Decompositions of Continuity

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Abstract

In 2004, Al-Hawary and Al-Omari introduced and explored the class of $\omega^o$—open sets which is strictly stronger than the class of $\omega$—open sets and weaker than that of open sets. In this paper, we introduce what we call $\omega^o$—continuity and $\omega^o_X$—continuity and we give several characterizations and two decompositions of $\omega^o$—continuity. Finally, new decompositions of continuity are provided.

Key Words: $\omega^o$—open, $\omega^o$—continuity, Continuity.

1. Introduction

Let $(X, \mathcal{T})$ be a topological space (or simply, a space). If $A \subseteq X$, then the closure of $A$ and the interior of $A$ will be denoted by $Cl_\mathcal{T}(A)$ and $Int_\mathcal{T}(A)$, respectively. If no ambiguity appears, we use $\overline{A}$ and $A^o$ instead. By $X$, $Y$ and $Z$ we mean topological spaces with no separation axioms assumed. $\mathcal{T}_{standard}, \mathcal{T}_{indiscrete}, \mathcal{T}_{leftray}$ and $\mathcal{T}_{counatble}$ stand for the standard, indiscrete, left ray and the cocountable topologies, respectively. A space $(X, \mathcal{T})$ is anti locally countable if all non-empty open subsets are uncountable.

In [3], the concept of $\omega$-closed subsets was explored where a subset $A$ of a space $(X, \mathcal{T})$ is $\omega$-closed if it contains all of its condensation points. In [4], several characterizations of $\omega$-continuity were provided where a map $f : X \rightarrow Y$ is $\omega$-continuous at $x \in X$ if for every open subset $V$ in $Y$ containing $f(x)$, there exists an $\omega$-open subset $U$ in $X$ containing $x$ such that $f(U) \subseteq V$. $f$ is $\omega$-continuous if it is $\omega$-continuous at every $x \in X$. Several properties of $\omega$-continuous mappings were also explored. Analogous to [4, 5, 8, 9], in

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Section 2 we introduce the relatively new notion of \( \omega^o \)-continuity, which is closely related to continuity and \( \omega \)-continuity. In fact, properly placed between them. Moreover, we show that \( \omega^o \)-continuity preserves Lindelof property and a space \( (X, \mathcal{T}) \) is Lindelof if and only if \( (X, \mathcal{T}_{\omega^o}) \) is Lindelof, where \( \mathcal{T}_{\omega^o} \) is the collection of all \( \omega^o \)-open subsets of \( X \). Sections 3 is devoted for studying four weaker notions of \( \omega^o \)-continuity by which we provide two decompositions of \( \omega^o \)-continuity. Finally, in Section 4 we give several decompositions of continuity which seem to be new.

Next, we recall several necessary definitions and results from [1].

**Definition 1** A subset \( A \) of a space \( (X, \mathcal{T}) \) is called \( \omega^o \)-open if for every \( x \in A \), there exists an open subset \( U_x \subseteq X \) containing \( x \) such that \( U_x \setminus A \) is countable. The complement of an \( \omega^o \)-open subset is called \( \omega^o \)-closed.

Clearly every open set is \( \omega^o \)-open and every \( \omega^o \)-open is \( \omega \)-open.

**Theorem 1** If \( (X, \mathcal{T}) \) is a space, then \( (X, \mathcal{T}_{\omega^o}) \) is a space such that \( \mathcal{T} \subseteq \mathcal{T}_{\omega^o} \subseteq \mathcal{T}_{\omega} \), where \( \mathcal{T}_{\omega} \) is the collection of all \( \omega \)-open subsets of \( X \).

**Corollary 1** If \( (X, \mathcal{T}) \) is anti locally countable and \( A \) is \( \omega^o \)-closed, then \( \text{Int}_{\mathcal{T}}(A) = \text{Int}_{\mathcal{T}_{\omega^o}}(A) \).

2. \( \omega^o \)-Continuous Mappings

We begin this section by introducing the notion of \( \omega^o \)-continuous mappings. Several characterizations of this class of mappings are also provided.

**Definition 2** A map \( f : X \to Y \) is \( \omega^o \)-continuous at \( x \in X \) if for every open subset \( V \) in \( Y \) containing \( f(x) \), there exists an \( \omega^o \)-open subset \( U \) in \( X \) containing \( x \) such that \( f(U) \subseteq V \). \( f \) is \( \omega^o \)-continuous if it is \( \omega^o \)-continuous at every \( x \in X \).

