Inequality for Ricci Curvature of Slant Submanifolds in Cosymplectic Space Forms

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Abstract

In this article, we establish inequalities between the Ricci curvature and the squared mean curvature, and also between the $k$-Ricci curvature and the scalar curvature for a slant, semi-slant and bi-slant submanifold in a cosymplectic space form of constant $\varphi$-sectional curvature with arbitrary codimension.

Key Words: Mean curvature, sectional curvature, $k$-Ricci curvature, slant submanifold, semi-slant submanifold, bi-slant submanifold, cosymplectic space form.

1. Introduction

Let $\tilde{M}$ be a $(2m + 1)$-dimensional almost contact manifold endowed with an almost contact structure $(\varphi, \xi, \eta)$, that is, $\varphi$ is a $(1,1)$ tensor field, $\xi$ is a vector field and $\eta$ is a 1-form such that

$$\varphi^2 = -I + \eta \otimes \xi \quad \text{and} \quad \eta(\xi) = 1.$$ 

Then, $\varphi(\xi) = 0$ and $\eta \circ \varphi = 0$. Let $g$ be a compatible Riemannian metric with $(\varphi, \xi, \eta)$, that is, $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$ or equivalent, $g(X, \varphi Y) = -g(\varphi X, Y)$ and $g(X, \xi) = \eta(X)$ for all $X, Y \in \tilde{M}$. Then, $\tilde{M}$ becomes an almost contact metric manifold equipped with an almost contact metric structure $(\varphi, \xi, \eta, g)$. An almost contact metric manifold is cosymplectic ([1]) if $\nabla_X \varphi = 0$, where $\nabla$ is the Levi-Civita connection of the Riemannian metric $g$. From the formula $\nabla_X \varphi = 0$ it follows that $\nabla_X \xi = 0$.

A plane section $\pi$ in $T_p \tilde{M}$ of an almost contact metric manifold $\tilde{M}$ is called a $\varphi$-section if $\pi \perp \xi$ and $\varphi(\pi) = \pi$. $\tilde{M}$ is of constant $\varphi$-sectional curvature if sectional curvature $\tilde{K}(\pi)$ does not depend on the choice of the $\varphi$-section $\pi$ of $T_p \tilde{M}$ and the choice of a point $p \in \tilde{M}$. A cosymplectic manifold $\tilde{M}$ is said to be a cosymplectic space form if the $\varphi$-sectional curvature is constant $c$ along $\tilde{M}$. A cosymplectic space form will be denoted by $\tilde{M}(c)$. Then the Riemannian curvature tensor $\tilde{R}$ on $\tilde{M}(c)$ is given by (1.1)

\[ 4\tilde{R}(X, Y, Z, W) = c\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W) + g(X, \varphi W)g(Y, \varphi Z) \]

\[ - g(X, \varphi Z)g(Y, \varphi W) - 2g(X, \varphi Y)g(Z, \varphi W) - g(X, W)\eta(Y)\eta(Z) \]

\[ + g(X, Z)\eta(Y)\eta(W) - g(Y, Z)\eta(X)\eta(W) + g(Y, W)\eta(X)\eta(Z) \}. \]

Let $M$ be an $n$-dimensional submanifold of a cosymplectic space form $\tilde{M}(c)$ equipped with a Riemannian metric $g$. The Gauss and Wiengarten formulas are given respectively by

\[ \nabla_X Y = \nabla_X Y + h(X, Y) \quad \text{and} \quad \nabla_X N = -A_N X + \nabla^\perp_X N \]

for all $X, Y \in TM$ and $N \in T^\perp M$, where $\nabla$, $\nabla^\perp$ and $\nabla^\perp$ are the Riemannian, induced Riemannian and induced normal connections in $\tilde{M}(c)$, $M$ and the normal bundle $T^\perp M$ of $M$ respectively, and $h$ is the second fundamental form related to the shape operator $A$ by $g(h(X, Y), N) = g(A_N X, Y)$. Then the equation of Gauss is given by

\[ \tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)), \quad (1.2) \]

for any vectors $X, Y, Z, W$ tangent to $M$.

