17]) *Let (X, d) be a complete separable metric spaces and $F : \Omega \to CL(X)$ a measurable map. Then F has a measurable selector.*

Lemma 2.2 (cf. Itoh 1977, [8]) *Suppose $\{T_n\}$ is a sequence of measurable multivalued operator from Ω to $CB(X)$ and $T : \Omega \to CB(X)$ is an operator. If, for each $\omega \in \Omega$, $H(T_n(\omega), T(\omega)) \to 0$, then T is measurable.*

Lemma 2.3 (cf. Tan and Yuan [16]) *Let X be a separable metric spaces and Y a metric spaces. If $f : \Omega \times X \to Y$ is measurable in $\omega \in \Omega$ and continuous in $x \in X$, and if $x : \Omega \to X$ is measurable, then $f(\cdot, x(\cdot)) : \Omega \to Y$ is measurable.*

As an easy application of Proposition 3 of Itoh[8] we have the following result.

Lemma 2.4 *Let E be s closed separable subset of a Banach space $X, T : \Omega \times E \to E$ a random continuous operator and $F : \Omega \to 2^E$ a measurable closed-valued operator. Then for any $s > 0$, the operator $G : \Omega \to 2^E$ given by*

$$G(\omega) = \{x \in F(\omega) : \|x - T(\omega, x)\| < s\}, \quad \omega \in \Omega$$

is measureble and so is the operator $cl\{G(\omega)\}$ (the closure of $G(\omega)$).

Lemma 2.5 (cf. Domínguez Benavidel and Lopez Acedo [5]) *Suppose E is a weakly closed nonempty separable subset of a Banach space $X; F : \Omega \to 2^X$ is a measurable mapping with weakly compact values, and $f : \Omega \times E \to \mathbb{R}$ is a measurable, continuous and weakly lower semicontinuous function. Then the marginal function $r : \Omega \to \mathbb{R}$ defined by*

$$r(\omega) := \inf_{x \in F(x)} f(\omega, x)$$

and the marginal map $R : \Omega \to X$ defined by

$$R(\omega) := \{x \in F(x) : f(\omega, x) = r(\omega)\}$$

are measurable.

Recall that the Kuratowski and Hausdorff measures of noncompactness of a nonempty bounded subset B of X are defined, respectively, as the numbers

$$\alpha(B) = \inf\{r > 0 : B \text{ can be covered by finitely many sets of diameter} \leq r\},$$

$$\chi(B) = \inf\{r > 0 : B \text{ can be covered by finitely many balls of radius} \leq r\}.$$

The separation measure of noncompacness of a nonempty bounded subset B of X is defined by

$$\beta(B) = \sup\{\varepsilon : \text{there exists a sequence } \{x_n\} \text{ in } B \text{ such that } sep(\{x_n\}) \geq \varepsilon\}.$$

Let X be a Banach space and $\phi = \alpha, \beta$ or χ. The modulus of noncompact convexity associated to ϕ is defined as

$$\Delta_{X,\phi}(\varepsilon) = inf\{1 - d(0, A) : A \subset B_X \text{ is convex}, \phi(A) \geq \varepsilon\},$$

where B_X is the unit ball of X.

The characteristic of noncompact convexity of X associated with the measure of noncompactness ϕ is defined by

$$\varepsilon_\phi(X) = \sup\{\varepsilon \geq 0 : \Delta_{X,\phi}(\varepsilon) = 0\}.$$

The following relationships among the different moduli are easy to obtain:

$$\Delta_{X,\alpha}(\varepsilon) \leq \Delta_{X,\beta}(\varepsilon) \leq \Delta_{X,\chi}(\varepsilon), \tag{2.1}$$

and consequently

$$\varepsilon_\alpha(X) \geq \varepsilon_\beta(X) \geq \varepsilon_\chi(X). \tag{2.2}$$

When X is a reflexive Banach space we have some alternative expressions for the moduli of noncompact convexity associated β and χ:

$$\Delta_{X,\beta}(\varepsilon) = \inf\{1 - \|x\| : \{x_n\} \subset B_X, x = w - \lim x_n, sep(\{x_n\}) \geq \varepsilon\},$$

$$\Delta_{X,\chi}(\varepsilon) = \inf\{1 - \|x\| : \{x_n\} \subset B_X, x = w - \lim x_n, \chi(\{x_n\}) \geq \varepsilon\}.$$

In order to study the fixed point theory for non-self mappings, we must introduce some terminology for boundary conditions. The inward set of E at $x \in E$ is defined by

$I_E(x) := \{x + \lambda(y - x) : \lambda \geq 0, y \in E\}$.

