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Formulas for the Fourier Coefficients of Cusp Form for Some Quadratic Forms

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Abstract

In this paper, representations of positive integers by certain quadratic forms Q_p defined for odd prime p are examined. The number of representations of positive integer n by the quadratic form Q_p, is denoted by $r(n;Q_p)$, obtained for $p = 3,5$ and 7. We prove that $r(n;Q_p) = \rho(n;Q_p) + \vartheta(n;Q_p)$ for $p = 3,5$ and 7, where $\rho(n;Q_p)$ is the singular series and $\vartheta(n;Q_p)$ is the Fourier coefficient of cusp form.

Key Words: representation of numbers, quadratic forms, generalized theta series, Fourier coefficient of cusp forms.

1. Introduction.

Let

$$Q = Q(x_1, x_2, ..., x_k) = \sum_{1\leq r \leq s\leq k} b_{rs}x_r x_s$$

be a positive quadratic form of discriminant Δ in k variables with integral coefficients b_{rs}. Let A be $k \times k$ symmetric matrix corresponding to Q such that $(r,s)-$ element of which is b_{rs} (but the diagonal elements are $2b_{rr}$). Define the determinant of A to be the discriminant of the quadratic form Q, i.e. $\det(A) = \Delta$.

Consider the quadratic form

$$2Q = \sum_{r,s=1}^{k} a_{rs}x_r x_s, \ (a_{rr} = 2b_{rr}, \ a_{rs} = a_{sr} = b_{rs}, \ r < s)$$

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of discriminant $\check{D}$. Then $\Delta = (-1)^{k/2}\check{D}$. Let A_{rs} be the algebraic cofactor of elements a_{rs} in $\check{D}$, $\delta = \gcd\left(\frac{A_{rr}}{2}, A_{rs}\right)$, $(r, s = 1, 2, ..., k)$, $N = \frac{\check{D}}{\delta}$ be the level of the form Q and $\chi(d)$ be the character of the form Q, i.e. $\chi(d) = 1$ if Δ is a perfect square; but if Δ is not a perfect square and $2 \nmid \Delta$, then $\chi(d) = \left(\frac{d}{|\Delta|}\right)$for $d > 0$ and $\chi(d) = (-1)^{k/2}\chi(-d)$ for $d < 0$, where $\left(\frac{d}{|\Delta|}\right)$ is the generalized Jacobi symbol.

A positive quadratic from in k variables of level N and character $\chi(d)$ is called a quadratic form of the type $\left(-\frac{k}{2}, N, \chi\right)$. Let $P_v = P_v(x_1, x_2, ..., x_k)$ be the spherical function of order v with respect to the quadratic form Q.

Let $\Gamma(1)$ denote a full modular group and Γ any subgroup of a finite index in $\Gamma(1)$. In particular,

$$\begin{aligned} \Gamma_0(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) : c \equiv 0 \pmod N \right\}, \\ \Gamma_1(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) : a \equiv d \equiv 1 \pmod N \right\}, \\ \Gamma(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N) : b \equiv 0 \pmod N \right\} \end{aligned}$$

for $N \in \mathbb{N}$.

Let $G_k(\Gamma, \chi)$ and $S_k(\Gamma, \chi)$ denote the space of entire modular and cusp forms, respectively, of the type (k, Γ, χ). If $F(\tau) \in G_k(\Gamma, \chi)$, then in the neighbourhood of the cusps $\zeta = i\infty$

$$F(\tau) = \sum_{m=m_0 \geqslant 0}^{\infty} a_m z^m, \ a_{m_0} \neq 0.$$

The order of an entire modular form $F(\tau) \neq 0$ of the type (k, Γ, χ) at the cusps $\zeta = i\infty$ with respect to Γ is

$$ord\,(F(\tau), i\infty, \Gamma) = m_0. \tag{1.1}$$

In this case we called a_{m_0} as the *coefficient of order* and denote by $a_{m_0}(F(\tau))$.

Let $F(\tau)$ be any function on the upper half plane $\mathbb{U}$ and $m \in \mathbb{Z}$. Then for any matrix

