Quasi-Dual Modules

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Abstract

Let $R$ be a ring, $M$ be a right $R$-module and $S = \text{End}_R(M)$. $M$ is called a quasi-dual module if, for every $R$-submodule $N$ of $M$, $N$ is a direct summand of $r_M(X)$ where $X \subseteq S$. In this article, we study and provide several characterizations of this module classes. We show that if $M$ is quasi-dual module, then, for all $m \in M$, $r_M \ell_S(m) = mR \oplus K$ for some submodule $K$ of $M$. We also show that every quasi-dual module is a Kasch module and $Z(sM) \subseteq \text{Rad}(M_R).

Key Words: Quasi-dual module, Kasch module, Ikeda-Nakayama module.

1. Introduction

Throughout this paper, $R$ is an associative ring with identity, modules are right and unitary over it and $S = \text{End}_R(M)$ is the ring of $R$-endomorphisms of $M$. Submodules of $M$ will be right $R$-modules unless specified otherwise. Clearly, the module $M$ is a left $S$ and right $R$-bimodule.

A ring $R$ is called a right dual ring if every right ideal of $R$ is an annihilator and $R$ is called right quasi-dual ring if every right ideal of $R$ is a direct summand of a right annihilator. Right dual and, as a generalization of right dual rings, right quasi-dual rings were discussed in detail in [4] and [9]. Some of the known results on right quasi-dual rings can be recalled as follows: $R$ is a right quasi-dual ring if and only if $r\ell(I) = I$ for every essential ideal $I$ of $R$; if $R$ is a right quasi-dual ring then, $R$ is a right Kasch ring and $r\ell(Soc(R_R)) = Soc(R_R)$ and $r\ell(J(R)) = J(R)$.

In this paper, the notion of a quasi-dual module is introduced as a generalization of quasi-dual rings to modules.
2. Preliminaries

Let $R$ and $S$ be rings and $SM_R$ be a bimodule. For any $X \leq M$ and $T \subseteq S$, denote $\ell_S(X) = \{ s \in S : sX = 0 \}$ and $r_M(T) = \{ m \in M : Tm = 0 \}$.

Lemma 2.1 For a right $R$-module $M$, let $S = End_R(M)$, $N \leq M$, $I \leq R$, $J \leq S$ and $0 \in S$; we then have

\[
\begin{align*}
r_M(0) &= M \\
\ell_S(0) &= S \\
r_M(S) &= \ell_S(S) = \ell_S(M) = 0 \\
\ell_M(r_R(\ell_M(I))) &= \ell_M(I) \\
\ell_S(r_M(\ell_S(N))) &= \ell_S(N) \\
r_R(\ell_M(r_R(N))) &= r_R(N) \\
r_M(\ell_S(r_M(J))) &= r_M(J) \\
\ell_S(\oplus_{i \in I} N_i) &= \cap_{i \in I} \ell_S(N_i).
\end{align*}
\]

Proof. See [2, 12].

Definition 2.2 A ring $R$ is said to be a right dual if every right ideal of $R$ is an annihilator ([4]).

Definition 2.3 A ring $R$ is called a right quasi-dual if every right ideal of $R$ is a direct summand of a right annihilator ([9]).

Definition 2.4 A module $M$ is called Ikeda-Nakayama module if

\[
\ell_S(A \cap B) = \ell_S(A) + \ell_S(B)
\]

for any submodules $A, B$ of $M_R$ (see [10]).

Definition 2.5 A module $M$ is called Kasch module if $\hat{M}$ is an (injective) cogenerator in $\sigma[M]$, where $\hat{M}$ is injective hull of $M$ in $\sigma[M]$ ([1]).
The notations, “≤” will denote a submodule, “≤ₚ” an essential submodule, and “<<” a small submodule.

We will refer to [2, 3, 4, 8, 9, 11] for all undefined notions used in the text, and also for basic facts concerning (quasi-)dual rings and annihilators.

3. Quasi-Dual Modules

In this paper, we shall introduce the notion of quasi-dual modules and try to give a module theoretic characterizations of quasi-dual ring.

