A Connected Sum of Knots and Fintushel-Stern Knot Surgery on 4-manifolds

Manabu Akaho

Abstract

We give some new examples of smooth 4-manifolds which are diffeomorphic although they are obtained by Fintushel-Stern knot surgeries on a smooth 4-manifold with different knots; the first such examples are given by Akbulut [1]. In the proof we essentially use the monodromy of a cusp.

1. Introduction

Let $X$ be a smooth 4-manifold. In [4] a cusp in $X$ is a PL embedded 2-sphere of self-intersection 0 with a single nonlocally flat point whose neighborhood is the cone on the right-hand trefoil knot. The regular neighborhood of a cusp is called a cusp neighborhood. It is fibered by smooth tori with one singular fiber, the cusp, and the monodromy is

$$
\begin{pmatrix}
1 & 1 \\
-1 & 0
\end{pmatrix}.
$$

If $T$ is a smoothly embedded torus representing a nontrivial homology class $[T]$, we say that $T$ is c-embedded if $T$ is a smooth fiber in a cusp neighborhood.

Consider an oriented knot $K$ in $S^3$, and let $m$ denote an oriented meridional circle to $K$; see Figure 1. Let $M_K$ be the 3-manifold obtained by performing 0-framed surgery on $K$. Then $m$ can also be viewed as a circle in $M_K$. In $M_K \times S^1$ we have a smooth torus

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$T_m = m \times S^1$ of self-intersection 0. Since a tubular neighborhood of $m$ has a canonical framing in $M_K$, a tubular neighborhood of the torus $T_m$ in $M_K \times S^1$ has a canonical identification with $T_m \times D^2$. Let $X_{(K,\phi)}$ denote the fiber sum

$$X_{(K,\phi)} := [X \setminus (T \times D^2)] \cup_{\phi} [(M_K \times S^1) \setminus (T_m \times D^2)],$$

where $T \times D^2$ is a tubular neighborhood of the torus $T$ in the manifold $X$ and $\phi : \partial(T \times D^2) \to \partial(T_m \times D^2)$ is a homeomorphism. In general, the diffeomorphism type of $X_{(K,\phi)}$ depends on $\phi$. If we fix an identification of $T$ with $S^1 \times S^1$ and a homeomorphism $\phi : \partial(T \times D^2) \to \partial(T_m \times D^2)$ such that

$$\phi(S^1 \times * \times *) = m \times * \times *,$$
$$\phi(* \times S^1 \times *) = * \times S^1 \times *,$$
$$\phi(* \times * \times \partial D^2) = * \times * \times \partial D^2,$$

where *'s are points, then we shall simply denote $X_{(K,\phi)}$ by $X_K$. We call this operation Fintushel-Stern knot surgery on a 4-manifold $X$ with $K$.

In case $H_1(X,\mathbb{Z})$ has no 2-torsion there is a natural identification of the spin$^c$ structures of $X$ with the characteristic elements of $H^2(X,\mathbb{Z})$. Recall that the Seiberg-Witten invariant $SW_X$ is a function

$$SW_X : \{k \in H^2(X,\mathbb{Z})|k \equiv w_2(TX) \mod 2\} \to \mathbb{Z}.$$

The function $SW_X$ has a compact support $B = \{\pm \beta_1, \ldots, \pm \beta_n\}$ which is called the set of basic classes. By setting $t_\beta := \exp \beta$ for each $\beta \in H^2(X,\mathbb{Z})$, the function $SW_X$ is usually written as a Laurent polynomial

$$SW_X = \sum_{\beta \in B} SW_X(\beta)t_\beta.$$
Then Fintushel and Stern [4] theorem says:

**Theorem 1.1** Let $X$ be a simply connected smooth 4-manifold with $b^+ > 1$. Suppose that $X$ contains a smoothly c-embedded torus $T$ such that $\pi_1(X \setminus T) = 1$. Then

$$SW_X = SW_X \cdot \Delta_K(t),$$

where $t = \exp 2[T]$ and $\Delta_K(t)$ is the Alexander polynomial of $K$.

To make sense of the statement of the theorem, we need to replace $[T]$ by its Poincaré dual.

Since the Seiberg-Witten invariant is a diffeomorphism invariant, if $SW_X$ and $\Delta_K(t)$ are nontrivial, then $X$ and $X_K$ are not diffeomorphic. Fintushel and Stern conjectured that if $X$ is the Kummer surface $K3$, then the association $K \mapsto X_K$ gives an injective map from the set of isotopy classes of knots in $S^3$ to the set of diffeomorphism classes of smooth structures on $X$. In [1] Akbulut gave first counterexamples to this conjecture:

**Theorem 1.2** Let $X$ be a smooth 4-manifold. Suppose that $X$ contains a smoothly c-embedded torus $T$. Fix an identification of $T$ with $S^1 \times S^1$. We denote the mirror reflection of an oriented knot $K$ by $K^*$, see Figure 2. Then

$$X_K = X_{K^*},$$

and this diffeomorphism leaves the core torus invariant.