For any vector $X$ tangent to $M$ we put $\varphi X = PX + FX$, where $PX$ and $FX$ are the tangential and the normal components of $\varphi X$, respectively. Given an orthonormal basis \{ $e_1, \ldots, e_n$ \} of $M$, we define the squared norm of $P$ by

\[ ||P||^2 = \sum_{i,j=1}^{n} g^2(\varphi e_i, e_j) \]

and the mean curvature vector $H(p)$ at $p \in M$ is given by

\[ H = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i). \]

We put

\[ h^r_{ij} = g(h(e_i, e_j), e_r) \quad \text{and} \quad ||h||^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)), \]
where \( \{e_{n+1}, \ldots, e_{2m+1}\} \) is an orthonormal basis of \( T^\perp_p M \) and \( \tau = n + 1, \ldots, 2m + 1 \). A submanifold \( M \) in \( \tilde{M}(c) \) is called \textit{totally geodesic} if the second fundamental form vanishes identically and \textit{totally umbilical} if there is a real number \( \lambda \) such that \( h(X, Y) = \lambda g(X, Y)H \) for any tangent vectors \( X, Y \) on \( M \).

For an \( n \)-dimensional Riemannian manifold \( M \), we denote by \( K(\pi) \) the sectional curvature of \( M \) associated with a plane section \( \pi \subset T_p M, p \in M \). For an orthonormal basis \( \{e_1, \ldots, e_n\} \) of the tangent space \( T_p M \), the scalar curvature \( \tau \) is defined by

\[
\tau = \sum_{i<j} K_{ij},
\]

(1.3)

where \( K_{ij} \) denotes the sectional curvature of the 2-plane section spanned by \( e_i \) and \( e_j \).

Suppose \( L \) is a \( k \)-plane section of \( T_p M \) and \( X \) a unit vector in \( L \). We choose an orthonormal basis \( \{e_1, \ldots, e_k\} \) of \( L \) such that \( e_1 = X \). Define the Ricci curvature \( \text{Ric}_L \) of \( L \) at \( X \) by

\[
\text{Ric}_L(X) = K_{12} + \cdots + K_{kk}.
\]

(1.4)

We simply call such a curvature a \( k \)-\textit{Ricci curvature}. The scalar curvature \( \tau \) of the \( k \)-plane section \( L \) is given by

\[
\tau(L) = \sum_{1 \leq i < j \leq k} K_{ij}.
\]

(1.5)

For each integer \( k, 2 \leq k \leq n \), the Riemannian invariant \( \Theta_k \) on an \( n \)-dimensional Riemannian manifold \( M \) is defined by

\[
\Theta_k(p) = \frac{1}{k-1} \inf_{L,X} \text{Ric}_L(X), \quad p \in M,
\]

(1.6)

where \( L \) runs over all \( k \)-plane sections in \( T_p M \) and \( X \) runs over all unit vectors in \( L \).

Recall that for a submanifold \( M \) in a Riemannian manifold, the relative null space of \( M \) at a point \( p \in M \) is defined by

\[
N_p = \{X \in T_p M | h(X, Y) = 0 \quad \text{for all} \quad Y \in T_p M\}.
\]

In [8], A. Lotta has introduced the following notion of slant submanifolds into almost contact metric manifolds. A submanifold \( M \) tangent to \( \xi \) is said to be \textit{slant} if for any
Let $p \in M$ and any $X \in T_pM$, linearly independent of $\xi$, the angle between $\varphi X$ and $T_pM$ is a constant $\theta \in [0, \pi/2]$, called the slant angle of $M$ in $\tilde{M}(\xi)$. Invariant and anti-invariant submanifolds of $\tilde{M}(\xi)$ are slant submanifolds with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively.