Definition 3.1 (See [5]) Let \( R \) be a ring, \( M \) be a right \( R \)-module and \( S = \text{End}_R(M) \). \( M \) is called a dual module if

1. \( r_M \ell_S(N) = N \) for every submodule \( N \) of \( M \);
2. \( \ell_Sr_M(I) = I \) for every right ideal \( I \) of \( S \).

Definition 3.2 Let \( R \) be a ring, \( M \) be a right \( R \)-module and \( S = \text{End}_R(M) \). We shall call \( M \) a quasi-dual module if, for every \( R \)-submodule \( N \) of \( M \), \( N \) is a direct summand of \( r_M(X) \), where \( X \subseteq S \) (compare with [6] and [7]). Trivially,

1. A right quasi-dual ring is a quasi-dual module as right module.
2. Every dual module is a quasi-dual module.
3. Every semisimple module is a quasi-dual module.

Lemma 3.3 The following conditions are equivalent for a right \( R \)-module \( M \).

1. \( M \) is a quasi-dual module.
2. For every essential submodule \( K \) of \( M \), \( r_M \ell_S(K) = K \)
3. For every submodule \( L \) of \( M \), \( L \) is a direct summand of \( r_M \ell_S(L) \).

Proof. \((1) \Rightarrow (2) \) Let \( M \) be a quasi-dual module and \( K \) an essential submodule of \( M \). Then \( K \) is a direct summand of \( r_M(Y) \) for some \( Y \subseteq S \). Let \( r_M(Y) = K \oplus K' \) for some \( K' \). Then \( K = r_M(Y) \). Note that \( \ell_S(K) = \ell_Sr_M(Y) \) implies \( r_M \ell_S(K) = r_M \ell_Sr_M(Y) = r_M(Y) = K \).

\((2) \Rightarrow (3) \) Let \( L \) be a submodule of \( M \). If \( L \) is essential in \( M \), \( r_M \ell_S(L) = L \) by \((2) \). Hence \( L \) is a direct summand of \( r_M \ell_S(L) \). Assume that \( L \) is not essential in \( M \). Then
$L \oplus L'$ is an essential for some submodule $L'$ of $M$. By (2), $r_M \ell_S(L \oplus L') = L \oplus L'$. Since $L \subseteq r_M \ell_S(L) \subseteq r_M \ell_S(L \oplus L')$, $L$ is a direct summand of $r_M \ell_S(L)$ by modularity.

(3) $\Rightarrow$ (1) clear. \hfill $\Box$

Following [10], $M$ is called almost principally injective (AP-injective for short) if, for any $m \in M$, there exists an $S$-submodule $K$ of $M$ such that $r_M \ell_S(m) = mR \oplus K$.

**Theorem 3.4** Every quasi-dual right $R$-module is an AP-injective module.

**Proof.** Clear. \hfill $\Box$

Let $N$ be any module. $N$ is said to be $M$-cyclic module if $N$ is isomorphic to $M/X$ for some $X \leq M$, and in case $N \leq M$ and $N$ is $M$-cyclic module then it is called $M$-cyclic submodule of $M$ and $N$ is called $M$-singular if $N \cong M/K$ with $K \leq_e M$.

**Proposition 3.5** Let $M$ be an $R$-module. Then

1. If, for every essential submodule $K$ of $M$, $r_M \ell_S(K) = K$ then, every $M$-cyclic singular $R$-module is cogenerated by $M$.

2. If every singular factor submodule (i.e. $M$-cyclic submodule) of $M$ is cogenerated by $M$, then $r_M \ell_S(K) = K$ for every essential submodule $K$ of $M$.

**Proof.** (1) Let $N$ be a singular $R$-module with $N \cong M/K$ and $K \leq_e M$. Since $K$ is essential in $M$, $r_M \ell_S(K) = K$ by assumption. Let $I = \ell_S(K)$. We define $\phi : M/K \rightarrow \Pi_{\alpha \in I} M_\alpha$ by $m + K \rightarrow \phi(m + K) = (\alpha m)_{\alpha \in I}$. Then $\alpha m = 0$ for all $\alpha \in I$. Hence $\alpha \in \ell_S(K)$ and so $m \in r_M \ell_S(K) = K$. Therefore $\phi$ is a monomorphism.