Clearly $E \subset I_E(x)$, and it is not hard to show that whenever $I_E(x)$ is a convex set, so is E. A multivalued mapping $T : E \to 2^X\{\emptyset\}$ is said to be inward on E if

$Tx \subset I_E(x)\ \ \forall x \in E$.

Let $\bar{I}_E(x) := x + \{\lambda(z - x) : z \in E, \lambda \geq 1\}$. Note that for a convex E, we have $\bar{I}_E(x) = \overline{I_E(x)}$, and T is said to be weakly inward on E if

$Tx \subset \bar{I}_E(x)\ \ \forall x \in E$.

Let E be a nonempty bounded closed subset of Banach space X and $\{x_n\}$ a bounded sequence in X: we use $r(E, \{x_n\})$ and $A(E, \{x_n\})$ to denote the asymptotic radius and the asymptotic center of $\{x_n\}$ in E, respectively, i.e.

$$\begin{aligned} r(E, \{x_n\}) &= \inf\left\{\limsup_n \|x_n - x\| : x \in E\right\}, \\ A(E, \{x_n\}) &= \left\{x \in E : \limsup_n \|x_n - x\| = r(E, \{x_n\})\right\}. \end{aligned}$$

If D is a bounded subset of X, the *Chebyshev radius* of D relative to E is defined by

$$r_E(D) := \inf\{\sup\{\|x-y\| : y \in D\} : x \in E\}.$$

Let S be a set $G \subset S$ and D be a directed set, we shall say that a net x_α in S eventually in G if there exist $\alpha_0 \in D$ such that $x_\alpha \in G$ for all $\alpha \geq \alpha_0$.

Definition 2.6 *A net $\{x_\alpha\}$ in a set S is called an ultranet if for each subset $E \subset S$, either $\{x_\alpha\}$ is eventually in G or $\{x_\alpha\}$ is eventually in $S \setminus G$.*

The following facts concerning ultranets can be found in [9]:

(a) Every net in a set has an ultranet.

(b) If $f : S_1 \to S_2$ is a map and if $\{x_\alpha\}$ is an ultranet in S_1, then $\{f(x_\alpha)\}$ is a ultranet in S_2.

(c) If S is compact and $\{x_\alpha\}$ is a ultranet in S, then $\lim_\alpha x_\alpha$ exists.

Obviously, the convexity of E implies that $A(E, \{x_\alpha\})$ is convex. Notice that $A(E, \{x_\alpha\})$ is a nonempty weakly compact set if E is weakly compact, or E is a closed convex subset of a reflexive Banach spaces X.

Let E be a nonempty bounded closed subset of Banach spaces X. Then $\{x_n\} \subset X$ is called *regular* with respect to E if $r(E, \{x_n\}) = r(E, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$.

Lemma 2.7 *(Geobel, Lim) Let $\{x_n\}$ and E be as above. Then, there always exists a subsequence of $\{x_n\}$ which is regular with respect to E.*

Moreover, we also need the following Lemma.

Theorem 2.8 (cf. T. D. Benavides and P. L. Ramírez; Theorem 4.3; [4].) *Let E be a closed convex subset of a reflexive Banach space X, and let $\{x_\beta : \beta \in D\}$ be a bounded ultranet. Then*

$$r_E(A(E, x_\beta)) \leq (1 - \Delta_{X,\alpha}(1^-))r(E, \{x_\beta\}). \tag{2.3}$$

The following results are now basic in fixed point theorems for multivalued mappings.

