On Linear the Homeomorphism Between Function Spaces $C_p(X)$ and $C_{p,A}(X) \times C_p(A)$

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Abstract

In this paper, we investigate a linear homeomorphism between function spaces $C_p(X)$ and $C_{p,A}(X) \times C_p(A)$, where $X$ is a normal space and $A$ is a neighborhood retraction of $X$.

Key Words: Function Spaces; linear homeomorphism, pointwise topology.

1. Introduction

In [2] Jan Baars and J. D. Groot derived an isomorphical classification of the spaces $C_p(X)$, where $X$ denotes any compact zero-dimensional space. In [6] J. Van Mill derived a isomorphical classification of the spaces $C_p(X)$, where $X$ denotes any metrizable space. It has been proved in [6] that for metrizable spaces, there always exist an extender which is both linear and continuous.

First we fix some notation and give some definitions.

For a space $X$, we define $C(X)$ to be the set real-valued continuous functions on $X$, and $C(X)$ is vector space with the natural addition and scalar multiplication. For a covering $\mathcal{K}$ of $X$, we define a topology on $C(X)$ by taking the family of all sets

$$\langle f, K, \delta \rangle = \{g \in C(X) : |f(x) - g(x)| < \delta, \text{ for every } x \in K\},$$

where $f \in C(X)$, $K \in \mathcal{K}$ and $\delta > 0$, as a subbase.

If $\mathcal{K}$ consists of all finite subsets of $X$, we denote $C(X)$ endowed with this topology
by \( C_p(X) \). The topology on \( C_p(X) \) is called the pointwise convergence topology. It is well known or easy to prove that \( C_p(X) \) is a topological vector space.

Let \( X \) be a space and \( A \subset X \) closed. By \( C_{p,A}(X) \), we denote the subspace of \( C_p(X) \) of all functions vanishing on \( A \). That is,

\[
C_{p,A}(X) = \{ f \in C_p(X) : f(A) = 0 \} .
\]

If \( A \) is singleton, say \( \{a\} \), then we denote \( C_{p,A}(X) \) simply by \( C_{p,a}(X) \). Let \( X/A \) be the quotient space obtained from \( X \) by identifying \( A \) to a single point, say \( \infty \). \( C_{p,\infty}(X/A) \) is the space of \( C_p(X/A) \) of all function vanishing at \( \infty \). That is,

\[
C_{p,\infty}(X/A) = \{ f \in C_p(X/A) : f(\infty) = 0 \}
\]

Let the constant function with value 0 be denoted by 0.

**Definition 1** Let \( X \) be a space with subspace \( A \). We say that \( A \) is a retract of \( X \) provided that there is a continuous function \( r : X \to A \) such that \( r \) restricted to \( A \) is the identity on \( A \). Such a function \( r \) is called a retraction.

**Lemma 1** [6] Let \( X \) be a Hausdorff space with subspace \( A \). If \( A \) is a retract of \( X \) then \( A \) is a closed subset of \( X \).

**Proof.** Let \( (X, \tau) \) be a Hausdorff space and \( r : X \to A \) be a retraction. We want to show that \( A \) is closed in \( X \). Take any point \( x_0 \) in \( X \setminus A \). Then \( r(x_0) = a \in A \). Since \( r \) is a retraction it comes to be \( x_0 \neq a \) and since \( X \) is a Hausdorff space, there are two open subsets \( U \in \tau \) \((x_0 \in U) \) and \( V \in \tau \) \((a \in V) \) such that \( U \cap V = \emptyset \). That \( A \) is a subspace of \( X \) makes \( A \cap V \) open in \( (A, \tau_A) \), and so long as \( r \) is a continuous function, \( r^{-1}(A \cap V) \) is open in \( (X, \tau) \) and \( x_0 \in r^{-1}(A \cap V) \). Let \( W = U \cap r^{-1}(A \cap V) \). The set \( W \) is open in \( (X, \tau) \), \( x_0 \in W \) and \( W \cap V = \emptyset \). Since \( r \) is a retraction, \( r(W) \subset V \). So for every \( x \in W \), we get \( r(x) \neq x \) and thus we have \( W \subset X \setminus A \). Thus \( X \setminus A \) is open in \( (X, \tau) \) and \( A \) is closed in \( X \). \( \square \)

