The Radius of Starlikeness \( p \)-Valently Analytic Functions in the Unit Disc

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Abstract

In the present paper we shall give the radius of starlikeness for the classes of \( p \)-valent analytic functions in the unit disc \( D = \{ z \mid |z| < 1 \} \).

Key Words: \( p \)-valent analytic functions, Radius of starlikeness, Radius of convexity.

1. Introduction

Let \( A_p \) the class of \( f(z) \) normalized by

\[
f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad p \in N = \{1, 2, 3, \ldots \} \tag{1.1}
\]

which are analytic and \( p \)-valent in \( D \). Further, let \( \Omega \) be the family of functions \( \omega(z) \) which are regular in \( D \) and satisfying the conditions \( \omega(0) = 0, \ |\omega(z)| < 1 \) for \( z \in D \).

Next, for arbitrary fixed numbers \( A, B, -1 \leq B < A \leq 1 \), denote by \( P(A, B) \) the family of functions

\[
p(z) = 1 + p_1 z + p_2 z^2 + \ldots \tag{1.2}
\]

which are regular in \( D \) such that \( p(z) \in P(A, B) \) if and only if

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\[ p(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)} \]  

(1.3)

for some function \( \omega(z) \in \Omega \) and every \( z \in D \). This class was introduced by W. Janowski [4].

Moreover, let \( S^*(A, B, b, p, q) \) denote the family of functions \( f(z) \in A_p \), and such that \( f(z) \) is in \( S^*(A, B, b, p, q) \) if and only if

\[ 1 + \frac{1}{b} \left( z: \frac{f^{(q+1)}(z)}{f^{(q)}(z)} - p + q \right) = p(z) \]  

(1.4)

for some functions \( p(z) \in P(A, B) \) and all \( z \in D \), and \( q \in N = N_0 = N \cup \{0\} \), whereas, as usual, \( f^{(0)}(z) \) denotes the derivative of \( f(z) \) with respect to \( z \) of order \( q \), and

\[ f^{(0)}(z) = f(z). \]

We note that by giving specific values to \( A, B, b, p \) and \( q \), we obtain the sub-classes of the class \( S^*(A, B, b, p, q) \) which were considered earlier by various authors [1], [2], [5], [6], [9], and [10].

We shall need the following definition and lemma.

**Definition 1.1** The radius for the property \( \Im \) in the class \( F \) is denoted by \( R_\Im(F) \) and is the largest \( R \) such that every function in the class \( F \) has the property \( \Im \) in each disc \( D_r \) for every \( r < R \).

### 2. New Results

In this section of this paper, we shall give the radius of starlikeness and the radius of convexity for the class \( S^*(A, B, b, p, q) \).

**Lemma 2.1** Let \( \omega(z) \) be regular in the unit disc with \( \omega(0) = 0 \). Then if \( |\omega(z)| \) attains its maximum value on the circle \( |z| = r \) at the point \( z_1 \), we can write \( z_1 \omega'(z_1) = k\omega(z_1) \), where \( k \) is real and \( k \geq 1 \).

This lemma was proved by I. S. Jack [3].
Lemma 2.2 The function

\[ w = \begin{cases} 
\frac{1 + A z}{1 + B z} , & B \neq 0 \\
1 + A z , & B = 0 
\end{cases} \]

maps \(|z| = r\) onto a disc centred at \(C(r)\), and having the radius \(\rho(r)\), viz.

\[ \begin{aligned}
C(r) &= \left( \frac{(1 - ABr^2)}{1 - B^2 r^2}, 0 \right) , \quad \rho(r) = \frac{(A-B)r}{1 - B^2 r^2} , \quad B \neq 0 \\
C(r) &= (0,0) , \quad \rho(r) = |A|.r , \quad B = 0.
\end{aligned} \]

Proof.

\[ \begin{aligned}
w = \frac{1 + A z}{1 + B z} \iff z = \frac{w-1}{A-Bw} \iff |z|^2 = r^2 = \frac{|w-1|^2}{|A-Bw|^2} \end{aligned} \quad , \quad B \neq 0 \]

\[ \begin{aligned}
w = 1 + A z \iff z = \frac{w-1}{A} \iff |z|^2 = r^2 = \frac{|w-1|^2}{|A|^2} \end{aligned} \quad , \quad B = 0. \tag{2.1}
\]

