Formulas for the Fourier Coefficients of Cusp Form for Some Quadratic Forms

Ahmet Tekcan

Abstract

In this paper, representations of positive integers by certain quadratic forms $Q_p$ defined for odd prime $p$ are examined. The number of representations of positive integer $n$ by the quadratic form $Q_p$, is denoted by $r(n; Q_p)$, obtained for $p = 3, 5$ and $7$. We prove that $r(n; Q_p) = \rho(n; Q_p) + \vartheta(n; Q_p)$ for $p = 3, 5$ and $7$, where $\rho(n; Q_p)$ is the singular series and $\vartheta(n; Q_p)$ is the Fourier coefficient of cusp form.

Key Words: representation of numbers, quadratic forms, generalized theta series, Fourier coefficient of cusp forms.

1. Introduction.

Let

$$Q = Q(x_1, x_2, \ldots, x_k) = \sum_{1 \leq r \leq s \leq k} b_{rs} x_r x_s$$

be a positive quadratic form of discriminant $\Delta$ in $k$ variables with integral coefficients $b_{rs}$. Let $A$ be $k \times k$ symmetric matrix corresponding to $Q$ such that $(r, s)$-element of which is $b_{rs}$ (but the diagonal elements are $2b_{rr}$). Define the determinant of $A$ to be the discriminant of the quadratic form $Q$, i.e. $\det(A) = \Delta$.

Consider the quadratic form

$$2Q = \sum_{r<s=1}^{k} a_{rs} x_r x_s, \ (a_{rr} = 2b_{rr}, \ a_{rs} = a_{sr} = b_{rs}, \ r < s)$$

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of discriminant \( \bar{D} \). Then \( \Delta = (-1)^{k/2} \bar{D} \). Let \( A_{rs} \) be the algebraic cofactor of elements \( a_{rs} \) in \( \bar{D} \), \( \delta = \gcd \left( \frac{A_{rr}}{c}, A_{rs} \right) \), \( N = \frac{\bar{D}}{\delta} \) be the level of the form \( Q \) and \( \chi(d) \) be the character of the form \( Q \), i.e. \( \chi(d) = 1 \) if \( \Delta \) is a perfect square; but if \( \Delta \) is not a perfect square and \( 2 \nmid \Delta \), then \( \chi(d) = \left( \frac{d}{\Delta} \right) \) for \( d > 0 \) and \( \chi(d) = (-1)^{k/2} \chi(-d) \) for \( d < 0 \), where \( \left( \frac{d}{\Delta} \right) \) is the generalized Jacobi symbol.

A positive quadratic form in \( k \) variables of level \( N \) and character \( \chi(d) \) is called a quadratic form of the type \((-k^2, N, \chi)\). Let \( P_v = P_v(x_1, x_2, ..., x_k) \) be the spherical function of order \( v \) with respect to the quadratic form \( Q \).

Let \( \Gamma(1) \) denote a full modular group and \( \Gamma \) any subgroup of a finite index in \( \Gamma(1) \). In particular,

\[
\begin{align*}
\Gamma_0(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) : c \equiv 0 \pmod{N} \right\}, \\
\Gamma_1(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) : a \equiv d \equiv 1 \pmod{N} \right\}, \\
\Gamma(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N) : b \equiv 0 \pmod{N} \right\}
\end{align*}
\]

for \( N \in \mathbb{N} \).

Let \( G_k(\Gamma, \chi) \) and \( S_k(\Gamma, \chi) \) denote the space of entire modular and cusp forms, respectively, of the type \((k, \Gamma, \chi)\). If \( F(\tau) \in G_k(\Gamma, \chi) \), then in the neighbourhood of the cusps \( \zeta = i\infty \)

\[
F(\tau) = \sum_{m=m_0 \geq 0}^{\infty} a_m z^m, \quad a_{m_0} \neq 0.
\]

The order of an entire modular form \( F(\tau) \neq 0 \) of the type \((k, \Gamma, \chi)\) at the cusps \( \zeta = i\infty \) with respect to \( \Gamma \) is

\[
\text{ord} (F(\tau), i\infty, \Gamma) = m_0.
\] (1.1)

In this case we called \( a_{m_0} \) as the coefficient of order and denote by \( a_{m_0} (F(\tau)) \).