Lemma 2.9 (cf. Deimling 1992, [2]). *Let X be a Banach space and $\emptyset \neq D \subset X$ be a closed bounded convex. Let $F : D \to 2^X$ be upper semicontinuous $\gamma-$condensing with closed convex values, where $\gamma(\cdot) = \alpha(\cdot) or \chi(\cdot)$. If $Fx \cap \overline{I_D(x)} \neq \emptyset$ for all $x \in E$, then F has a fixed point. (Here, $I_D(x)$ is called the inward set at x defined by $I_D := \{x + \lambda(y - x) : \lambda \geq 0, y \in D\}$)*

Proposition 2.10 (cf. Kirk-Massa Theorem [10]) *Let E be a nonempty weakly compact separable subset of a Banach space X, $T : E \to KC(E)$ a nonexpansive mapping, and $\{x_n\}$ a sequence in E such that* $\lim_n d(x_n - Tx_n) = 0$. *Then, there exists a subsequence $\{z_n\}$ of $\{x_n\}$ such that*

$$Tx \cap A \neq \emptyset, \forall x \in A := A(E, \{z_n\})$$

3. The Main Results

The following states the main result of this paper, and is the random version of theorem 3.4 of Domínguez Benavides and Lorenzo Ramírez ([4]).

Theorem 3.1 *Let E be a nonempty closed bounded, convex and separable subset of a Banach space X such that $\epsilon_\alpha(X) < 1$, and $T : \Omega \times E \to KC(X)$ be a nonexpansive random operator and 1-χ-contractive mapping, such that for each $\omega \in \Omega$, $T(\omega, E)$ is a bounded set, which satisfies the inwardness condition. Then T has a random fixed point.*

Proof. For each $\omega \in \Omega$, and for every $n \geq 1$, we set

$$F(\omega) = \{x \in E : x \in T(\omega, x)\},$$

and

$$F_n(\omega) = \{x \in E : d(x, T(\omega, x)) \leq \frac{1}{n} diamC\}.$$

It follows from Theorem 3.4 of Benavides-Ramírez's [4] that $F(\omega)$ is nonempty. Clearly $F(\omega) \subseteq F_n(\omega)$, and $F_n(\omega)$ is closed and convex. Furthermore, by Lemma 2.4, each F_n is measurable. Then, by Lemma 2.1, each F_n admits a measurable selector $x_n(\omega)$ and $d(x_n(\omega), T(\omega, x_n(\omega))) \leq \frac{1}{n} diamE \to 0$ as $n \to \infty$.

Let $\{n_\alpha\}$ be an ultranet of the positive integer $\{n\}$. Defined a function $f : \Omega \times E \to \mathbb{R}^+ := [0, \infty)$ by

$$f(\omega, x) = \limsup_\alpha \|x_{n_\alpha}(\omega) - x\|, \;\; x \in E.$$

Since $\{x_{n_\alpha}(\omega)\}$ is countable, it is easily see that $f(\omega,\cdot)$ is measurable and $f(\omega,\cdot)$ is continuous and convex, therefore it is a weakly lower semicontinuous function. Hence by Lemma 2.5, the marginal functions

$$r(\omega) := \inf_{x\in E} f(\omega,x)$$

and

$$A(\omega) := \{x \in E : f(\omega,x) = r(\omega)\}$$

are measurable, and $A(\omega)$ is a weakly compact convex subset of E. Note that $A(\omega) = A(E,\{x_{n_\alpha}(\omega)\})$, and $r(\omega) = r(E,\{x_{n_\alpha}(\omega)\})$. Moreover, we can apply Lemma 2.8 to obtain

$$r_E(A(\omega)) \leq \lambda r(E,\{x_{n_\alpha}(\omega)\}), \tag{3.1}$$

where $\lambda = 1 - \Delta_{X,\alpha}(1^-) < 1$, since $\varepsilon_\alpha(X) < 1$. By Lemma 2.1 we can take $x_0(\omega)$ as a measurable selector of $A(\omega)$. For each $\omega \in \Omega$ and $n \geq 1$, we define the contraction $T_n(\omega,\cdot) : A(\omega) \to KC(X)$ defined by

$$T_n(\omega,x) = \frac{1}{n}x_0 + (\frac{n-1}{n})T(\omega,x),$$

for each $(\omega,x) \in \Omega\times E$. We shall prove that the inwardness of T on E implies a weaker inwardness of T on A, i.e.,

$$T(\omega,x) \cap I_{A(\omega)}(x) \neq \emptyset, \forall x \in A(\omega). \tag{3.2}$$

Indeed, the compactness of $T(\omega,x_{n_\alpha}(\omega))$ implies that for each n_α fixed $\omega \in \Omega$, we can take $y_{n_\alpha}(\omega) \in T(\omega,x_{n_\alpha}(\omega))$ such that

$$\|x_{n_\alpha}(\omega) - y_{n_\alpha}(\omega)\| = d(x_{n_\alpha}(\omega),T(\omega,x_{n_\alpha}(\omega))).$$