$L=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$, let $F(\tau)|_m L = (c\tau+d)^{-m}F(L\tau)$ and

$$\wp(\tau;Q,P_v(x),h)=\sum_{n_i\equiv h_i(\operatorname{mod} N)} P_v(n_1,n_2,....,n_k)z_N^{\frac{1}{N}Q(n_1,n_2,....,n_k)} \tag{1.2}$$

and

$$\wp(\tau;Q,P_v(x))=\sum_{n=1}^{\infty}\left(\sum_{Q(x)=n}P_v(x)\right)z^n, \tag{1.3}$$

where $Q(x) = \frac{1}{2}\sum\limits_{r,s=1}^{k} a_{rs}x_r x_s$ is a quadratic form of the type $\left(\frac{k}{2},\, N,\, \chi\right)$, $P_v(x)$ is a spherical function of order v with respect to the Q ; $n_1,n_2,....,n_k$ are integers and $h=(h_1,h_2,....,h_k)$, where h_i are integers such that

$$\sum_{s=1}^{k} a_{rs}h_s \equiv 0 \ (\operatorname{mod} N),\ \ (r=1,2,...,k). \tag{1.4}$$

It is well known, to each positive quadratic form Q, there corresponds the theta series

$$\wp(\tau;Q)=1+\sum_{n=1}^{\infty} r(n;Q)z^n, \tag{1.5}$$

where $r(n;Q)$ the number of representations of positive integer n by the quadratic form Q.

Any positive quadratic form Q of the type $(-k,q,1), k>2, 2|k,\ z=e^{2\pi i\tau}, Im(\tau)>0$, corresponds to one and the same Eisenstein series defined by

$$E(\tau;Q) \;=\; 1+\sum_{n=1}^{\infty}(\alpha\sigma_{k-1}(n)z^n+\beta\sigma_{k-1}(n)z^{qn}) = 1+\sum_{n=1}^{\infty}\rho(n;Q)z^n \tag{1.6}$$

for

$$\alpha=\frac{i^k}{\rho_k}.\frac{q^{k/2}-i^k}{q^k-1},\ \beta=\frac{1}{\rho_k}.\frac{q^k-i^kq^{k/2}}{q^k-1},\ \ \rho_k=(-1)^{k/2}\frac{(k-1)!}{(2\pi)^k}\zeta(k), \tag{1.7}$$

where $\zeta(k)$ is the Riemann zeta function, $\sigma_{k-1}(n)=\sum\limits_{d|n} d^{k-1}$ and $\rho(n;Q)$ is the singular series defined in following lemma.

Lemma 1.1 *1. If* $2|k,\ v=\prod\limits_{p|n,\,p\nmid 2\Delta}p^{w},\ \Delta=r^{2}w$, *($w$ is a square-free number), then*

$$\begin{aligned}\rho(n;Q) &= \frac{\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2})\Delta^{\frac{1}{2}}}n^{\frac{k}{2}-1}\chi_2\prod_{p|\Delta,p>2}\chi_p \times\\ &\times\prod_{p|r,p>2}\left(1-\left(\frac{(-1)^{\frac{k}{2}}w}{p}\right)p^{-\frac{k}{2}}\right)^{-1}\times\\ &\times L^{-1}\left(\frac{k}{2},(-1)^{\frac{k}{2}}w\right)\sum_{d|v}\left(\frac{(-1)^{\frac{k}{2}}\Delta}{d}\right)d^{1-\frac{k}{2}}\end{aligned}$$

2. If $2\nmid k$, $\Delta n=2^{\alpha+\gamma}v_1v_2=r^2w,\ 2^{\alpha}||n,\ 2^{\gamma}||\Delta,\ p^{l}||\Delta,\ p^{w}||n,(p>2),\ v_1=\prod\limits_{p|n,\,p\nmid 2\Delta}p^{w}=r_1^2w_1,\ v_2=\prod\limits_{p|n\Delta,\ p|\Delta,\ p>2}p^{w+l}=r_2^2w_2$, *($w, w_1$ and w_2 are square-free integers). Then*

$$\begin{aligned}\rho(n;Q) &= \frac{r_1^{2-k}n^{\frac{k}{2}-1}(k-1)!}{\Gamma(\frac{k}{2})2^{k-2}\pi^{\frac{k}{2}-1}|B_{k-1}|\Delta^{\frac{1}{2}}}\chi_2\prod_{p|\Delta,\,p>2}\chi_p\times\\ &\times\prod_{p|2\Delta}(1-p^{1-k})^{-1}L\left(\frac{k-1}{2},(-1)^{\frac{k-1}{2}}w\right)\times\\ &\times\prod_{p|r_2,\,p>2}\left(1-\left(\frac{(-1)^{\frac{k-1}{2}}w}{p}\right)p^{\frac{1-k}{2}}\right)\times\\ &\times\sum_{d|r_1}d^{k-2}\prod_{p|d}\left(1-\left(\frac{(-1)^{\frac{k-1}{2}}w}{p}\right)p^{\frac{1-k}{2}}\right),\end{aligned}$$

where B_{k-1} are Bernoulli's numbers, $(\frac{\cdot}{p})$ is Jacobi symbol. [2]