Lemma follows from (2.1). \(\square\)

Lemma 2.3 The function

\[ w = \begin{cases} 
\frac{(A-B)z}{1 + B z} , & B \neq 0 \\
A z , & B = 0 
\end{cases} \]

maps \(|z| = r\) onto the disc centred at \(C(r)\), and having radius \(\rho(r)\)

\[ \begin{aligned}
C(r) &= \left( - \frac{B(A-B)r^2}{1 - B^2 r^2}, 0 \right) , \quad \rho(r) = \frac{(A-B)r^2}{1 - B^2 r^2} , \quad B \neq 0 \\
C(r) &= (0,0) , \quad \rho(r) = |A|.r , \quad B = 0.
\end{aligned} \]
Proof.

\[
\begin{aligned}
    w &= \frac{(A-B)z}{1+Bz} \iff z = \frac{(A-B)w}{1+Bw} \iff |z|^2 = r^2 = \frac{|w|^2}{|A|^2} \\
    \Rightarrow u^2 + v^2 + \frac{2B(A-B)uw^2}{1-B^2v^2}u + \frac{(A-B)^2v^2}{1-B^2v^2} = 0
\end{aligned}
\]

(2.2)

Lemma follows from (2.2).

**Theorem 2.1** Let \( f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \ldots \) be an analytic function in the unit disc \( D \). If \( f(z) \) satisfies

\[
\frac{1}{b}(z f^{(q+1)}(z) - p + q) < \begin{cases} 
\frac{(A-B)z}{1+Bz} = F_1(z), & B \neq 0 \\
A.z = F_2(z), & B = 0,
\end{cases}
\]

(2.3)

then \( f(z) \in S^*(A, B, b, p, q) \), and this result is as sharp as the function \( \frac{1+Az}{1+Bz} \).

**Proof.** We define the function \( w(z) \) by

\[
\frac{f^{(q)}(z)}{z^{p-q}} = \begin{cases} 
(1 + Bw(z)) \frac{w(A-B)}{w}, & B \neq 0 \\
e^{A/w(z)}, & B = 0,
\end{cases}
\]

(2.4)

where \( (1 + Bw(z))^{\frac{w(A-B)}{w}} \) and \( e^{A/w(z)} \) have the values 1 at the origin.

Then \( w(z) \) is analytic in \( \mathbb{D} \) and \( w(0) = 0 \). If we take the logarithmic derivative from the equality (2.4) and after the brief calculations we get

\[
\frac{1}{b}(z f^{(q+1)}(z) - p + q) < \begin{cases} 
\frac{(A-B)zw'(z)}{1+Bw(z)}, & B \neq 0 \\
A.zw'(z), & B = 0.
\end{cases}
\]

(2.5)
Now it is easy to realize that subordination (2.3) is equivalent to $|w(z)| < 1$ for all $z \in D$. Indeed, assume the contrary: there exists a $z_1 \in D$ such that $|w(z_1)| = 1$. Then by the Lemma of I. S. Jack, $z_1 w'(z_1) = kw(z_1)$ and $k \geq 1$ for such $z_1 \in D$ (using Lemma 2.3), and we have

$$\frac{1}{b} \frac{f^{(q+1)}(z_1)}{f^{(q)}(z_1)} - p + q = \begin{cases} \frac{(A-B)kw(z_1)}{1+Bw(z_1)} = F_1(w(z_1)) \notin F_1(D) \ , \ B \neq 0 \\ A,k,w(z_1) = F_2(w(z_1)) \notin F_2(D) \ , \ B = 0. \end{cases} \quad (2.6)$$

But this is a contradiction of (2.3) of this theorem; so our assumption is wrong, i.e., $|w(z)| < 1$ for all $z \in D$. By using condition (2.5), we get

$$1 + \frac{1}{b} \frac{f^{(q+1)}(z)}{f^{(q)}(z)} - p + q = \begin{cases} \frac{1+Bw(z)}{1+Bw(z)} \ , \ B \neq 0 \\ 1 + Aw(z) \ , \ B = 0. \end{cases} \quad (2.7)$$