(2) Let $M/K$ be a singular module for some $K \leq_e M$. By hypothesis, there exists a monomorphism $\sigma : M/K \rightarrow \Pi_{\alpha \in I} M_\alpha$ for some index set $I$ with $M_\alpha = M$ for all $\alpha \in I$. We consider the natural epimorphism $\pi : M \rightarrow M/K$ and canonical projection $p_\alpha : \Pi_{\alpha \in I} M_\alpha \rightarrow M_\alpha$. Then $p_\alpha \sigma \pi \in \ell_S(K)$. Let $m \in r_M \ell_S(K)$. Then $p_\alpha \sigma \pi(m) = 0$ for all $\alpha \in I$. Therefore $\sigma \pi(m) \in \text{Ker}(p_\alpha)$ for all $\alpha \in I$ and so $\sigma \pi(m) \in \cap_{\alpha \in I} \text{Ker}(p_\alpha)$. Since $\cap_{\alpha \in I} \text{Ker}(p_\alpha) = 0$, $\sigma \pi(m) = 0$. But $\sigma$ is a monomorphism, so $\pi(m) = 0$. Therefore $m \in K$. Other side is obvious. Hence $r_M \ell_S(K) = K$. \hfill $\Box$
σ[M] will denote the full subcategory of left $R$-modules whose objects are the submodules of $M$-generated modules. Hence

$$σ[M] = \{N ∈ R-Mod : N ≅ K/L ≤ M^{(A)}/L \text{ for some } A\}.$$ 

Following [1], a module $M$ is called Kasch module if $\hat{M}$ is an (injective) cogenerator in $σ[M]$, where $\hat{M}$ is injective hull of $M$ in $σ[M]$.

**Proposition 3.6** For a module $M$, the following are equivalent:

1. $M$ is a Kasch module;
2. Any simple module in $σ[M]$ can be embedded in $M$;
3. Any simple module in $σ[M]$ is cogenerated by $M$;
4. $\text{Hom}(C, M) ≠ 0$ for any nonzero (cyclic) $R$-module $C$ from $σ[M]$;
5. $\ell_S(N) ≠ 0$ for every proper submodule $N$ of $M$;
6. $r_M\ell_S(N) = N$ for every maximal submodule $N$ of $M$.

**Proof.** $1 ⇔ 2 ⇔ 3 ⇔ 4$ by [1, Proposition 2.6], the other equivalences follows from Lemma 3.3 and Proposition 3.5.

**Theorem 3.7** Let $M$ be a quasi dual module.

1. $r_M\ell_S(\text{Soc}(M)) = \text{Soc}(M)$.
2. For every maximal submodule $N$ of $M$, $r_M\ell_S(N) = N$. Therefore, $M$ is a Kasch module and $r_M\ell_S(\text{Rad}(M)) = \text{Rad}(M)$.
3. If $L$ is a submodule of $M$, then $r_M\ell_S(L) = L ⊕ L'$ for a submodule $L'$ with $\ell_S(L) ≤ \ell_S(L')$.

**Proof.** (1) Let $M$ be a quasi dual module. Then, for each essential submodule $K$ of $M$, $r_M\ell_S(K) ≅ K$ by Lemma 3.3. By Proposition 3.5, $M/K$ is cogenerated by $M$. Since $\text{Soc}(M)$ is the intersection of all essential submodules, $M/\text{Soc}(M)$ is cogenerated by $M$. Since $\text{Soc}(M)$ is an essential submodule of $M$ and $M/\text{Soc}(M)$ is singular factor module, so $r_M\ell_S(\text{Soc}(M)) = \text{Soc}(M)$ by Lemma 3.3.