We say that a submanifold $M$ tangent to $\xi$ is a bi-slant submanifold in $\tilde{M}(\xi)$ if there exist two orthogonal distributions $D_1$ and $D_2$ on $M$ such that

1. $TM$ admits the orthogonal direct decomposition $TM = D_1 \oplus D_2 \oplus \{\xi\}$
2. For any $i = 1, 2$, $D_i$ is slant distribution with slant angle $\theta_i$.

On the other hand, $CR$-submanifolds of $\tilde{M}(\xi)$ are bi-slant submanifolds with $\theta_1 = 0$, $\theta_2 = \pi/2$.

Let $2d_1 = \dim D_1$ and $2d_2 = \dim D_2$.

**Remark.** If either $d_1$ or $d_2$ vanishes, the bi-slant submanifold is a slant submanifold. Thus, slant submanifolds are particular cases of bi-slant submanifolds.

A submanifold $M$ tangent to $\xi$ is called a semi-slant submanifold in $\tilde{M}(\xi)$ if there exist two orthogonal distributions $D_1$ and $D_2$ on $M$ such that

1. $TM$ admits the orthogonal direct decomposition $TM = D_1 \oplus D_2 \oplus \{\xi\}$.
2. The distribution $D_1$ is an invariant distribution, i.e., $\varphi(D_1) = D_1$.
3. The distribution $D_2$ is slant with angle $\theta \neq 0$.

**Remark.** The invariant distribution of a semi-slant submanifold is a slant distribution with zero angle. Thus, it is obvious that in fact, semi-slant submanifolds are particular cases of bi-slant submanifolds.

1. If $d_2 = 0$, then $M$ is an invariant submanifold.
2. If $d_1 = 0$ and $\theta = \pi/2$, then $M$ is an anti-invariant submanifold.

For the other properties and examples of slant, bi-slant and semi-slant submanifolds in almost contact metric manifold, we refer to the reader [2], [3].

2. **Ricci Curvature and Squared Mean Curvature**

B.Y. Chen established a sharp relationship between the Ricci curvature and the squared mean curvature for submanifolds in real space forms (see [6]). We prove similar inequalities for slant, bi-slant and semi-slant submanifolds in a cosymplectic space form.
\( \tilde{M}(c) \). We consider submanifolds \( M \) tangent to the vector field \( \xi \).

**Theorem 2.1** Let \( M \) be an \( n \)-dimensional \( \theta \)-slant submanifold tangent to \( \xi \) into a \((2m + 1)\)-dimensional cosymplectic space form \( \tilde{M}(c) \). Then, we have

1. For each unit vector \( X \in T_pM \) orthogonal to \( \xi \)
   \[
   \text{Ric}(X) \leq \frac{1}{4} \left( (n - 1)c + \frac{1}{2}(3\cos^2 \theta - 2)c + n^2\|H\|^2 \right). \tag{2.1}
   \]

2. If \( H(p) = 0 \), then a unit tangent vector \( X \) orthogonal to \( \xi \) at \( p \) satisfies the equality case of \( (2.1) \) if and only if \( X \in N_p \).

3. The equality case of \( (2.1) \) holds identically for all unit tangent vectors orthogonal to \( \xi \) at \( p \) if and only if \( p \) is a totally geodesic point.

**Proof.** Let \( X \in T_pM \) be a unit tangent vector at \( p \) orthogonal to \( \xi \). We choose an orthonormal basis \( e_1, \ldots, e_n = \xi, e_{n+1}, \ldots, e_{2m+1} \) such that \( e_1, \ldots, e_n \) are tangent to \( M \) at \( p \) with \( e_1 = X \). Then, from the equation of Gauss, we have

\[
 n^2\|H\|^2 = 2\tau + \|h\|^2 - \{n(n - 1) + 3(n - 1)\cos^2 \theta - 2n + 2\} \frac{c}{4}. \tag{2.2}
\]