Since $T(\omega,x(\omega))$ is compact, for each $x(\omega) \in A(\omega)$, we can find $z_{n_\alpha}(\omega) \in T(\omega,x(\omega))$ such that

$$\begin{aligned}\|y_{n_\alpha}(\omega) - z_{n_\alpha}(\omega)\| &= d(y_{n_\alpha}(\omega),T(\omega,x(\omega)))\\ &\leq H(T(\omega,x_{n_\alpha}(\omega)),T(\omega,x(\omega)))\\ &\leq \|x_{n_\alpha}(\omega) - x(\omega)\|.\end{aligned}$$

Let $z(\omega) = \lim_\alpha z_{n_\alpha}(\omega) \in T(\omega, x(\omega))$. It should remain to prove $z(\omega) \in I_{A(\omega)}(x)$. It follows that

$$\begin{aligned} \limsup_\alpha \|x_{n_\alpha}(\omega) - z(\omega)\| &= \limsup_\alpha \|y_{n_\alpha}(\omega) - z_{n_\alpha}(\omega)\| \\ &\leq \limsup_\alpha \|x_{n_\alpha}(\omega) - x(\omega)\| \\ &= r(\omega). \end{aligned}$$

Since $z(\omega) \in T(\omega, x(\omega)) \subseteq I_E(x)$, for each fixed $\omega \in \Omega$ there exist $\lambda \geq 0$ and $v(\omega) \in E$ such that

$$z(\omega) = x(\omega) + \lambda(v(\omega) - x(\omega)).$$

If $\lambda \leq 1$, then by the convexity of E, $z(\omega) \in E$ and hence $z(\omega) \in A(\omega) \subseteq I_{A(\omega)}(x)$ and we are done. So assume that $\lambda > 1$. Then we can write

$$v(\omega) = \mu z(\omega) + (1-\mu)x(\omega) \text{ with } \mu = \frac{1}{\lambda} \in (0,1).$$

It follows that

$$\begin{aligned} f(\omega, v(\omega)) &= \limsup_\alpha \|x_{n_\alpha}(\omega) - v(\omega)\| \\ &\leq \mu \limsup_\alpha \|x_{n_\alpha}(\omega) - z(\omega)\| + (1-\mu)\limsup_\alpha \|x_{n_\alpha}(\omega) - x(\omega)\| \\ &\leq r(\omega). \end{aligned}$$

Therefore $v(\omega) \in A(\omega)$ and thus $z(\omega) = x(\omega) + \lambda(v(\omega) - x(\omega))$ belong to $I_{A(\omega)}(x)$. That is $T(\omega, x(\omega)) \cap I_{A(\omega)}(x) \neq \emptyset \ \forall x(\omega) \in A(\omega)$.

Now, we have a mapping $T(\omega, \cdot) : A(\omega) \to KC(X)$ which satisfies the boundary condition (3.2). Consequently, since $T_n(\omega, \cdot)$ is 1-χ-contractive mapping, it is easily seen that $T_n(\omega, \cdot)$ is χ-condensing (see [3]). By lemma 2.9, $T_n(\omega, \cdot)$ has a fixed point $x_n(\omega) \in A(\omega), i.e. F(\omega) \cap A(\omega) \neq \emptyset$. Also, we have

$$dist(x_n(\omega), T(\omega, x_n(\omega))) \leq \frac{1}{n} diamE \to 0 \text{ as } n \to \infty.$$

Thus, $F_n^1(\omega) := \{x \in A(\omega) : d(x, T(\omega, x)) \leq \frac{1}{n} diamE\} \neq \emptyset$ for each $n \geq 1$ is closed and measurable. Hence, by Lemma 2.1, we can choose x_n^1 a measurable selector of F_n^1, and from definition of it we have $x_n^1(\omega) \in A(\omega)$ and $d(x_n^1(\omega), T(\omega, x_n^1(\omega))) \to 0$ as $n \to \infty$. Consider the function $f_2 : \Omega \times E \to \mathbb{R}^+$ defined by

$$f_2(\omega, x) = \limsup_\alpha \|x_{n_\alpha}^1(\omega) - x\|, \ \forall \omega \in \Omega.$$