As every open set is \( \omega^o \)-open and every \( \omega^o \)-open set is \( \omega \)-open, every continuous map is \( \omega^o \)-continuous and every \( \omega^o \)-continuous map is \( \omega \)-continuous. The converses need not be true.

**Example 1** Let \( X = \{a, b\} \), \( \mathcal{T}_1 = \{\emptyset, X, \{a\}\} \) and \( \mathcal{T}_2 = \{\emptyset, X, \{b\}\} \). Then the identity map \( \text{id} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2) \) is \( \omega^o \)-continuous but not continuous.
Example 2 Let $Y = \{0, 1\}$ and $\mathfrak{T} = \{\emptyset, Y, \{0\}\}$. Then the map $f : (\mathbb{R}, \mathfrak{T}_{\text{standard}}) \to (Y, \mathfrak{T})$ defined by $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$ is $\omega$-continuous but not $\omega^o$-continuous.

The proofs of the following three results are similar to those for $\omega$-continuous maps given in [4] and are thus omitted.

Lemma 1 Let $X$, $Y$ and $Z$ be spaces. Then

(1) If $f : X \to Y$ is $\omega^o$-continuous surjection and $g : Y \to Z$ is continuous surjection, then $g \circ f$ is $\omega^o$-continuous.

(2) If $f : X \to Y$ is $\omega^o$-continuous surjection and $A \subseteq X$, then $f|_A$ is $\omega^o$-continuous.

(3) If $f : X \to Y$ is a map such that $X = X_1 \cup X_2$ where $X_1$ and $X_2$ are closed and both $f|_{X_1}$ and $f|_{X_2}$ are $\omega^o$-continuous, then $f$ is $\omega^o$-continuous.

(4) If $f_1 : X \to X_1$ and $f_2 : X \to X_2$ are maps and $g : X \to X_1 \times X_2$ is the map defined by $g(x) = (f_1(x), f_2(x))$ for all $x \in X$, then $g$ is $\omega^o$-continuous if and only if $f_1$ and $f_2$ are $\omega^o$-continuous.

Lemma 2 For a map $f : X \to Y$, the following are equivalent:

(1) $f$ is $\omega^o$-continuous.

(2) The inverse image of every open subset of $Y$ is $\omega^o$-open in $X$.

(3) The inverse image of every closed subset of $Y$ is $\omega^o$-closed in $X$.

(4) The inverse image of every basic open subset of $Y$ is $\omega^o$-open in $X$.

(5) The inverse image of every subbasic open subset of $Y$ is $\omega^o$-open in $X$.

Lemma 3 A space $(X, \mathfrak{T}_X)$ is Lindelof if and only if $(X, \mathfrak{T}_{\omega^o})$ is Lindelof.

Next we show that being Lindelof is preserved under $\omega^o$-continuity.

Theorem 2 If $f : (X, \mathfrak{T}_X) \to (Y, \mathfrak{T}_Y)$ is $\omega^o$-continuous and $X$ is Lindelof, then $Y$ is Lindelof.

Proof. Let $\mathfrak{B} = \{V_\alpha : \alpha \in \nabla\}$ be an open cover of $Y$. Since $f$ is $\omega^o$-continuous, $\mathfrak{A} = \{f^{-1}(V_\alpha) : \alpha \in \nabla\}$ is a cover of $X$ by $\omega^o$-open subsets and as $X$ is Lindelof, by Lemma 3, $\mathfrak{A}$ has a countable subcover $\{f^{-1}(V_{\alpha_n}) : n \in \mathbb{N}\}$. Now $Y = f(X) = f(\bigcup\{f^{-1}(V_{\alpha_n}) : n \in \mathbb{N}\}) \subseteq \bigcup\{V_{\alpha_n} : n \in \mathbb{N}\}$. Therefore $Y$ is Lindelof. $\square$
If $X$ is a countable space, then every subset of $X$ is $\omega^o$-open and hence every map $f : X \to Y$ is $\omega^o$-continuous. Next, we show that if $X$ is uncountable such that every $\omega^o$-continuous map $f : X \to Y$ is a constant map, then $X$ has to be connected.

**Theorem 3** If $X$ is uncountable space such that every $\omega^o$-continuous map $f : X \to Y$ is a constant map, then $X$ is connected.