Let \( F(\tau) \) be any function on the upper half plane \( U \) and \( m \in \mathbb{Z} \). Then for any matrix
\[
L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1), \text{ let } F(\tau)|_m L = (c\tau + d)^{-m} F(L\tau) \text{ and }
\]

\[
\varphi(\tau; Q, P_v(x), h) = \sum_{n_i \equiv h_i \pmod{N}} P_v(n_1, n_2, \ldots, n_k) z^{\frac{1}{N} Q(n_1, n_2, \ldots, n_k)} (1.2)
\]

and

\[
\varphi(\tau; Q, P_v(x)) = \sum_{n=1}^{\infty} \left( \sum_{Q(x) = n} P_v(x) \right) z^n, \quad (1.3)
\]

where \(Q(x) = \frac{1}{2} \sum_{r,s=1}^{k} a_{rs} x_r x_s\) is a quadratic form of the type \((\frac{1}{2}, N, \chi)\), \(P_v(x)\) is a spherical function of order \(v\) with respect to the \(Q\); \(n_1, n_2, \ldots, n_k\) are integers and \(h = (h_1, h_2, \ldots, h_k)\), where \(h_i\) are integers such that

\[
\sum_{s=1}^{k} a_{rs} h_s \equiv 0 \pmod{N}, \quad (r = 1, 2, \ldots, k). \quad (1.4)
\]

It is well known, to each positive quadratic form \(Q\), there corresponds the theta series

\[
\varphi(\tau; Q) = 1 + \sum_{n=1}^{\infty} t(n; Q) z^n, \quad (1.5)
\]

where \(r(n; Q)\) the number of representations of positive integer \(n\) by the quadratic form \(Q\).

Any positive quadratic form \(Q\) of the type \((-k, q, 1)\), \(k > 2\), \(2|k\), \(z = e^{2\pi i \tau}, \text{Im}(\tau) > 0\), corresponds to one and the same Eisenstein series defined by

\[
E(\tau; Q) = 1 + \sum_{n=1}^{\infty} (\alpha \sigma_{k-1}(n) z^n + \beta \sigma_{k-1}(n) z^{\sigma_{k-1}(n)}) = 1 + \sum_{n=1}^{\infty} \rho(n; Q) z^n \quad (1.6)
\]

for

\[
\alpha = \frac{i^k}{\rho_k} q^{k/2} - \frac{i^k}{q^k - 1}, \quad \beta = \frac{1}{\rho_k} q^k - \frac{k}{\rho_k} q^{k/2}, \quad \rho_k = (-1)^{k/2} \frac{(k - 1)!}{(2\pi)^k} \zeta(k), \quad (1.7)
\]

where \(\zeta(k)\) is the Riemann zeta function, \(\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}\) and \(\rho(n; Q)\) is the singular series defined in following lemma.
Lemma 1.1 1. If $2 | k$, $v = \prod_{p | n, p \neq 2} p^w$, $\Delta = r^2w$, ($w$ is a square-free number), then

$$
\rho(n; Q) = \frac{\pi^k}{\Gamma(\frac{k}{2})} \frac{1}{\Delta^\frac{k}{2}} n^\frac{k}{2} \chi_2 \prod_{p | n, p > 2} \chi_p \times
\prod_{p | n, p > 2} \left( 1 - \left( \frac{(-1)^{\frac{k}{2}w}}{p} \right)^{-1} \right) 
\times L^{-1} \left( \frac{k}{2}, (-1)^{\frac{k}{2}w} \right) \sum_{d | v} \left( \frac{-1}{d} \right) d^{1-\frac{k}{2}}
$$

2. If $2 \nmid k$, $\Delta n = 2^{\alpha+\gamma} v_1 v_2 = r^2w$, $2^\alpha | n$, $2^\gamma | \Delta$, $p^\gamma | \Delta$, $p^w | n, (p > 2)$, $v_1 = \prod_{p | n, p \neq 2} p^w = r_1^2 w_1$, $v_2 = \prod_{p | n, p \neq 2} p^w = r_2^2 w_2$, ($w, w_1$ and $w_2$ are square-free integers). Then