(2) Let $N$ be a maximal submodule of $M$. Assume that $r_M\ell_S(N) ≠ N$. By maximality
of \( N \), \( r_M \ell_S(N) = M \). Note that, for \( x \in \ell_S(N) \), \( xN = 0 \) implies \( xM = 0 \). Since \( M \) is a quasi-dual module, \( N \) is a direct summand of \( r_M \ell_S(N) \) by Lemma 3.3, and so of \( M \). Let \( M = N \oplus N’ \) for some submodule \( N’ \) of \( M \). We consider the canonical projection \( \pi \) on \( N’ \). Since \( \pi(N) = 0 \) implies \( \pi(M) = 0 \), we have \( M = N \). It is a contradiction by maximality of \( N \). Hence \( r_M \ell_S(N) = N \). So, \( M \) is a Kasch module by Proposition 3.6. Let \( x \in r_M \ell_S(\text{Rad}(M)) \). Then \( \ell_S(\text{Rad}(M))x = 0 \). Note that \( M/\text{Rad}(M) = M/\cap_{N \leq_{\text{max}} M} N \). We consider

\[
M \xrightarrow{\pi} M/\text{Rad}(M) = M/\cap_{N \leq_{\text{max}} M} N \xrightarrow{\sigma} \Pi_{N \leq_{\text{max}} M} M/N \xrightarrow{\beta} \prod_{\alpha \in I} M_{\alpha} \xrightarrow{\rho_{\alpha}} M_{\alpha} = M.
\]

We know that \( \sigma \) and \( \beta \) are one to one. Since \( p_{\alpha} \beta \sigma \pi \in \ell_S(\text{Rad}(M)) \), we have \( (p_{\alpha} \beta \sigma \pi)(x) = 0 \) for all \( \alpha \in I \). Then \( \beta \sigma \pi(x) = 0 \) and so \( \pi(x) = 0 \). This implies that \( x \in \text{Rad}(M) \). Other side is obvious.

(3) Let \( L \) be a submodule of \( M \). Then \( r_M \ell_S(L) = L \oplus L’ \) for a submodule \( L’ \) by Lemma 3.3. Note that \( \ell_S(r_M \ell_S(L)) = \ell_S(L \oplus L’) = \ell_S(L) \cap \ell_S(L’) \) by Lemma 2.1. Hence \( \ell_S(L) \leq \ell_S(L’) \), as required.

Recall that;

(C1) Every complement submodule is a direct summand.
(C2) If every submodule isomorphic to a direct summand of \( M \) is itself a direct summand. 
(C3) If \( N \) and \( K \) are direct summands of \( M \) and \( N \cap K = 0 \), then \( N \oplus K \) is a direct summand of \( M \).

\( M \) is called a \textit{continuous} (or a \textit{quasi-continuous}) module if \( M \) has C1 and C2 (or C1 and C3).

\textbf{Theorem 3.8} Let \( M \) be a finitely generated Kasch module such that, any complement submodule \( N \) of \( M \), \( r_M \ell_S(N) = N \). Then \( M \) is quasi-continuous.

\textbf{Proof.} Let \( N_1 \) and \( N_2 \) be submodules of \( M \) such that they are complements of each other in \( M \). Then \( N_1 \cap N_2 = 0 \). So \( 0 = N_1 \cap N_2 = r_M \ell_S(N_1) \cap r_M \ell_S(N_2) = r_M (\ell_S(N_1) + \ell_S(N_2)) \). Since \( M \) is a Kasch module, by Proposition 3.6, \( \ell_S(N_1) + \ell_S(N_2) = M \). Hence \( M \) is a quasi-continuous by [11, Theorem 8].

\textbf{Question :} When \( M \) is a semiperfect module with essential socle in \( \sigma[M] \) under the conditions of Theorem 3.8 ?
Proposition 3.9 The following conditions are equivalent for a right $R$-module $M$.

1. $M$ is a quasi-dual module and, for every right ideal $I$ of $S$, $I$ is a direct summand of $\ell_S(K)$ where $K \subseteq M$.

2. (a) For every essential submodule $K$ of $M$, $r_M \ell_S(K) = K$
   (b) For every essential right ideal $I$ of $S$, $\ell_S r_M(I) = I$

3. (a) For every submodule $L$ of $M$, $L$ is a direct summand of $r_M \ell_S(L)$
   (b) For every essential right ideal $I$ of $S$, $I$ is a direct summand of $\ell_S r_M(I)$.