**Proof.** If $X$ is disconnected, then there exists a non-empty proper subset $A$ of $X$ which is both open and closed. Let $Y = \{a, b\}$ and $\mathcal{T}_Y = \{\emptyset, Y, \{b\}\}$ and $f : X \to Y$ defined by $f(A) = \{a\}$ and $f(X \setminus A) = \{b\}$. Then $f$ is a non-constant $\omega^o$-continuous map. □

The converse of the preceding result need not be true even when $X$ is uncountable.

**Example 3** The identity map $id : (\mathbb{R}, \mathcal{T}_{leftray}) \to (\mathbb{R}, \mathcal{T}_{indiscrete})$ is a non-constant $\omega^o$-continuous.

3. Decompositions of $\omega^o$-Continuity

We begin by recalling the following well-known two definitions.

**Definition 3** A map $f : X \to Y$ is weakly continuous at $x \in X$ if for every open subset $V$ in $Y$ containing $f(x)$, there exists an open subset $U$ in $X$ containing $x$ such that $f(U) \subseteq \overline{V}$. $f$ is weakly continuous if it is weakly continuous at every $x \in X$.

**Definition 4** A map $f : X \to Y$ is $W^*$-continuous if for every open subset $V$ in $Y$, $f^{-1}(\text{Fr}(V))$ is closed in $X$, where $\text{Fr}(V) = \overline{V} \setminus \text{int}(V)$.

Weakly continuity and $W^*$-continuity are independent notions that are weaker than continuity and the two together characterize continuity (see for example [7]). Next we give two relatively new such definitions.

**Definition 5** A map $f : X \to Y$ is weakly $\omega^o$-continuous at $x \in X$ if for every open subset $V$ in $Y$ containing $f(x)$, there exists an $\omega^o$-open subset $U$ in $X$ containing $x$ such that $f(U) \subseteq \overline{V}$. $f$ is weakly $\omega^o$-continuous if it is weakly $\omega^o$-continuous at every $x \in X$.

Clearly, every $\omega^o$-continuous and every weakly continuous map is weakly $\omega^o$-continuous. Non of the converses need be true as shown next.
Example 4 Let \( Y = \{a, b, c\} \) and \( \mathcal{T} = \{\emptyset, Y, \{a\}, \{c\}, \{a, c\}\} \). Then the map \( f : (\mathbb{R}, \mathcal{T}_{\text{co-countable}}) \to (Y, \mathcal{T}) \) defined by \( f(x) = a \) for all \( x \in \mathbb{R} \). Then \( f \) is weakly \( \omega^o \)-continuous but not \( \omega^o \)-continuous.

Example 5 Let \( Y = \{a, b, c\} \) and \( \mathcal{T} = \{\emptyset, Y, \{a\}, \{c\}, \{a, c\}\} \). Then the map \( f : (\mathbb{R}, \mathcal{T}_{\text{co-countable}}) \to (Y, \mathcal{T}) \) defined by \( f(x) = \begin{cases} a & x \in \mathbb{Q} \\ b & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \) for all \( x \in \mathbb{R} \). Then \( f \) is weakly continuous and hence weakly \( \omega^o \)-continuous but not \( \omega^o \)-continuous.

Definition 6 A map \( f : X \to Y \) is coweakly \( \omega^o \)-continuous if for every open subset \( V \) in \( Y \), \( f^{-1}(\text{Fr}(V)) \) is \( \omega^o \)-closed in \( X \), where \( \text{Fr}(V) = \overline{V} \setminus \overline{V} \).

Clearly, every \( \omega^o \)-continuous is coweakly \( \omega^o \)-continuous. The converse need not be true.

Example 6 Let \( X = Y = \{a, b\} \), \( \mathcal{T}_X = \{\emptyset, X\} \) and \( \mathcal{T}_Y = \{\emptyset, Y, \{a\}, \{b\}\} \). Then the identity map \( \text{id} : X \to Y \) is coweakly \( \omega^o \)-continuous but not \( \omega^o \)-continuous.

Our first characterization of \( \omega^o \)-continuity in terms of the preceding two notions of continuity is given next.

Theorem 4 The following are equivalent for a map \( f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y) \):

1. \( f \) is \( \omega^o \)-continuous.
2. \( f : (X, \mathcal{T}_{\omega^o}) \to (Y, \mathcal{T}_Y) \) is continuous.
3. \( f : (X, \mathcal{T}_{\omega^o}) \to (Y, \mathcal{T}_Y) \) is weakly continuous and W*-continuous.