$$
\rho(n; Q) = \frac{r_1^{2-k} n^\frac{k}{2}(k-1)!}{\Gamma(\frac{k}{2})} \frac{1}{\Delta^\frac{k}{2}} \frac{1}{B_{k-1}} \chi_2 \prod_{p | n, p > 2} \chi_p \times
\prod_{p | n, p > 2} (1 - p^{1-k})^{-1} L \left( \frac{k-1}{2}, (-1)^{\frac{k-1}{2}w} \right) 
\times \prod_{p | n, p > 2} \left( 1 - \left( \frac{(-1)^{\frac{k-1}{2}w}}{p} \right)^{\frac{k-1}{2}} \right) 
\times \sum_{d | v_1} d^{k-2} \prod_{p | d} \left( 1 - \left( \frac{(-1)^{\frac{k-1}{2}w}}{p} \right)^{\frac{k-1}{2}} \right),
$$

where $B_{k-1}$ are Bernoulli’s numbers, $\left( \frac{-1}{p} \right)$ is Jacobi symbol. [2]

The values of $\chi_2$ are given as

$$
\chi_2 = 1 \text{ for } 2 \nmid k, \alpha = 0, \text{ or for } 2 | k, \alpha = 0, u \equiv 1(\text{mod } 4) \text{ or } 2 | k, \alpha = 1,
\chi_2 = 1 + (-1)^{\frac{k-1}{2}-2^\frac{k}{2}}, \text{ for } 2 | k, \alpha = 0, u \equiv 3(\text{mod } 4),
$$

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\[
1 + \frac{2^{k-3}(1 - 2^{\frac{n}{2}} \cdot 63)}{31}, \text{ for } 2 | k, 2 | \alpha, \ u \equiv 1(\text{mod } 4),
\]
\[
1 + \frac{2^{k-3}(1 - 2^{\frac{n}{2}} + (-1)^{\frac{n-1}{2}}2^{-\frac{n}{2}} \cdot 31)}{31}, \text{ for } 2 | k, 2 | \alpha, \ u \equiv 3(\text{mod } 4),
\]
\[
1 + \frac{2^{k-3}(1 - 2^{\frac{n}{2}} \cdot 63)}{31}, \text{ for } 2 | k, 2 \nmid \alpha, \alpha > 1,
\]
\[
1 + \frac{2^{k-3}(1 - 2^{\frac{n}{2}} \cdot 63)}{31}, \text{ for } 2 | k, 2 | \alpha, \ u \equiv 1(\text{mod } 4),
\]
\[
1 + \frac{2^{k-3}(1 - 2^{\frac{n}{2}} - \frac{1}{2} \cdot 63)}{31}, \text{ for } 2 | k, 2 \nmid \alpha, u \equiv 3(\text{mod } 4).
\]

**Lemma 1.2** If \( \varphi(\tau; Q, P_v(x), h) \) is not identically equal to zero, then
\[
\varphi(\tau; Q, P_v(x), h) \in G_{v\tau}(\Gamma(N)). [1]
\]

**Lemma 1.3** If \( Q \) is a quadratic form of the type \((k, q, 1)\) or \((k, q, \chi)\), then
\[
\varphi(\tau; Q) - E(\tau; Q)
\]
is a cusp form of the type \((k, \Gamma_0(q), 1)\) or \((k, \Gamma_0(q), \chi)\), respectively. [1]

Let \( r(n; Q) \) denote the number of representations of positive integer \( n \) by the quadratic form \( Q \) in \( k \) variables. Then it is well known that \( r(n; Q) \) can be represented as
\[
r(n; Q) = \rho(n; Q) + \vartheta(n; Q), \tag{1.8}
\]
where \( \rho(n; Q) \) is the singular series and \( \vartheta(n; Q) \) is the Fourier coefficient of cusp form.

This can be represented in terms of the theory of modular forms by stating that
\[
\varphi(\tau; Q) = E(\tau; Q) + X(\tau; Q), \tag{1.9}
\]
where \( E(\tau; Q) \) is the Eisenstein series defined in (1.6) and \( X(\tau; Q) \) is a cusp form.