**Remark 1.** The statement of Lemma 1 is not necessarily true if \( X \) is not Hausdorff. For instance, the subset \( Z_e \) of all even integers of the cofinite topology defined on \( Z \) is an example of a non-closed retract under the continuous function \( f : Z \to Z_e \) where \( f(2k - 1) = 2k = f(2k) \) for each \( k \in Z \). Notice that \( f^{-1}(F) \) is finite whenever \( F \subset Z_e \).
is finite and thus $f$ is continuous and furthermore cofinite topology determines a $T_1$ topological space on $Z$ which is not $T_2$ (Hausdorff).

We say that $A$ is a neighborhood retract of $X$ provided that there exists a neighborhood $U$ of $A$ in $X$ such that $A$ is a retract of $U$.

Now we prove theorem 1 which will be used in the proof of the theorem 2.

**Theorem 1** Let $X$ be a normal space and $A$ be a neighborhood retract of $X$. Then there is a continuous linear and one to one function $\Phi : C_p(A) \to C_p(X)$ such that for each $f \in C(A)$, $\Phi(f)|_A = f$.

**Proof.** Let $U$, including $A$, be an open subset of $X$ and $r : U \to A$ be a retraction. Since $X$ is a normal space, for an open subset $W$ of $X$,

$$A \subseteq W \subseteq clW \subseteq U$$

$A$ is closed in $U$ because $A$ is a retract of $U$. Then $A$ is a closed subset of $clW$ and also a closed subset of $X$. Hence $A$ and $X\setminus W$ are two disjoint closed subset of $X$. Then for a continuous function

$$f_0 : X \to [0, 1]$$

we get $f_0(A) = \{1\}$ and $f_0(X\setminus W) = \{0\}$. Define

$$\Phi(f)(x) = \begin{cases} 0 & \text{if } x \in X\setminus W \\ f_0(x) f(r(x)) & \text{if } x \in W \end{cases}$$

for $f \in C_p(A)$. We want to show that $\Phi(f) \in C_p(X)$. In other words,

$$\Phi(f) : X \to \mathbb{R}$$

is continuous. If $x = a \in A$ then

$$\Phi(f)(a) = f_0(a) f(r(a)) = 1 f(a) = f(a).$$

From this, we get $\Phi(f)|_A = f$. $\Phi(f)$ is continuous on $W$ as $\Phi(f)(x) = f_0(x) f(r(x))$ and $W$ is open. Now take $x \in X\setminus W$. We claim that $\Phi(f)$ is continuous at $X\setminus W$. \square
We prove this latter claim via the following two cases.

Case 1. Let \( x \in clW \setminus W \) and let \( \{x_\mu\}_{\mu \in \Gamma} \), which converges to element, \( x \) be a net. We want to show that
\[
(\Phi (f) (x_\mu))_{\mu \in \Gamma} \to \Phi (f) (x),
\]
since \( x \) is an element of \( U \) and \( U \) is open; a tail of this net will be in \( U \). For this reason, without lose of generality, we can assume that all the elements of this net are in \( U \). As \( x_\mu \to x \) and \( r : U \to A \) are continuous,
\[
(r (x_\mu))_{\mu \in \Gamma} \to r (x)
\]
in \( A \). Furthermore,
\[
(f (r (x_\mu)))_{\mu \in \Gamma} \to f (r (x))
\]
due to the continuity of \( f : A \to R \). Since \( x \in X\setminus W \), \( \Phi (f) (x) = 0 \) and \( f_0 (x) = 0 \). Then we get
\[
(f_0 (x_\mu))_{\mu \in \Gamma} \to 0.
\]
On the other hand,
\[
\Phi (f) (x_\mu) = \begin{cases} 
0 & \text{if } x_\mu \in U \setminus W \\
f_0 (x_\mu) f (r (x_\mu)) & \text{if } x_\mu \in W
\end{cases}
\]
In every case, \( \Phi (f) (x_\mu) = f_0 (x_\mu) f (r (x_\mu)) \). Then it is seen that
\[
(\Phi (f) (x_\mu))_{\mu \in \Gamma} \to \Phi (f) (x) = 0.
\]