From (2.2) we get

\[
 n^2\|H\|^2 = 2\tau + \sum_{r=n+1}^{2m+1} [(h_{11}^r)^2 + (h_{22}^r + \cdots + h_{nn}^r)^2 + 2 \sum_{1 \leq i < j \leq n} (h_{ij}^r)^2]
\]

\[
 - 2 \sum_{r=n+1}^{2m+1} \sum_{2 \leq i < j \leq n} h_{ii}^r h_{jj}^r - \frac{c}{4}[n(n - 1) + 3(n - 1)\cos^2 \theta - 2n + 2]
\]

\[
 n^2\|H\|^2 = 2\tau + \frac{1}{2} \sum_{r=n+1}^{2m+1} [(h_{11}^r + h_{22}^r + \cdots + h_{nn}^r)^2 + (h_{11}^r - h_{22}^r - \cdots - h_{nn}^r)^2]
\]

\[
 + 2 \sum_{r=n+1}^{2m+1} \sum_{1 \leq i < j \leq n} (h_{ij}^r)^2 - 2 \sum_{r=n+1}^{2m+1} \sum_{2 \leq i < j \leq n} h_{ii}^r h_{jj}^r
\]

\[
 - \frac{c}{4}[n(n - 1) + 3(n - 1)\cos^2 \theta - 2n + 2].
\]
By using the equation of Gauss, we have
\[ \sum_{2 \leq i < j \leq n} K_{ij} = \sum_{r=n+1}^{2m+1} \sum_{2 \leq i < j \leq n} [h_{ij}^r h_{ij}^r - (h_{ij}^r)^2] + (n-1)(n-2) \frac{c}{8} \]
[2.4]
\[ + [3(n-2) \cos^2 \theta - 2n + 4] \frac{c}{8}. \]

Substituting (2.4) in (2.3), we get
\[ \frac{1}{2} n^2 ||H||^2 \geq 2 \text{Ric}(X) - 2(n - 1) \frac{c}{4} - (3 \cos^2 \theta - 2) \frac{c}{4}, \]
which is equivalent to (2.1).

(2) Assume \( H(P) = 0 \). Equality holds in (2.1) if and only if
\[
\begin{aligned}
& h_{ij}^r = 0, \quad i \neq j, \\
& h_{ij}^r = h_{ij}^r + \cdots + h_{ij}^r, \quad r \in \{n + 1, \cdots, 2m + 1\}.
\end{aligned}
\]

Then \( h_{ij}^r = 0 \) for all \( j \in \{1, \cdots, n\}, r \in \{n + 1, \cdots, 2m + 1\} \), that is, \( X \in N_p \).

(3) Then equality case of (2.1) holds for all unit tangent vectors orthogonal to \( \xi \) at \( p \) if and only if
\[
\begin{aligned}
& h_{ij}^r = 0, \quad i \neq j, \quad r \in \{n + 1, \cdots, 2m + 1\}, \\
& h_{ij}^r + \cdots + h_{ij}^r - 2h_{ij}^r = 0, \quad r \in \{n + 1, \cdots, 2m + 1\},
\end{aligned}
\]

In this case, it follows that \( p \) is a totally geodesic point. The converse is trivial.

Theorem 2.2 Let \( M \) be an \( n \)-dimensional bi-slant submanifold satisfying \( g(X, \varphi Y) = 0 \), for any \( X \in D_1 \) and any \( Y \in D_2 \), tangent to \( \xi \) in a \((2m + 1)\)-dimensional cosymplectic space form \( \bar{M}(c) \). Then,

(1) For each unit vector \( X \in T_p M \) orthogonal to \( \xi \) and if
(i) \( X \) is tangent to \( D_1 \), we have
\[ \text{Ric}(X) \leq \frac{1}{4} \left\{ (n-1)c + \frac{1}{2}(3 \cos^2 \theta - 2)c + n^2 ||H||^2 \right\}; \]  
(2.5)

and if
(ii) $X$ is tangent to $D_2$, we have

$$\text{Ric}(X) \leq \frac{1}{4} \left\{ (n-1)c + \frac{1}{2} (3\cos^2 \theta_2 - 2)c + n^2 ||H||^2 \right\}.$$  \hspace{1cm} (2.6)

(2) If $H(p) = 0$, then a unit tangent vector $X$ orthogonal to $\xi$ at $p$ satisfies the equality case of (2.5) and (2.6) if and only if $X \in N_p$.