If the genus of the quadratic form \( Q \) contains one class, then from Siegel’s Theorem \( \varphi(\tau; Q) = E(\tau; Q) \); but if the genus of the quadratic form \( Q \) contains more than one class, then we need to find a cusp form \( X(\tau; Q) \).
In [3], Vepkhvadze constructed generalized theta functions with characteristic and spherical functions

\[ \varphi_{gh}(\tau; P_v, Q) = \sum_{x \equiv g \pmod{N}} (-1)^{\frac{\Delta_A(x - g)}{N}} P_v(x)e^{\frac{2\pi i x A}{N}}. \] (1.10)

Here \( g \) and \( h \) are special vectors with respect to the matrix \( A \) of form \( Q \), i.e. \( Ag \equiv 0 \pmod{N} \), \( Ah \equiv 0 \pmod{N} \), where \( N \) is a level of the form \( Q \), \( P_v = P_v(x) = (x_1, ..., x_k) \) is a spherical function of order \( v \) with respect to \( Q \).

**Lemma 1.4** Let \( K \) be an arbitrary integral vector, and \( L \) be a special vector with respect to the matrix \( A \) of the form \( Q \). Then the equalities

\[ \varphi_{g+ NK, h}(\tau; P_v, Q) = (-1)^{\frac{kA}{N}} \varphi_{gh}(\tau; P_v, Q), \]

\[ \varphi_{g, h+2L}(\tau; P_v, Q) = \varphi_{gh}(\tau; P_v, Q) \]

are satisfied.

For \( M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(N) \) denote

\[ v(M) = \left( i \frac{\gamma}{\delta} \right)^{k+2v} (\text{sgn} \delta)^v \left( \frac{2\Delta(\text{sgn} \delta) \beta}{|\delta|} \right) \left( -\frac{1}{|\delta|} \right), \]

where \( \eta(\gamma) = 1 \) for \( \gamma \geq 0 \), \( \eta(\gamma) = -1 \) for \( \gamma < 0 \). By \( v_0(M) \) we denote \( v(M) \) for \( v = 0 \).

**Lemma 1.5** Let \( Q_s = Q_s(x) \) \((s = 1, 2, ..., j)\) be an integral positive quadratic form with \( k \) variables, \( P^{(s)}_v = P^{(s)}_v(x) \) the corresponding spherical functions, \( A_s \) is a matrix of the form \( Q_s(x) \), \( \Delta_s \) be the discriminant of the matrix \( A_s \), and \( N_s \) the level of the form \( Q_s \). Moreover let \( g^{(s)} \) and \( h^{(s)} \) be vectors with even components and \( B_s \) be arbitrary complex number. Then the function

\[ X(\tau; Q_s) = \sum_{s=1}^j B_s \varphi_{g^{(s)}, h^{(s)}}(\tau; P^{(s)}_v, Q_s) \] (1.11)

is an integral modular form of the type \( -\left( \frac{k}{2} + v \right), N, v_0(M) \) iff the conditions

\[ N_s | N, \quad N_s^2 | Q_s(g^{(s)}) \quad \text{and} \quad 4N_s \frac{N}{N_s} Q_s(h^{(s)}) \]
are satisfied and for all \( \alpha \) and \( \delta \) satisfying the condition \( \alpha \delta \equiv 1 \pmod{N} \) we get

\[
\sum_{s=1}^{j} B_s \Psi_{\alpha g^{(s)}, -h^{(s)}}(\tau; P_v^{(s)}, Q_s)(\text{sgn}\delta)^v \left( \frac{(-1)^{\frac{1+v}{2}} \Delta_s}{|\delta|} \right)
\]

\[
= \left( \frac{(-1)^{\frac{1+v}{2}} + v \Delta}{|\delta|} \right) \sum_{s=1}^{j} B_s \Psi_{\alpha g^{(s)}, -h^{(s)}}(\tau; P_v^{(s)}, Q_s).
\]

**Lemma 1.6** If all conditions of Lemma 1.5 are satisfied and \( v > 0 \) then \( X(\tau; Q_s) \) defined in (1.11) is a cusp form of the type \( -(\frac{1}{2} + v), N, v_0(M) \) [2].

2. **Formulas for the Fourier Coefficients of Cusp Form for Some Quadratic Forms**

In the present paper, we obtain the formulas for the Fourier coefficient of cusp form for the quadratic form

\[
Q_p = p \sum_{1 \leq i < j \leq p-2} x_i x_j + p \sum_{1 \leq i \leq p-2} x_i x_{p-1} + \frac{p-1}{2} x_{p-1}^2
\]

with \( p-1 \) variables.