Case 2. Let \( x \in X \setminus clW \). Since \( X \setminus clW \) is open, \( \Phi (f) = 0 \) is continuous on \( X \setminus clW \). We show that \( \Phi \) is a linear. Let \( f, g \in C_p (A) \), \( \alpha, \beta \in R \)
\[
\Phi (\alpha f + \beta g) (x) = \begin{cases} 
\alpha 0 + \beta 0 = 0 & \text{if } x \in X \setminus W \\
f_0 (x) (\alpha f + \beta g) (r (x)) & \text{if } x \in W
\end{cases}
\]
\[
= \begin{cases} 
\alpha 0 & \text{if } x \in X \setminus W \\
f_0 (x) (\alpha f) (r (x)) & \text{if } x \in W
\end{cases} + \begin{cases} 
\beta 0 & \text{if } x \in X \setminus W \\
f_0 (x) (\beta g) (r (x)) & \text{if } x \in W
\end{cases}
\]
\[
= \alpha \Phi (f) (x) + \beta \Phi (g) (x)
\]
Thus $\Phi$ is linear. Now let us show that $\Phi : C_p (A) \to C_p (X)$ is continuous. Since $\Phi$ is linear, $C_p (A)$ and $C_p (X)$ are topological vector spaces, it is sufficient to prove that $\Phi$ is continuous at 0.

\[
\langle 0, \{x_0, x_1, \ldots, x_n \} \rangle = \bigcap_{i=0}^{n} \langle 0, \{x_i \} \rangle.
\]

Let us choose $x_0 \in X$ and consider the open set

\[
\langle 0, \{x_0 \} \rangle = \{ f \in C_p (X) : |f(x_0)| < \varepsilon \} = T
\]

We want to show that

\[
\Phi (\langle 0, \{a \}, \delta \rangle) = \Phi (\{ g \in C_p (A) : |g(a)| < \delta \}) \subseteq T
\]

for $a \in A$ and $\delta > 0$. Let us assume that $a \in A$ and $g \in \langle 0, \{a \}, \delta \rangle$. Then

\[
\Phi (g) (x_0) = \begin{cases} 
0 & \text{if } x_0 \in X \setminus W \\
 f_0 (x_0) g (r(x_0)) & \text{if } x_0 \in W
\end{cases}
\]

If $x_0 \in W$, then take $a = r(x_0)$ and $0 < \delta = \varepsilon / (f_0 (x_0) + 1)$. Then we have $a$ and $\delta > 0$. Because,

\[
|\Phi (g) (x_0)| = |f_0 (x_0)||g(a)| < f_0 (x_0) / (f_0 (x_0) + 1) < 1.
\]

Hence, $\Phi (g) \in T$. If $x_0 \in X \setminus W$ then $|\Phi (g) (x_0)| = 0 < 1$ for any $a$ which is chosen from $A$. This implies $\Phi (g) \in T$. Therefore since $\Phi (g) \in T$ for $g \in \langle 0, \{a \}, \delta \rangle$, $\Phi$ is continuous on each two cases. $\Phi$ is one to one, it is seen by definition of $\Phi$ easily.

We now come to the following important theorem.