(3) The equality case of (2.5) and (2.6) holds identically for all unit tangent vectors orthogonal to $\xi$ at $p$ if and only if $p$ is a totally geodesic point.

**Proof.** Let $X \in T_p M$ be a unit tangent vector at $p$ orthogonal to $\xi$. We choose an orthonormal basis $e_1, \cdots, e_n = \xi, e_{n+1}, \cdots, e_{2m+1}$ such that $e_1, \cdots, e_n$ are tangent to $M$ at $p$ with $e_1 = X$. Then, from the equation of Gauss, we have

$$n^2 ||H||^2 = 2\tau + ||h||^2 - \{n(n-1) + 6(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2) - 2n + 2\} \frac{c}{4},$$  \hspace{1cm} (2.7)

where $2d_1 = \dim D_1$ and $2d_2 = \dim D_2$.

From (2.7) we get

$$n^2 ||H||^2 = 2\tau + \sum_{r=n+1}^{2m+1} [(h_{11}^r)^2 + (h_{22}^r + \cdots + h_{nn}^r)^2 + 2 \sum_{1 \leq i < j \leq n} (h_{ij}^r)^2]$$

$$- 2 \sum_{r=n+1}^{2m+1} \sum_{1 \leq i < j \leq n} h_{ij}^r h_{jj}^r - \frac{c}{4} [n(n-1) + 6(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2) - 2n + 2]$$

$$= 2\tau + \frac{1}{2} \sum_{r=n+1}^{2m+1} [(h_{11}^r + h_{22}^r + \cdots + h_{nn}^r)^2 + (h_{11}^r - h_{22}^r - \cdots - h_{nn}^r)^2]$$

$$+ 2 \sum_{r=n+1}^{2m+1} \sum_{1 \leq i < j \leq n} (h_{ij}^r)^2 - 2 \sum_{r=n+1}^{2m+1} \sum_{1 \leq i < j \leq n} h_{ij}^r h_{jj}^r$$

$$- \frac{c}{4} [n(n-1) + 6(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2) - 2n + 2].$$  \hspace{1cm} (2.8)

We distinguish two cases:
(i) if $X$ is tangent to $D_1$, then we have
\[
\sum_{2 \leq i < j \leq n} K_{ij} = \sum_{r=n+1}^{2m+1} \sum_{2 \leq i < j \leq n} [h_{ii} h_{jj}^r - (h_{ij}^r)^2] + (n-1)(n-2)\frac{c}{8}
\]
\[
+ [6(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2) - 3 \cos^2 \theta_1 - 2n + 4] \frac{c}{8}.
\]
(2.9)

Substituting (2.9) in (2.8), one gets
\[
\frac{1}{2} n^2 \|H\|^2 \geq 2\text{Ric}(X) - 2(n-1)\frac{c}{4} - (3 \cos^2 \theta_1 - 2)\frac{c}{4},
\]
which is equivalent to (2.5).

(ii) if $X$ is tangent to $D_2$, then we have
\[
\sum_{2 \leq i < j \leq n} K_{ij} = \sum_{r=n+1}^{2m+1} \sum_{2 \leq i < j \leq n} [h_{ii} h_{jj}^r - (h_{ij}^r)^2] + (n-1)(n-2)\frac{c}{8}
\]
\[
+ [6(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2) - 3 \cos^2 \theta_2 - 2n + 4] \frac{c}{8}.
\]
(2.10)

Substituting (2.10) in (2.8), one gets
\[
\frac{1}{2} n^2 \|H\|^2 \geq 2\text{Ric}(X) - 2(n-1)\frac{c}{4} - (3 \cos^2 \theta_2 - 2)\frac{c}{4},
\]
which is equivalent to (2.6).