**Theorem 2.1** Let \( Q_p \) be the quadratic form defined in (2.1). Then the discriminant of \( Q_p \) is

\[
\begin{cases} 
-3 & \text{if } p = 3 \\
\frac{p-2}{2} & \text{if } p > 3.
\end{cases}
\]

**Proof.** For \( p = 3 \) we obtain the form

\[
Q_3 = 3x_1^2 + 3x_1x_2 + x_2^2
\]

which is a binary quadratic form. The discriminant of \( Q_3 \) is \(-3\).

We know that for \( p \geq 3 \) the discriminant of \( Q_p \) is the determinant of the matrix \( A_p \) which corresponds to \( Q_p \). The matrix \( A_p \) is

\[
A_p = \left( \begin{array}{ccccccc}
p & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & p & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & p & \cdots & \frac{1}{2} & \frac{1}{2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\
\end{array} \right)_{(p-1) \times (p-1)}
\]

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We want to find the determinant of $A_p$. To get this, using row operations, in the first step we obtain

\[
\begin{bmatrix}
p & \frac{p}{2} & \frac{p}{2} & \cdots & \frac{p}{2} \\
p & \frac{p}{2} & \frac{p}{2} & \cdots & \frac{p}{2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p & \frac{p}{2} & \frac{p}{2} & \cdots & \frac{p}{2} \\
p & \frac{p}{2} & \frac{p}{2} & \cdots & \frac{p}{2} \\
\end{bmatrix}
\]

\[
+ \frac{p}{2} (-1)^{1+1}
\]

\[
\begin{bmatrix}
p & \frac{p}{2} & \frac{p}{2} & \cdots & \frac{p}{2} \\
p & \frac{p}{2} & \frac{p}{2} & \cdots & \frac{p}{2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p & \frac{p}{2} & \frac{p}{2} & \cdots & \frac{p}{2} \\
p & \frac{p}{2} & \frac{p}{2} & \cdots & \frac{p}{2} \\
\end{bmatrix}
\]

\[
+ \frac{p}{2} (-1)^{1+3}
\]

\[
\begin{bmatrix}
p & \frac{p}{2} & \frac{p}{2} & \cdots & \frac{p}{2} \\
p & \frac{p}{2} & \frac{p}{2} & \cdots & \frac{p}{2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p & \frac{p}{2} & \frac{p}{2} & \cdots & \frac{p}{2} \\
p & \frac{p}{2} & \frac{p}{2} & \cdots & \frac{p}{2} \\
\end{bmatrix}
\]

\[
+ \frac{p}{2} (-1)^{1+(p-2)}
\]

\[
\begin{bmatrix}
p & \frac{p}{2} & \frac{p}{2} & \cdots & \frac{p}{2} \\
p & \frac{p}{2} & \frac{p}{2} & \cdots & \frac{p}{2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p & \frac{p}{2} & \frac{p}{2} & \cdots & \frac{p}{2} \\
p & \frac{p}{2} & \frac{p}{2} & \cdots & \frac{p}{2} \\
\end{bmatrix}
\]

If we continue in the same way we obtain

\[
\begin{bmatrix}
p & \frac{p}{2} & \frac{p}{2} & \cdots & \frac{p}{2} \\
p & \frac{p}{2} & \frac{p}{2} & \cdots & \frac{p}{2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p & \frac{p}{2} & \frac{p}{2} & \cdots & \frac{p}{2} \\
p & \frac{p}{2} & \frac{p}{2} & \cdots & \frac{p}{2} \\
\end{bmatrix}
\] = \frac{p^{p-3}}{2^{p-3}},

\[
\begin{bmatrix}
p & \frac{p}{2} & \frac{p}{2} & \cdots & \frac{p}{2} \\
p & \frac{p}{2} & \frac{p}{2} & \cdots & \frac{p}{2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p & \frac{p}{2} & \frac{p}{2} & \cdots & \frac{p}{2} \\
p & \frac{p}{2} & \frac{p}{2} & \cdots & \frac{p}{2} \\
\end{bmatrix}
\] = \frac{p^{p-3}}{2^{p-2}},
i.e. the determinant of the first matrix is \( \frac{p^{p-3}}{2^{p-2}} \), the determinant of the second, third, 
............(p–2)--th matrix are same and is \( \pm \frac{p^{p-3}}{2^{p-2}} \), and the determinant of the last ((p–1)--th) 
matrix is \( \frac{p^{p-2}}{2^{p-2}} \). Hence 