Proof. Similar to Lemma 3.3. \qed

Definition 3.10 We shall call $M$ a strongly quasi-dual module if, for every $R$-submodule $N$ of $M$ and for every right ideal $I$ of $S$, $N$ is a direct summand of $r_M \ell_S(N)$ and $I$ is a direct summand of $\ell_S(K)$ where $X \subseteq S$ and $K \subseteq M$.

Let $R$ and $S$ be any rings and $M$ be an $S - R$-bimodule. Following [6,7], if $M$ is strongly quasi-dual module, then $M$ is called quasi-dual bimodule

Proposition 3.11

1. Let $M$ be a quasi-dual module and $A$ be a submodule of $M$. Then we have:
   (i) If $\ell_S(A) = 0$, then $A = M$.
   (ii) If $M$ is an IN-module and $\ell_S(A) << S$, then $A \leq_e M$.

2. Let $M$ be a strongly quasi-dual module and $I$ be a right ideal of $S$. Then we have:
   (i) If $r_M(I) = 0$, then $I = S$.
   (ii) If $\ell_S(A) \leq_e S$, then $A << M$.
   (iii) If $M$ is indecomposable and $A \leq_e M$, then $\ell_S(A) << S$.
   (iv) If $r_M(I) \leq_e M$, then $I << S$.

Proof. 1.(i) Assume that $A$ is an essential submodule of $M$. By Lemma 3.3, $r_M \ell_S(A) = A$. But $\ell_S(A) = 0$ and $M$ is a quasi-dual module, we have $M = A$. If $A$ is not essential submodule of $M$, then there exists a submodule $B$ of $M$ such that $A \oplus B$ is essential. So $M = A \oplus B$. Let $\pi_B$ projection on $B$. Then $\pi_B(A) = 0$, and so $\pi_B \in \ell_S(A)$. Therefore $B = 0$. 

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(ii) Assume that $A$ is not essential in $M$. Then there exist a non-zero submodule $K$ of $M$ such that $A \cap K = 0$. Hence $\ell_S(A \cap K) = S$. Since $M$ is an IN-module, $\ell_S(A \cap K) = \ell_S(A) + \ell_S(K) = S$. Then $\ell_S(K) = S$. Therefore, $K = 0$.

2. (i) Similar to 1.(i).

(ii) Let $A + B = M$ for some submodule $B$ of $M$. Then $\ell_S(A + B) = \ell_S(A) \cap \ell_S(B) = 0$. By assumption, $\ell_S(B) = 0$. By 1.(i), we have $B = M$.

(iii) Let $\ell_S(A) + X = S$ for $X \subseteq S$. Then $r_M(\ell_S(A) + X) = r_M(S) = 0$. But $r_M(\ell_S(A) + X) = r_M\ell_S(A) \cap r_M(X) = A \cap r_M(X) = 0$. Since $A$ is an essential submodule of $M$, $r_M(X) = 0$. Then $X = S$ by 2.(i).

(iv) Let $I + J = S$ for some $J \subseteq S$. Then $0 = r_M(I + J) = r_M(I) \cap r_M(J)$. Since $r_M(I)$ is essential in $M$, $r_M(J) = 0$ and so $J = S$ by 2.(i).

In Theorem 3.4, shown that every quasi-dual module is AP-injective. Following [9], we have $Z(R_R) = J(R)$, where $J(R)$ and $Z(M_R)$ denote Jacobson radical of $R$ and the singular submodule of an $R$-module $M$, respectively. Therefore,

**Theorem 3.12** Let $M$ be a quasi-dual module. Then $Z(SM) \subseteq \text{Rad}(M_R)$.

**Proof.** If $x \in Z(SM)$, then $xR$ is small in $M$ by Proposition 3.12 and hence $x \in \text{Rad}(M_R)$.

**Question:** Let $M$ be a quasi-dual module. When $Z(SM) \subseteq \text{Rad}(M_R)$?

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