Proof. (1) \( \Rightarrow \) (2): Obvious.

(2) \( \Rightarrow \) (3): Follows from Theorem 1.

(3) \( \Rightarrow \) (1): Since \( f : (X, \mathcal{T}_{\omega^o}) \to (Y, \mathcal{T}_Y) \) is W*-continuous, it is coweakly \( \omega^o \)-continuous and as it is weakly-continuous, it is weakly \( \omega^o \)-continuous. Thus by Theorem 1, \( f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y) \) is \( \omega^o \)-continuous.

We show that weakly \( \omega^o \)-continuity and coweakly \( \omega^o \)-continuity are independent notions, but together they characterize \( \omega^o \)-continuity. This will be our first decomposition of \( \omega^o \)-continuity which is analogous to the result that can be found in [2] for \( \omega \)-continuity.
Example 7 The map $id$ in Example 6 is coweakly $\omega^o$-continuous but not weakly $\omega^o$-continuous.

Example 8 Let $Y = \{a, b\}$ and $\mathfrak{T} = \{\emptyset, Y, \{a\}\}$. Then the map $f : (\mathbb{R}, \mathfrak{T}_{\text{co-countable}}) \to (Y, \mathfrak{T})$ defined by $f(x) = \begin{cases} a & x \in \mathbb{Q} \\ b & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ for all $x \in \mathbb{R}$. Then $f$ is weakly $\omega^o$-continuous but not coweakly $\omega^o$-continuous.

Theorem 5 A map $f : X \to Y$ is $\omega^o$-continuous if and only if $f$ is both weakly and coweakly $\omega^o$-continuous.

Proof. $\omega^o$-continuity implies weakly and coweakly $\omega^o$-continuity is obvious. Conversely, suppose $f : X \to Y$ is both weakly and coweakly $\omega^o$-continuous and let $x \in X$ and $V$ be an open subset of $Y$ such that $f(x) \in V$. Then as $f$ is weakly $\omega^o$-continuous, there exists a $\omega^o$-open subset $U$ of $X$ containing $x$ such that $f(U) \subseteq V$. Now $Fr(V) = V \setminus V$ and hence $f(x) \notin Fr(V)$. So $x \in U \setminus f^{-1}(Fr(V))$ which is $\omega^o$-open in $X$ since $f$ is coweakly $\omega^o$-continuous. For every $y \in f(U \setminus f^{-1}(Fr(V)))$, $y = f(a)$ for some $a \in U \setminus f^{-1}(Fr(V))$ and hence $f(a) = y \in f(U) \subseteq V$ and $y \notin Fr(V)$. Thus $f(a) = y \notin Fr(V)$ and thus $f(a) \in V$. Therefore, $f(U \setminus f^{-1}(Fr(V))) \subseteq V$ and hence $f$ is $\omega^o$-continuous. \qed

Next, we define a new class of open sets that is independent of $\omega$-open class, but together they characterize $\omega^o$-open.

Definition 7 For a space $(X, \mathfrak{T})$, let $\omega^o_\omega : = \{A \subseteq X : \text{Int}_{\mathfrak{T}_{\omega}}(A) = \text{Int}_{\mathfrak{T}}(A)\}$. A is $\omega^o_\omega$-set if $A \in \omega^o_\omega$.

Clearly every $\omega^o$-open set is $\omega^o_\omega$-set, but the converse need not be true.

Example 9 Consider $\mathbb{R}$ with the standard topology $\mathfrak{T}_{\text{standard}}$. Then $\mathbb{Q}$ is an $\omega^o_\omega$-set which is neither $\omega^o$-open nor $\omega$-open.

Even an $\omega$-open subset need not be an $\omega^o_\omega$-set.

Example 10 Consider $\mathbb{R}$ with the standard topology $\mathfrak{T}_{\text{standard}}$. Then $\mathbb{R} \setminus \mathbb{Q}$ is an $\omega$-open which is not an $\omega^o_\omega$-set.

Theorem 6 A subset $A$ of a space $X$ is $\omega^o$-open if and only if $A$ is $\omega$-open and an $\omega^o_\omega$-set.
Proof. Trivially every $\omega^o$-open is $\omega$-open and an $\omega^o$-set. Conversely, let $A$ be an $\omega$-open set that is $\omega^o$-set. Then $A = \text{Int}_{\omega}(A) = \text{Int}_{\omega^o}(A)$ and therefore $A$ is $\omega^o$-open.