\[
\det(A_p) = p \left( \frac{p^{p-3}}{2^{p-2}} \right) - p \left( \frac{p^{p-3}}{2^{p-2}} \right) + p \left( \frac{p^{p-3}}{2^{p-2}} \right) - p \left( \frac{p^{p-3}}{2^{p-2}} \right) + \ldots - \frac{p}{2} \left( \frac{p^{p-2}}{2^{p-2}} \right) \\
= \frac{p^{p-2}}{2^{p-3}} + (p-3) \frac{p^{p-2}}{2^{p-1}} - p^{p-1} \\
= \frac{2^2p^{p-2} + (p-3)p^{p-2} - p^{p-1}}{2^{p-1}}
\]
\[ \begin{align*}
&= \frac{4p^{p-2} + (\rho - 4)p^{p-2} + p^{p-2} - p^{p-1}}{2^{p-1}} \\
&= \frac{p^{p-2}(4 + \rho - 4) + p^{p-2} - p^{p-1}}{2^{p-1}} \\
&= \frac{p^{p-1} + p^{p-2} - p^{p-1}}{2^{p-1}} \\
&= \frac{p^{p-2}}{2^{p-1}}.
\end{align*} \]

Therefore the discriminant of \( Q_p \) is \( \frac{p^{p-2}}{2^{p-1}} \).

Now we obtain the formulas for the Fourier coefficients of cusp form for the quadratic form \( Q_p \) for \( p = 3, 5 \) and 7.

Let \( r(n; Q) \) denote the number of representations of positive integer \( n \) by the quadratic form \( Q \) in \( k \) variables. Then it is well known that \( r(n; Q) \) can be represented as

\[ r(n; Q) = \rho(n; Q) + \vartheta(n; Q), \]

where \( \rho(n; Q) \) is the singular series and \( \vartheta(n; Q) \) is the Fourier coefficient of cusp form.

**Theorem 2.2** For the quadratic form \( Q_3 \) the equality

\[ r(n; Q_3) = \rho(n; Q_3) + \vartheta(n; Q_3) \]

is satisfied, where \( r(n; Q_3) \) denote the number of representations of positive integer \( n \) by the quadratic form \( Q_3 \), \( \rho(n; Q_3) \) is the singular series and

\[ \vartheta(n; Q_3) = \begin{cases} 12 & \text{for } n = 1, \\ 18 & \text{for } n = 2, \\ 30 & \text{for } n = 3, \\ 48 & \text{for } n = 4, \\ 36 & \text{for } n = 5. \end{cases} \]
Proof. For the quadratic form $Q_3$, we get from (1.7) that $\alpha = -6$ for $\rho_2 = -\frac{1}{24}$.

Therefore from (1.6) we obtain

$$E(\tau; Q_3) = 1 + \sum_{n=1}^{\infty} \left( \alpha \sigma_{k-1}(n) z^n + \beta \sigma_{k-1}(n) z^{\eta n} \right)$$

$$= 1 + \sum_{n=1}^{\infty} \rho(n; Q_3) z^n$$

$$= 1 - 6 \left( z + 3z^2 + 4z^3 + 7z^4 + 6z^5 + \ldots \right). \quad (2.2)$$

Now consider the equation

$$Q_3(x_1, x_2) = n$$

for positive integer $n$.

This equation

1. has six integral solutions $(-1, 1), (-1, 2), (0, -1), (0, 1), (1, -2), (1, 1)$ for $n = 1$,
2. has no integral solution for $n = 2$ and $n = 5$,
3. has six integral solutions $(-2, 3), (-1, 0), (-1, 3), (1, -3), (1, 0), (2, -3)$ for $n = 3$,
4. has six integral solutions $(-2, 2), (-2, 4), (0, -2), (0, 2), (2, -4), (2, -2)$ for $n = 4$.