(2) Assume $H(p) = 0$. Equality holds in (2.5) and (2.6) if and only if
\[
\begin{aligned}
&h_{i2} = \cdots = h_{in} = 0, \\
h_{i1} = h_{i2} + \cdots + h_{in}, & \quad r \in \{n+1, \cdots, 2m+1\}.
\end{aligned}
\]
Then $h_{ij}^r = 0$ for all $j \in \{1, \cdots, n\}, r \in \{n+1, \cdots, 2m+1\}$, that is, $X \in N_p$.

(3) Then equality case of (2.5) and (2.6) holds for all unit tangent vectors orthogonal to $\xi$ at $p$ if and only if
\[
\begin{aligned}
h_{ij}^r = 0, & \quad i \neq j, & r \in \{n+1, \ldots, 2m+1\}; \\
h_{i1}^r + \cdots + h_{nn}^r - 2h_{ii}^r = 0, & \quad i \in \{1, \cdots, n\}, & r \in \{n+1, \cdots, 2m+1\}.
\end{aligned}
\]
Corollary 2.3 Let $M$ be an $n$-dimensional semi-slant submanifold in a $(2m+1)$-dimensional cosymplectic space form $\tilde{M}(c)$. Then,

(1) For each unit vector $X \in T_pM$ orthogonal to $\xi$ and if
   (i) $X$ is tangent to $D_1$ we have
   \[
   \text{Ric}(X) \leq \frac{1}{4} \left\{ (n-2)c + n^2\|H\|^2 \right\}.
   \]
   and if
   (ii) $X$ is tangent to $D_2$ we have
   \[
   \text{Ric}(X) \leq \frac{1}{4} \left\{ (n-1)c + \frac{1}{2}(3\cos^2 \theta - 2)c + n^2\|H\|^2 \right\}. \tag{2.12}
   \]

(2) If $H(p) = 0$, then a unit tangent vector $X$ orthogonal to $\xi$ at $p$ satisfies the equality case of (2.11) and (2.12) if and only if $X \in N_p$.

(3) The equality case of (2.11) and (2.12) holds identically for all unit tangent vectors orthogonal to $\xi$ at $p$ if and only if $p$ is a totally geodesic point.

Corollary 2.4 Let $M$ be an $n$-dimensional invariant submanifold in a $(2m+1)$-dimensional cosymplectic space form $\tilde{M}(c)$. Then,

(1) For each unit vector $X \in T_pM$ orthogonal to $\xi$
   \[
   \text{Ric}(X) \leq \frac{1}{4} \left\{ (n-1)c + n^2\|H\|^2 \right\}. \tag{2.13}
   \]

(2) If $H(p) = 0$, then a unit tangent vector $X$ orthogonal to $\xi$ at $p$ satisfies the equality case of (2.13) if and only if $X \in N_p$.

(3) The equality case of (2.13) holds identically for all unit tangent vectors orthogonal to $\xi$ at $p$ if and only if $p$ is a totally geodesic point.

Corollary 2.5 Let $M$ be an $n$-dimensional anti-invariant submanifold in a $(2m+1)$-dimensional cosymplectic space form $\tilde{M}(c)$. Then,

(1) For each unit vector $X \in T_pM$ orthogonal to $\xi$
   \[
   \text{Ric}(X) \leq \frac{1}{4} \left\{ (n-2)c + n^2\|H\|^2 \right\}. \tag{2.14}
   \]
(2) If $H(p) = 0$, then a unit tangent vector $X$ orthogonal to $\xi$ at $p$ satisfies the equality case of (2.14) if and only if $X \in N_p$.

(3) The equality case of (2.14) holds identically for all unit tangent vectors orthogonal to $\xi$ at $p$ if and only if $p$ is a totally geodesic point.

3. $k$-Ricci Curvature and Squared Mean Curvature

In this section, we prove the relationship between the $k$-Ricci curvature and the squared mean curvature for slant, bi-slant and semi-slant submanifolds in a cosymplectic space form $\tilde{M}(c)$. We state an inequality between the scalar curvature and the squared mean curvature for submanifolds $M$ tangent to the vector field $\xi$.