Then we obtain from equality (2.7)

$$1 + \frac{1}{b} \frac{f^{(q+1)}(z)}{f^{(q)}(z)} - p + q = \begin{cases} \frac{1+Bw(z)}{1+Bw(z)} \ , \ B \neq 0 \\ 1 + A,w(z) \ , \ B = 0. \end{cases} \quad (2.8)$$

From equality (2.8), we get $f(z) \in S^*(A,B,b,p,q)$. \qed

**Corollary 2.1** Let $f(z) \in S^*(A,B,b,p,q)$. Then $f(z)$ can be written in the form

$$f^{(q)}_*(z) = \begin{cases} z^{B-q}(1+Bw(z))^{\frac{B(B-D)}{B}} \ , \ B \neq 0 \\ z^{B-q}e^{Aw(z)} \ , \ B = 0. \end{cases}$$
Theorem 2.2 The radius of starlikeness and the radius of convexity of the class $S^*(A, B, b, p, q)$ is

$$R_{sc} = \frac{2(p - q)}{|b|(A - B) + \sqrt{|b|^2(A - B)^2 - 4(p - q)[(B^2 - AB)Re b + (q - p)B^2]}}$$  \hspace{1cm} (2.9)$$

This radius is sharp because the extremal function is

$$f_{\alpha}^{(q)}(z) = \begin{cases} 
z^{p-q}(1 + Bw(z))^\frac{\Delta(A-B)}{B}, & B \neq 0 \\
z^{p-q}e^{Abw(z)}, & B = 0
\end{cases}$$

Proof. By using Lemma 2.2, set of values $(z \cdot \frac{f^{(q+1)}(z)}{f^{(q)}(z)})$ is obtained which comprises the closed disc with centre $C(r)$ and the radius $\rho(r)$, where

$$C(r) = \frac{(p - q) - [(AB - B^2)b + (p - q)B^2].r^2}{1 - B^2.r^2},$$

$$\rho(r) = \frac{|b|(A - B)r}{1 - B^2.r^2}.$$ 

Therefore, by using the definition of the class $S^*(A, B, b, p, q)$, we have

$$\left| z \cdot \frac{f^{(q+1)}(z)}{f^{(q)}(z)} - C(r) \right| \leq \rho(r).$$

This gives

$$\text{Re}(z \cdot \frac{f^{(q+1)}(z)}{f^{(q)}(z)}) \geq \frac{(p - q) - |b|(A - B)r + [(B^2 - AB)Re b + (q - p)B^2].r^2}{1 - B^2.r^2}.$$  \hspace{1cm} (2.10)$$

Hence for $r < R_{sc}$ the first hand side of the preceeding inequality is positive, implying that

$$R_{sc} = \frac{2(p - q)}{|b|(A - B) + \sqrt{|b|^2(A - B)^2 - 4(p - q)[(B^2 - AB)Re b + (q - p)B^2]}}$$  \hspace{1cm} (2.11)$$

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Also note that inequality (2.9) becomes an equality for the function $f^{(q)}_*(z)$; it follows that

$$R_{sc} = \frac{2(p - q)}{|b| (A - B) + \sqrt{|b|^2 (A - B)^2 - 4(p - q) [(B^2 - AB)Reb + (q - p)B^2]}}$$

□

**Remark 2.3** (i) By taking $q = 0$, $p = 1$, $A = 1$, and $B = -1$ in (2.9), we obtain

$$R_s = \frac{1}{|b| + \sqrt{|b|^2 - 2Reb + 1}}$$

This is the radius of starlikeness for the class of starlike functions of complex order which was obtained by M. A. Nasr and M. K. Aouf [6].

(ii) By setting $q = 0$ in (2.9), then we obtain the radius of starlikeness for the class $S^*(A, B, b, p, 0)$

$$R_s = \frac{2p}{|b| (A - B) + \sqrt{|b|^2 (A - B)^2 - 4p [(B^2 - AB)Reb + -pB^2]}}$$

(iii) By letting $q = 1$ in (2.9), we also obtain the radius of convexity for the class $S^*(A, B, b, p, 1)$

$$R_c = \frac{2(p - 1)}{|b| (A - B) + \sqrt{|b|^2 (A - B)^2 - 4(p - 1) [(B^2 - AB)Reb + (1 - p)B^2]}}$$

**References**


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