Therefore from (1.5) we obtain

$$\varphi(\tau; Q_3) = 1 + 6z + 6z^3 + 6z^4 + \ldots \quad (2.3)$$

Using (2.2) and (2.3) we get

$$X(\tau; Q_3) = \varphi(\tau; Q_3) - E(\tau; Q_3)$$

$$= 12z + 18z^2 + 30z^3 + 48z^4 + 36z^5 + \ldots \quad (2.4)$$

is a cusp form of the type $(1, \Gamma_0(3), \chi)$. Therefore from (2.4) it is clear that

$$\vartheta(n; Q_3) = \begin{cases} 
12 & \text{for } n = 1, \\
18 & \text{for } n = 2, \\
30 & \text{for } n = 3, \\
48 & \text{for } n = 4, \\
36 & \text{for } n = 5.
\end{cases}$$

\[\square\]
Theorem 2.3 For the quadratic form $Q_5$, the equality

\[ r(n; Q_5) = \rho(n; Q_5) + \vartheta(n; Q_5) \]

is satisfied, where $r(n; Q_5)$ denote the number of representations of positive integer $n$ by the quadratic form $Q_5$, $\rho(n; Q_5)$ is the singular series and

\[ \vartheta(n; Q_5) = \frac{1}{15881} \begin{cases} 
61440 & \text{for } n = 1, \\
394150 & \text{for } n = 2, \\
1402700 & \text{for } n = 3, \\
4485120 & \text{for } n = 4, \\
7423820 & \text{for } n = 5.
\end{cases} \]

Proof. For the quadratic form $Q_5$ we get from (1.7) that $\alpha = \frac{61440}{15881}$ for $\rho_4 = \frac{1}{240}$.
Therefore from (1.6) we obtain

\[ E(\tau; Q_5) = 1 + \sum_{n=1}^{\infty} (\alpha \sigma_{k-1}(n)z^n + \beta \sigma_{k-1}(n)z^{q_n}) \]

\[ = 1 + \sum_{n=1}^{\infty} \rho(n; Q_5)z^n \]

\[ = 1 + \frac{61440}{15881} \left( z + 9z^2 + 28z^3 + 73z^4 + 126z^5 + \ldots \right). \quad (2.5) \]

Now consider the equation

\[ Q_5(x_1, x_2, x_3, x_4) = n \]

for positive integer $n$.

This equation

1. has no integral solution for $n = 1$ and $n = 4$,
2. has ten integral solutions $(-1, -1, -1, 4), (-1, 0, 0, 1), (0, -1, 0, 1), (0, 0, -1, 1), (0, 0, 0, -1), (0, 0, 1, -1), (0, 1, 0, -1), (1, 0, 0, -1), (1, 1, 1, -4)$ for $n = 2$,
3. has twenty integral solutions $(-1, -1, -1, 3), (-1, -1, 0, 2), (-1, -1, 0, 3), (-1, 0, -1, 2), (-1, 0, -1, 3), (0, -1, -1, 2), (0, -1, -1, 3), (0, -1, 0, 2), (0, 0, -1, 2), (0, 0, 1, -2), (0, 1, 0, -2), (0, 1, 1, -3), (0, 1, 1, -2), (1, 0, 0, -2), (1, 0, 1, -3), (1, 0, 1, -2), (1, 1, 0, -3), (1, 1, 0, -2), (1, 1, 1, -3)$ for $n = 3$,
4. has twenty integral solutions $(-2, -1, -1, 5), (-1, -2, -1, 5), (-1, -1, -2, 5), (-1, -1, -1, 5), (-1, 0, 0, 0), (-1, 0, 1, 0), (-1, 1, 0, 0), (0, -1, 0, 0)$, for $n = 5$.
(0, 0, −1, 0), (0, 0, 1, 0), (0, 1, −1, 0), (0, 1, 0, 0), (1, −1, 0, 0), (1, 0, −1, 0), (1, 0, 0, 0),
(1, 1, 1, −5), (1, 1, 2, −5), (1, 2, 1, −5), (2, 1, 1, −5) for \( n = 5 \).

Therefore from (1.5) we obtain

\[
\varphi(\tau; Q_5) = 1 + 10z^2 + 20z^3 + 20z^5 + \ldots \quad (2.6)
\]

Using (2.5) and (2.6) we get

\[
X(\tau; Q_5) = -\frac{1}{15881} \left( 61440z + 394150z^2 + 1402700z^3 + 4485120z^4 + 7423820z^5 + \ldots \right)
\]

is a cusp form of the type \((2, \Gamma_0(5), \chi)\). Therefore from (2.7) it is clear that

\[
\vartheta(n; Q_5) = -\frac{1}{15881} \begin{cases} 
61440 & \text{for } n = 1, \\
394150 & \text{for } n = 2, \\
1402700 & \text{for } n = 3, \\
4485120 & \text{for } n = 4, \\
7423820 & \text{for } n = 5.
\end{cases}
\]