**Theorem 3.1** Let $M$ be an $n$-dimensional $\theta$-slant submanifold tangent to $\xi$ into a $(2m + 1)$-dimensional cosymplectic space form $\tilde{M}(c)$. Then we have

$$||H||^2 \geq \frac{2\tau}{n(n-1)} - \frac{[n(n-1) + 3(n-1)\cos^2 \theta - 2n + 2]c}{4n(n-1)},$$

equality holding at a point $p \in M$ if and only if $p$ is a totally umbilical point.

**Proof.** Let $p$ be a point of $M$. We choose an orthonormal basis $\{e_1, e_2, \cdots, e_n = \xi\}$ for the tangent space $T_pM$ and $\{e_{n+1}, \cdots, e_{2m+1}\}$ for the normal space $T_p^\bot M$ at $p$ such that the normal vector $e_{n+1}$ is in the direction of the mean curvature vector and $e_1, e_2, \cdots, e_n$ diagonalize the shape operator $A_{n+1}$. Then we have

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix},$$

$$A_r = (h_{ij}^r), \quad \sum_{i=1}^{n} h_{ij}^r = 0, \quad n + 2 \leq r \leq 2m + 1.$$
From the equation of Gauss

\[ n^2||H||^2 = 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n} (h_{ij}^r)^2 - [n(n-1) + 3(n-1) \cos^2 \theta - 2n + 2]c^2, \tag{3.3} \]

On the other hand,

\[ \sum_{i<j} (a_i - a_j)^2 = (n-1) \sum_{i=1}^{n} a_i^2 - 2 \sum_{i<j} a_i a_j. \tag{3.4} \]

Therefore, from the above equation we have

\[ n^2||H||^2 = (\sum_{i=1}^{n} a_i)^2 = \sum_{i=1}^{n} a_i^2 + 2 \sum_{i<j} a_i a_j \leq n \sum_{i=1}^{n} a_i^2. \tag{3.5} \]

Combining (3.3) and (3.5), we get

\[ n(n-1)||H||^2 \geq 2\tau + \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n} (h_{ij}^r)^2 - [n(n-1) + 3(n-1) \cos^2 \theta - 2n + 2]c^2, \tag{3.6} \]

which implies inequality (3.1). If the equality sign of (3.1) holds at a point \( p \in M \) then from (3.4) and (3.6), we get \( A_r = 0 \) for \( r = n + 2, \ldots, 2m + 1 \) and \( a_1 = \cdots = a_n \). Consequently, \( p \) is a totally umbilical point. The converse is trivial. \( \Box \)

**Theorem 3.2** Let \( M \) be an \( n \)-dimensional bi-slant submanifold satisfying \( g(X, \varphi Y) = 0 \), for any \( X \in D_1 \) and any \( Y \in D_2 \), tangent to \( \xi \) into a \((2m + 1)\)-dimensional cosymplectic space form \( \hat{M}(c) \). Then we have

\[ ||H||^2 \geq \frac{2\tau}{n(n-1)} - \frac{[n(n-1) + 6(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2) - 2n + 2]c}{4n(n-1)}, \]

where \( 2d_1 = \dim D_1 \) and \( 2d_2 = \dim D_2 \).

**Theorem 3.3** Let \( M \) be an \( n \)-dimensional semi-slant submanifold tangent to \( \xi \) into a \((2m + 1)\)-dimensional cosymplectic space form \( \hat{M}(c) \). Then we have

\[ ||H||^2 \geq \frac{2\tau}{n(n-1)} - \frac{[n(n-1) + 6(d_1 + d_2 \cos^2 \theta) - 2n + 2]c}{4n(n-1)}. \]
where \(2d_1 = \dim D_1\) and \(2d_2 = \dim D_2\).