As above, f_2 is a measurable function and weakly lower semicontunuous function. Then the marginal function

$$r_2(\omega) := \inf_{x \in A(\omega)} f_2(\omega, x)$$

and

$$A^1(\omega) := \{x \in A(\omega) : f_2(\omega, x) = r_2(\omega)\}$$

are measurable. Since $A^1(\omega) = A(A(\omega), \{x^1_{n_\alpha}(\omega)\})$, it follows that $A^1(\omega)$ is a weakly compact and convex. Also $r_2(\omega) = r(A(\omega), \{x^1_n(\omega)\})$. We proceed as before to obtain that

$$T(\omega, x(\omega)) \cap I_{A^1}(x(\omega)) \neq \emptyset \ \ \forall x(\omega) \in A^1 = A(A(\omega), \{x^1_{n_\alpha}(\omega)\}),$$

and by (3.1) we obtain that

$$r_E(A^1) \leq \lambda r(A(\omega), \{x^1_{n_\alpha}(\omega)\}) \leq \lambda r_E(A(\omega)). \tag{3.3}$$

By induction, for each $m \geq 1$, we take a sequence $\{x^m_n(\omega)\}_n \subseteq A^{m-1}$ such that $\lim_n d(x^m_n(\omega), T(\omega, x^m_n(\omega))) = 0$ for each fixed $\omega \in \Omega$. By means of the ultranet $\{x^m_{n_\alpha}(\omega)\}_\alpha$ we construct the set $A^m := A(E, \{x^m_{n_\alpha}(\omega)\})$ such that

$$r_E(A^m) \leq \lambda^m r_E(A(\omega)). \tag{3.4}$$

Choose $x_m(\omega)$ as a measurable selector of A^m. We shall prove that $\{x_m(\omega)\}_m$ is a Cauchy sequence. For each $m \geq 1$, we have

$$\begin{aligned} \|x_{m-1}(\omega) - x_m(\omega)\| &\leq \|x_{m-1}(\omega) - x^m_n(\omega)\| + \|x^m_n(\omega) - x_m(\omega)\| \\ &\leq diamA_{m-1}(\omega) + \|x^m_n(\omega) - x_m(\omega)\|. \end{aligned}$$

Since $distA^m \leq 2r_E(A^m)$, taking upper limit as $n \to +\infty$, we have

$$\begin{aligned} \|x_{m-1}(\omega) - x_m(\omega)\| &\leq diamA^{m-1} + \limsup_n \|x^m_n(\omega) - x_m(\omega)\| \\ &= diamA^{m-1} + r(E, \{x^m_n(\omega)\}) \\ &\leq diamA^{m-1} + r_E(A^{m-1}\}) \\ &\leq 2r_E(A^{m-1}) + r_E(A^{m-1}) \\ &= 3\lambda^{m-1} r_E(A(\omega)). \end{aligned}$$

Since $\lambda < 1$, hence $\{x_m(\omega)\}_{m\geq 1}$ is a Cauchy sequence we conclude that there exists $x(\omega) \in E$ such that $x_m(\omega)$ converges to $x(\omega)$. Finally, we will show that $x(\omega)$ is a random fixed point of T. Indeed, for each $m \geq 1$, we have

$$\begin{aligned} d(x_m(\omega), T(\omega, x_m(\omega)) &\leq \|x_m(\omega) - x_n^m(\omega)\| + d(x_n^m(\omega), T(\omega, x_n^m(\omega))) \\ &+ H(T(\omega, x_n^m(\omega)), T(\omega, x_m(\omega))) \\ &\leq 2\|x_m(\omega) - x_n^m(\omega)\| + d(x_n^m(\omega), T(\omega, x_n^m(\omega))). \end{aligned}$$

Taking the upper limit as $n \to +\infty$,

$$\begin{aligned} d(x_m(\omega), T(\omega, x_m(\omega)) &\leq 2 \limsup_n \|x_m(\omega) - x_n^m(\omega)\| \\ &\leq 2\lambda^{m+1} r_E(A(\omega)). \end{aligned}$$