Definition 8 A map $f : X \to Y$ is $\omega^o$-continuous if the inverse image of every open subset of $Y$ is an $\omega^o$-set.

Clearly every $\omega^o$-continuous map is $\omega^o$-continuous, but the converse need not be true as not every $\omega^o$-set is $\omega^o$-open. An immediate consequence of Theorem 6 is the following decomposition of $\omega^o$-continuity.

Theorem 7 A map $f : X \to Y$ is $\omega^o$-continuous if and only if $f$ is $\omega$-continuous and $\omega^o$-continuous.

4. Decompositions of Continuity

We begin this section by introducing the notion of an $\omega^o_X$-set. We then introduce the notion of $\omega^o_X$-continuity which gives an immediate decomposition of continuity.

Definition 9 For a space $(X, \mathcal{T})$, let $\omega^o_X = \{ A \subseteq X : \text{Int}_{\omega}(A) = \text{Int}(A) \}$. $A$ is an $\omega^o_X$-set if $A \in \omega^o_X$.

The proof of the following result follows immediately from Corollary 1.

Corollary 2 If $(X, \mathcal{T})$ is anti locally countable, then $\omega^o_X$ contains all $\omega^o$-closed subsets of $X$.

We remark that, in general, an $\omega^o$-closed set need not be an $\omega^o_X$-set as shown in the next example.

Example 11 Let $X = \{a, b\}$ and $\mathcal{T} = \{\emptyset, X, \{a\}\}$. Set $A = \{b\}$. Then $A$ is $\omega^o$-closed but not an $\omega^o_X$-set.

As every open set is $\omega^o$-open, every open set is an $\omega^o_X$-set but the converse need not be true.

Example 12 Consider $\mathbb{R}$ with the standard topology $\mathcal{T}_{\text{standard}}$. Then $\mathbb{Q}$ is an $\omega^o_X$-set which is not open.
Next, we show that the notions of $\omega_X$-set and $\omega^\alpha$-open are independent, but together they characterize open sets.

**Example 13** In Example 11, $A$ is $\omega^\alpha$-open but not an $\omega_X$-set.

**Example 14** In Example 12, $Q$ is an $\omega_X$-set which is not $\omega^\alpha$-open.

**Theorem 8** A subset $A$ of a space $X$ is open if and only if $A$ is $\omega^\alpha$-open and an $\omega_X$-set.

**Proof.** Trivially every open set is $\omega^\alpha$-open and an $\omega_X$-set. Conversely, let $A$ be an $\omega^\alpha$-open set that is $\omega_X$-set. Then $A = \text{Int}_{\omega^\alpha}(A) = \text{Int}_T(A)$ and therefore $A$ is open. \(\Box\)

In a similar manner, for a space $(X, \mathcal{T})$ let $\omega_X = \{A \subseteq X : \text{Int}_{\omega^\alpha}(A) = \text{Int}_T(A)\}$ and call a subset $A$ is $\omega_X$-set if $A \in \omega_X$. Then we have the following result.

**Theorem 9** A subset $A$ of a space $X$ is open if and only if $A$ is $\omega$-open and an $\omega_X$-set.

**Definition 10** A map $f : X \to Y$ is $\omega_X$-continuous (respectively, $\omega_X$-continuous) if the inverse image of every open subset of $Y$ is an $\omega_X$-set (respectively, $\omega_X$-set).

Clearly every continuous map is $\omega_X$-continuous, but the converse need not be true as not every $\omega_X$-set is open. An immediate consequence of Theorems 5, 7, 8 and 9 are the following decompositions of continuity, which seem to be new.

**Theorem 10** For a map $f : X \to Y$, the following are equivalent:

1. $f$ is continuous.
2. $f$ is $\omega^\alpha$-continuous and $\omega_X$-continuous.
3. $f$ is $\omega$-continuous and $\omega_X$-continuous.
4. $f$ is both weakly $\omega^\alpha$-continuous, coweakly $\omega^\alpha$-continuous and $\omega_X$-continuous.
5. $f$ is $\omega$-continuous, $\omega^\alpha$-continuous and $\omega_X$-continuous.

**References**


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