**Theorem 3.4** Let \(M\) be an \(n\)-dimensional \(\theta\)-slant submanifold tangent to \(\xi\) into a \((2m+1)\)-dimensional cosymplectic space form \(\tilde{M}(c)\). Then, for any integer \(k\) \((2 \leq k \leq n)\) and any point \(p \in M\), we have

\[
||H||^2 \geq \Theta_k(p) = \frac{n(n-1) + 3(n-1)\cos^2 \theta - 2n + 2}{4n(n-1)} c.
\]

**Proof.** Let \(\{e_1, \cdots, e_n\}\) be an orthonormal basis of \(T_pM\). Denote by \(L_{i_1\cdots i_k}\) the \(k\)-plane section spanned by \(e_{i_1}, \cdots, e_{i_k}\). It follows from (1.4) and (1.5) that

\[
\tau(L_{i_1\cdots i_k}) = \frac{1}{2} \sum_{i \in \{i_1, \cdots, i_k\}} \text{Ric}_{L_{i_1\cdots i_k}}(e_i),
\]

\[
\tau(p) = \frac{1}{\binom{n-2}{k-2}} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \tau(L_{i_1\cdots i_k}).
\]

Combining (1.6), (3.7) and (3.8), we obtain

\[
\tau(p) \geq \frac{n(n-1)}{2} \Theta_k(p).
\]

Therefore, by using (3.1) and (3.9) we can obtain the inequality in Theorem 3.4. \(\square\)

**Theorem 3.5** Let \(M\) be an \(n\)-dimensional bi-slant submanifold tangent to \(\xi\) into a \((2m+1)\)-dimensional cosymplectic space form \(\tilde{M}(c)\). Then, for any integer \(k\) \((2 \leq k \leq n)\) and any point \(p \in M\), we have

\[
||H||^2 \geq \Theta_k(p) = \frac{n(n-1) + 6d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2 - 2n + 2}{4n(n-1)} c,
\]

where \(2d_1 = \dim D_1\) and \(2d_2 = \dim D_2\).

**Theorem 3.6** Let \(M\) be an \(n\)-dimensional semi-slant submanifold tangent to \(\xi\) into a \((2m+1)\)-dimensional cosymplectic space form \(\tilde{M}(c)\). Then, for any integer \(k\) \((2 \leq k \leq n)\) and any point \(p \in M\), we have

\[
||H||^2 \geq \Theta_k(p) = \frac{n(n-1) + 6(d_1 + d_2 \cos^2 \theta) - 2n + 2}{4n(n-1)} c,
\]
where $2d_1 = \dim D_1$ and $2d_2 = \dim D_2$.

**Corollary 3.7** Let $M$ be an $n$-dimensional invariant submanifold tangent to $\xi$ into a $(2m+1)$-dimensional cosymplectic space form $\mathcal{M}(\mathfrak{c})$. Then, for any integer $k$ $(2 \leq k \leq n)$ and any point $p \in M$, we have

$$\|H\|^2 \geq \Theta_k(p) - \frac{(n+1)\mathfrak{c}}{4n}.$$ 

**Corollary 3.8** Let $M$ be an $n$-dimensional anti-invariant submanifold tangent to $\xi$ into a $(2m+1)$-dimensional cosymplectic space form $\mathcal{M}(\mathfrak{c})$. Then, for any integer $k$ $(2 \leq k \leq n)$ and any point $p \in M$, we have

$$\|H\|^2 \geq \Theta_k(p) - \frac{(n-2)\mathfrak{c}}{4n}.$$ 

**Corollary 3.9** Let $M$ be an $n$-dimensional contact $CR$-submanifold tangent to $\xi$ into a $(2m+1)$-dimensional cosymplectic space form $\mathcal{M}(\mathfrak{c})$. Then, for any integer $k$ $(2 \leq k \leq n)$ and any point $p \in M$, we have

$$\|H\|^2 \geq \Theta_k(p) - \frac{n(n-1) + 6d_1 - 2n + 2n}{4n(n-1)} \mathfrak{c}.$$ 

**References**


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