**Theorem 2** Let $X$ be a normal space and let $A$ be a neighborhood retract of $X$. Then

\[
C_p (X) \cong C_{p,A} (X) \times C_p (A).
\]

**Proof.** Define $G : C_p (X) \to C_p (A)$ by $G (f) = f|_A$. Notice that $G$ is a continuous linear function. For $f \in C_p (X)$, $f \in C_{p,A} (X)$, if and only if $G (f) = 0$. By theorem 1, there is a continuous linear function $\Phi : C_p (A) \to C_p (X)$ such that for each $f \in C_p (A)$,

\[
\Phi (f)|_A = f. \text{ Notice that } G \circ \Phi = id_{C_p (A)}.
\]

Now define $\theta : C_p (X) \to C_{p,A} (X) \times C_p (A)$ by

\[
\theta (f) = (f - (\Phi \circ G) (f), G (f)).
\]

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We have to prove that \( \theta \) is well-defined. Take an arbitrary \( f \in C_p(X) \). It is obvious that \( G(f) \in C_p(A) \) and that \( f - (\Phi \circ G)(f) \in C_p(X) \). Furthermore,

\[
G(f - (\Phi \circ G)(f)) = G(f) - (G \circ \Phi \circ G)(f) = G(f) - G(f) = 0
\]

so \( f - (\Phi \circ G)(f) \in C_{p,A}(X) \). That \( \theta \) is continuous and linear is a triviality. We show that \( \theta \) is a linear homeomorphism. For that, define

\[
\Gamma : C_{p,A}(X) \times C_p(A) \to C_p(X)
\]

By \( \Gamma(f, h) = f + \Phi(h) \) it is trivial that \( \Gamma \) is well defined, continuous and linear. Furthermore, as is easily seen, \( \Gamma \circ \theta = id_{C_p(X)} \) and we show that \( \theta \circ \Gamma = id_{C_{p,A}(X) \times C_p(A)} \).

Take \( f \in C_{p,A}(X) \) and \( h \in C_p(A) \). Notice that \( G(f) = 0 \) hence by linearity of \( \Phi \),

\[
(\Phi \circ G)(f) = \Phi(0) = 0,
\]

so

\[
(\theta \circ \Gamma)(f, h) = \theta(f + \Phi(h)) = (f + \Phi(h) - (\Phi \circ G)(f + \Phi(h)))G(f + \Phi(h))
\]

\[
= (f + \Phi(h) - 0 - \Phi(h))G(f + \Phi(h))
\]

\[
= (f, h).
\]

Hence \( \theta \circ \Gamma = id_{C_{p,A}(X) \times C_p(A)} \), i.e., \( \theta \) is a linear homeomorphism.

\[
\Box
\]

**Lemma 2** Let \( X \) be a normal space and \( A \) be a neighborhood retract of \( X \). Then

\[
C_{p,A}(X) \approx C_{p,\infty}(X/A).
\]

**Proof.** Let \( p : X \to X/A \) be the quotient map between \( X \) and \( X/A \). For every function \( f \in C_{p,A}(X) \) there is a unique function \( g \in C_{p,\infty}(X/A) \) such that \( g \circ p = f \). If we now define \( \theta : C_{p,A}(X) \to C_{p,\infty}(X/A) \) by \( \theta(f) = g \), then \( \theta \) is a well-defined linear bijection.

Since for \( f \in C_{p,A}(X), y_1, ..., y_n \in X/A, \delta > 0 \) and \( x_i \in p^{-1}(y_i), (i \leq n) \) it is easily seen that

\[
\theta((f, \{x_1, ..., x_n\}, \delta)) = (\theta(f), \{y_1, ..., y_n\}, \delta),
\]

and it follows that \( \theta \) is linear homeomorphism.

\[
\Box
\]

From the last lemma and theorem 2, we have the useful following corollary.
Corollary 1 Let $X$ be a normal space and let $A$ be a neighborhood retract of $X$. Then

$$C_p(X) \cong C_{p,\infty}(X/A) \times C_{p,A}(X).$$

Proof. By lemma 2 and theorem 2

$$C_p(X) \cong C_{p,\infty}(X/A) \times C_{p,A}(X).$$

References


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