Now taking the limit $m \to +\infty$ on both sides we get $d(x_m(\omega), T(\omega, x_m(\omega))) = 0$, and the continuity of $T(\omega, \cdot)$ implies that $d(x(\omega), T(\omega, x(\omega))) = 0$, that is, $x(\omega) \in T(\omega, x(\omega))$. This completes the proof. □

Corollary 3.2 (Domínguez Benavides and Lorenzo Ramírez [4, Theorem 3.4]) *Let X be a Banach space such that $\varepsilon_\alpha(X) < 1$, and E be a nonempty closed bounded convex subset of X. If $T : E \to KC(X)$ is a nonexpansive and $1-\chi$-contractive nonexpansive mapping, such that $T(E)$ is a bounded set, and which satisfies $Tx \subset I_E(x)$ $\forall x \in E$, then T has a fixed point.*

Proof. Define a random operator $S : \Omega \times E \to KC(X)$ by $S(\omega, x) = T(x)$ for all $\omega \in \Omega$ and for all $x \in E$. Thus $S(\omega, \cdot)$ is a nonexpansive random operator and $1 - \chi$-contractive mapping such that $S(\omega, E)$ is bounded for all $\omega \in \Omega$. Hence, by Theorem 3.1, $S(C_i)$ has a random fixed point $x(\omega) \in S(\omega, x) = T(x)$ for all $\omega \in \Omega$. Thus is completes of the proof. □

Next we prove the random version of the following celebrated deterministic result due to Xu ([20, Theorem 3.4]). The proof below is inspired by same ideas in the proof of [20].

Theorem 3.3 *Let E be a nonempty closed, bounded and convex separable subset of a uniformly convex Banach space X and $T : \Omega \times E \to KC(X)$ be a multivalued nonexpansive random operator such that for each $\omega \in \Omega$, $T(\omega, E)$ is a bounded set, which satisfies the inwardness condition, i.e., for each $\omega \in \Omega$, $T(\omega, x) \subset \bar{I}_E(x)$, $\forall x \in E$. Then T has a random fixed point.*

Proof. Fix $x_0 \in E$ for each $n \geq 1$, define the mapping $T_n : E \to KC(X)$ by

$$T_n(\omega, x) = \frac{1}{n}x_0 + (1 - \frac{1}{n})T(\omega, x), \ \ \omega \in \Omega, x \in E.$$

Then T_n is a multivalued random contraction satisfying the same boundary condition as T does, i.e. we have, $T_n(\omega, x) \subset \bar{I}_E(x)$ for all $x \in E$. Hence, by [20, Theorem 1.4], $T_n(\omega, \cdot)$ has a random fixed point denoted $x_n(\omega)$. Also it is easily seen that we have $dist(x_n(\omega), T(\omega, x_n(\omega))) \leq \frac{1}{n} diamE \to 0$ as $n \to \infty$. Let $\{n_\alpha\}$ be a universal subnet of the positive integers $\{n\}$. Define a function $f : \Omega \times E \to \mathbb{R}^+$ by

$$f(\omega, x) = \limsup_n \|x_{n_\alpha}(\omega) - x\|, \ \ \forall \omega \in \Omega.$$

Since $\{x_{n_\alpha}\}$ is countable, it is easily seen that for each $x \in E, f(\cdot, x) : \Omega \to \mathbb{R}^+$ is measurable and each $\omega \in \Omega$, $f(\omega, \cdot) : E \to \mathbb{R}^+$ is continuous and convex (and hence weakly lower semicontinuous (w-l.s.c.)). Since the space X is uniformly convex and E is weakly compact and convex for each $\omega \in \Omega$, there exists exactly a point $x(\omega) \in E$ such that

$$f(\omega, x(\omega)) = \inf_{x \in E} f(\omega, x) =: r(\omega).$$

Note that $x(\omega)$ is an asymptotic center of the net $\{x_{n_\alpha}(\omega)\}$ with respect to E. Lim [12], and Kirk and Massa [10] actually proved that for each $\omega \in \Omega$, $x(\omega)$ is a fixed point of the map $T(\omega, \cdot)$. By using the same argument as in the proof of Xu ([20, p.1091]), we obtain $x(\omega)$ is measurable. Therefore $x(\omega)$ is a random fixed point of T. The proof of the theorem, is complete. □

Corollary 3.4 (Xu cf. [20]) *Assume X is a uniformly convex Banach space, E is a closed bounded convex subset of X, and $T : E \to KC(X)$ is a nonexpansive mapping satisfying the inwardness condition,i.e., $Tx \subset \bar{I}_E(x)$, $x \in E$. Then T has a fixed point.*