**Theorem 2.4** For the quadratic form \( Q_7 \) the equality

\[
\varrho(n; Q_7) = \rho(n; Q_7) + \vartheta(n; Q_7)
\]

is satisfied, where \( r(n; Q_7) \) denote the number of representations of positive integer \( n \) by the quadratic form \( Q_7 \), \( \rho(n; Q_7) \) is the singular series and

\[
\vartheta(n; Q_7) = \frac{1}{4747561247799} \begin{cases} 
-132120576 & \text{for } n = 1, \\
-435979008 & \text{for } n = 2, \\
664332048642 & \text{for } n = 3, \\
-139651448832 & \text{for } n = 4, \\
1989845623486982 & \text{for } n = 5.
\end{cases}
\]

**Proof.** For the quadratic form \( Q_7 \), we get from (1.7) that \( \alpha = \frac{132120576}{4747561247799} \) for \( \rho_6 = -\frac{1}{504} \). Therefore from (1.6) we obtain

\[
E(\tau; Q_7) = 1 + \sum_{n=1}^{\infty} (\alpha \sigma_{k-1}(n)z^n + \beta \sigma_{k-1}(n)z^n)
\]

\[
= 1 + \sum_{n=1}^{\infty} \rho(n; Q_7)z^n
\]

\[
\quad = 1 + \frac{132120576}{4747561247799} \left( z + 33z^2 + 244z^3 + 1057z^4 + 3126z^5 + \ldots \right). \quad (2.8)
\]

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Now consider the equation

$$Q_7(x_1, x_2, x_3, x_4, x_5, x_6) = n$$

for positive integer $n$.

This equation

1. has no integral solution for $n = 1, 2$ and 4,

2. has fourteen integral solutions $(-1, -1, -1, -1, -1, 6), (-1, 0, 0, 0, 0, 1)$,

3. has fortytwo integral solutions $(-1, -1, -1, -1, 5), (-1, -1, 0, -1, -1, 5)$,

4. has no integral solution for

$$Q(x) = 1 + 14z^3 + 42z^5 + \cdots$$

Using (2.8) and (2.9) we get

$$X(\tau; Q_7) = \varphi(\tau; Q_7) - E(\tau; Q_7)$$

is a cusp form of the type $(3, \Gamma_0(7), \chi)$. Therefore from (2.10) it is clear that

$$\vartheta(n; Q_7) = \begin{cases} 
-132120576 & \text{for } n = 1, \\
-4359979008 & \text{for } n = 2, \\
66433620048642 & \text{for } n = 3, \\
-139651448832 & \text{for } n = 4, \\
198984563486982 & \text{for } n = 5.
\end{cases}$$
Theorem 2.5  For the quadratic form $Q_p$, we get

$$\text{ord} (\varphi(\tau; Q_p), i\infty, \Gamma_0(p)) = \frac{p - 1}{2}$$

and

$$a_{\frac{p-1}{2}}(Q_p) = 2p$$

for $p = 3, 5$ and 7.

Proof.  We know from (2.3), (2.6) and (2.9) that

\begin{align*}
\varphi(\tau; Q_3) &= 1 + 6z + 6z^3 + 6z^4 + \ldots \\
\varphi(\tau; Q_5) &= 1 + 10z^2 + 20z^4 + 20z^5 + \ldots \\
\varphi(\tau; Q_7) &= 1 + 14z^3 + 42z^5 + \ldots \\
\end{align*}

(2.11)

Therefore

$$\text{ord} (\varphi(\tau; Q_p), i\infty, \Gamma_0(p)) = \frac{p - 1}{2}$$

by (1.1).

Using (2.11) it is clear that

\begin{align*}
a_1(Q_3) &= 6, \\
a_2(Q_5) &= 10, \\
a_3(Q_7) &= 14. \\
\end{align*}

Therefore

$$a_{\frac{p-1}{2}}(Q_p) = 2p.$$

References


Ahmet TEKCAN
Faculty of Science
Department of Mathematics
University of Uludag
Görükle, 16059 Bursa-TURKEY
e-mail: fahmet@uludag.edu.tr

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