We denote an oriented meridional circle to $K^*$ by $m'$. In the Alexander polynomials $K$ is equal to $K^*$, i.e., $\Delta_K(t) = \Delta_{K^*}(t)$. In Section 2 we give a simple proof of Theorem 1.2.

Next we give a relation between a connected sum of knots and Fintushel-Stern knot surgery; this observation is given by S. Finashin, see Lemma 3.1 in [3]. Let $T_1$ and $T_2$ be regular fibers in a cusp neighborhood in $X$. We fix common identifications of $T_1$ and $T_2$ with $S^1 \times S^1$ by holonomy. Let $K_1$ and $K_2$ be oriented knots in $S^3$. We construct $X_{K_1}$ by using $T_1$ and $K_1$. Since $X_{K_1}$ also has a cusp neighborhood which has $T_2$ as a smooth fiber, we can construct $(X_{K_1})_{K_2}$ by using $T_2$ and $K_2$.

**Theorem 1.3**

$$(X_{K_1})_{K_2} = X_{K_1 \sharp K_2},$$

where $K_1 \sharp K_2$ is the connected sum of $K_1$ and $K_2$. 

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Figure 2

Note that \( \Delta_{K_1}(t) \cdot \Delta_{K_2}(t) = \Delta_{K_1 \# K_2}(t) \). Because the core torus is invariant with respect to the diffeomorphism \( X_{K_1} = X_{K_1^*} \), we obtain the following corollary:

**Corollary 1.4**

\[
(X_{K_1})_{K_2} = X_{(K_1 \# K_2)}.
\]

Finally these claims give us new counterexamples to the conjecture:

**Corollary 1.5**

\[
X_{K_1 \# K_2} = X_{(K_1^* \# K_2)}.
\]

In section 3 we prove Theorem 1.3.

2. A simple proof of Theorem 1.2

In this section we give a simple proof of Theorem 1.2.

We denote the oppositely oriented circle to an oriented \( S^1 \) by \( \overline{S^1} \). Let \( Y \) denote the fiber sum

\[
Y := [X \setminus (T \times D^2)] \cup \psi [(M_{K^*} \times S^1) \setminus (T_{m^*} \times D^2)],
\]
where \( \psi : \partial (T \times D^2) \to \partial (T_m \times D^2) \) is a homeomorphism such that
\[
\begin{align*}
\psi(\overline{S^1} \times \ast \times \ast) &= m' \times \ast \times \ast, \\
\psi(\ast \times \overline{S^1} \times \ast) &= \ast \times S^1 \times \ast, \\
\psi(\ast \times \ast \times \partial D^2) &= \ast \times \ast \times \partial D^2.
\end{align*}
\]

Since the third power of the monodromy of the cusp is \(-1\) on a smooth fiber \(T\), \(Y\) is diffeomorphic to \(X_K\). Let \(f : M_K \to M_K\) be an orientation reversing diffeomorphism which maps the points to their mirror reflection points and \(f \times (-\text{id}_{S^1}) : M_K \times S^1 \to M_K \times S^1\) an orientation preserving diffeomorphism, where \(-\text{id}_{S^1}\) is the orientation reversing diffeomorphism of \(S^1\). Then we can construct a diffeomorphism \(F : X_K \to Y\) by
\[
F(x) := \begin{cases} 
(f \times (-\text{id}_{S^1}))(x), & \text{for } x \in (M_K \times S^1) \setminus (T_m \times D^2) \\
x, & \text{for } x \in X \setminus (T \times D^2),
\end{cases}
\]
and \(F\) maps the core torus to itself. Hence \(X_K = Y = X_K\) and we finish proving the theorem. \(\square\)

3. Proof of Theorem 1.3

We define an oriented link as in Figure 3; let \(N\) be the 3-manifold obtained by performing 0-framed surgery on each component of the link. Let \(W\) denote the fiber

![Figure 3](image-url)
sum

\[ W := [X \setminus (T \times D^2)] \cup_0 [(N \times S^1) \setminus (T_m \times D^2)], \]

where \(m_1\) is an oriented meridional circle to \(K_1\). Because \(T_2\) is ambient isotopic to \(m_1 \times S^1\), we can easily see that \(W\) is diffeomorphic to \((X_{K_1})_{K_2}\). Now we shall play Kirby calculus on the 3-manifold \(N\) as in Figure 4. The last step of vanishing components can be found in example 5.2 of [6]. Hence \(N \setminus (m_1 \times D^2)\) is diffeomorphic to \(M_{K_1K_2} \setminus (m_1 \times D^2)\), and \(W\) is diffeomorphic to \(X_{K_1K_2}\). We finish proving Theorem 1.3.

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References