Corollary 3.5 (Xu cf. [18]) *Let (Ω, Σ) be a measurable spaces with Σ a sigma-algebra of subsets of Ω. Let E be a nonempty, bounded, closed, convex and separable subset of a uniformly convex Banach space X, and let $T : \Omega \times E \to KC(E)$ be a multivalued nonexpansive random operator. Then T has a random fixed point.*

Proof. It follows from Theorem 3.3, since every self multivalued mappings satisfies the inwardness condition. □

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References

[1] Ayerbe Toledano, J. M., Domínguez Benavides, T. and López Acedo G.: "Measures of noncompactness in metric fixed point theory": Advances and Applications Topics in metric fixed point theory, Birkhuser-Verlag, Basel, **99** (1997).

[2] Deimling, K.: "Nonlinear Functional Analysis", Springer-Verlag, Berlin (1974).

[3] Domínguez Benavides, T. and Lorenzo Ramírez, P.: "Fixed point theorem for Multivalued nonexpansive mapping without uniform convexity", Abstr. Appl. Anal. 6, 375-386 (2003).

[4] Domínguez Benavides, T. and Lorenzo Ramírez, P.: "Fixed point theorem for multivalued nonexpansive mapping satisfying inwardness conditions", J. Math. Anal. Appl. 291, 100-108 (2004).

[5] Domínguez Benavides, T., Lopez Acedo G. and Xu, H.-K.: "Random fixed point of set-valued operator", Proc. Amer. Math. Soc. 124, 838-838 (1996).

[6] Hans, O.: "Random fixed point theorems", Transac. of the 1st Prague Conference, pp. 105-125 (1957).

[7] Hans, O.: "Random operator equations", Proc. of the 4th Berkeley Symposium, vol.2, pp. 185-201 (1960).

[8] Itoh, S.: "Random fixed point theorem for a multivalued contraction mapping", Pacific J. Math. 68, 85-90 (1977).

[9] Kelly, J.L.: "General Topology", van Nostrand, Princeton, NJ, 1955.

[10] Kirk, W.A. and Massa, S.: "Remarks on asymptotic and Chebyshev centers", Houston J. Math. 16, 357-364 (1990).

[11] Kumam, P. and Plubtieng, S.: "The Characteristic of noncompact convexity and random fixed point theorem for set-valued operators", (Accepted) Czechoslovak Math. J.

[12] Lim, T.C.: "A fixed point theorem for multivalued nonexpansive mappings in a uniformly convex Banach space", Bull. Amer. Math. Soc. 80, 1123-1126 (1974).

[13] Lorenzo Ramírez, P.: "Some random fixed point theorems for nonlinear mappings", Nonlinear Anal. 50, 265-274 (2002).

[14] Lorenzo Ramírez, P.: "Random fixed point of uniformly Lipschitzian mappings", Nonlinear Anal. 57, 23-34 (2004).

[15] Shahzad, N. and Latif, S.: "Random fixed points for several classes of 1-ball-contractive and 1-set-contractive random maps", J. Math. Anal. Appl. 237, 83-92 (1999).

[16] Tan, K.-K. and Yuan, X.Z.: "Some Random fixed point theorem ", in: K.-K. Tan (Ed), Fixed Point Theory and Applications, Wold Sciedtific, Singapro, 334-345 1992.

[17] Wagner, D.-H.: "Survey of measurable selection theorems", SIAM J. Control Optim. 15, 859-903 (1977).

[18] Xu, H. K.: "A Random fixed point theorem for multivalued nonexpansive operators in Uniformly convex Banach spaces ",Proc. Amer. Math. Soc. 117, 4, 1089-1092 (1993).

[19] Xu, H. K.: "Random fixed point theorems for nonlinear Uniform Lipschitzian mappings ",Nonlinear Anal. 26, 7, 1301-1311 (1996).

[20] Xu, H. K.: "Multivalued nonexpansive mappings in Banach spaces. Nonlinear Anal. 43, 693-706 (2001).

[21] Yuan, X. and Yu, J.: "Random fixed point theorems for nonself mappings ",Nonlinear Anal. 26, 6, 1097-1102 (1996).

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