The values of χ_2 are given as

$$\begin{aligned}\chi_2 &= 1 \text{ for } 2\nmid k,\ \alpha=0, \text{ or for } 2|k,\ \alpha=0, u\equiv 1(\operatorname{mod}4) \text{ or } 2|k,\ \alpha=1,\\ &= 1+(-1)^{\frac{u^2-1}{6}}2^{\frac{k}{2}-5},\ \text{ for } 2|k,\ \alpha=0,\ u\equiv 3(\operatorname{mod}4),\end{aligned}$$

$$= \quad 1+\frac{2^{\frac{k}{2}-3}(1-2^{\frac{-5\alpha}{2}}.63)}{31}, \text{ for } 2|k,\ 2|\alpha,\ u\equiv 1(\operatorname{mod}4),$$

$$= \quad 1+\frac{2^{\frac{k}{2}-3}(1-2^{\frac{-5\alpha}{2}}+(-1)^{\frac{u^2-1}{8}}2^{\frac{-5\alpha}{2}-2}.31)}{31}, \text{ for } 2|k,\ 2|\alpha,\ u\equiv 3(\operatorname{mod}4),$$

$$= \quad 1+\frac{2^{\frac{k}{2}-3}(1-2^{\frac{-5\alpha}{2}+\frac{5}{2}}.63)}{31}, \text{ for } 2|k,\ 2\nmid\alpha, \alpha>1,$$

$$= \quad 1+\frac{2^{\frac{k}{2}-\frac{1}{2}}(1-2^{\frac{-5\alpha}{2}}.63)}{31}, \text{ for } 2\nmid k,\ 2|\alpha,\ \alpha>0,$$

$$= \quad 1+\frac{2^{\frac{k}{2}-\frac{1}{2}}(1-2^{\frac{-5\alpha}{2}-\frac{5}{2}}.63)}{31}, \text{ for } 2\nmid k,\ 2\nmid\alpha, u\equiv 1(\operatorname{mod}4),$$

$$= \quad 1+\frac{2^{\frac{k}{2}-\frac{1}{2}}(1-2^{\frac{-5\alpha}{2}-\frac{5}{2}}+(-1)^{\frac{u^2-1}{8}}2^{\frac{-5\alpha}{2}-\frac{9}{2}}.31)}{31}, \text{ for } 2\nmid k,\ 2\nmid\alpha, u\equiv 3(\operatorname{mod}4).$$

Lemma 1.2 *If $\wp(\tau;Q,P_v(x),h)$ is not identically equal to zero, then*

$$\wp(\tau;Q,P_v(x),h)\in G_{v+\frac{k}{2}}\left(\Gamma(N)\right).\ [1]$$

Lemma 1.3 *If Q is a quadratic form of the type $(k,\ q,\ 1\ )$ or $(k,\ q,\ \chi)$, then*

$$\wp(\tau;Q)-E(\tau;Q)$$

is a cusp form of the type $(k,\ \Gamma_0(q),\ 1\ )$ or $(k,\ \Gamma_0(q),\ \chi)$, respectively.[1]

Let $r(n;Q)$ denote the number of representations of positive integer n by the quadratic form Q in k variables. Then it is well known that $r(n;Q)$ can be represented as

$$r(n;Q)=\rho(n;Q)+\vartheta(n;Q), \tag{1.8}$$

where $\rho(n;Q)$ is the singular series and $\vartheta(n;Q)$ is the Fourier coefficient of cusp form.

This can be represented in terms of the theory of modular forms by stating that

$$\wp(\tau;Q)=E(\tau;Q)+X(\tau;Q), \tag{1.9}$$

where $E(\tau;Q)$ is the Eisenstein series defined in (1.6) and $X(\tau;Q)$ is a cusp form.

If the genus of the quadratic form Q contains one class, then from Siegel's Theorem $\wp(\tau;Q)=E(\tau;Q)$; but if the genus of the quadratic form Q contains more than one class, then we need to find a cusp form $X(\tau;Q)$.

In [3], Vepkhvadze constructed generalized theta functions with characteristic and spherical functions

$$\wp_{gh}(\tau;Pv,Q)=\sum_{x\equiv g\,(\mathrm{mod}\,N)}(-1)^{\frac{h^t A(x-g)}{N^2}}P_v(x)e^{\frac{\pi i\tau x^t Ax}{N^2}}. \tag{1.10}$$

Here g and h are special vectors with respect to the matrix A of form Q, i.e. $Ag\equiv 0\ (\mathrm{mod}\,N)$, $Ah\equiv 0\,(\mathrm{mod}N)$, where N is a level of the form Q, $P_v=P_v(x)=(x_1,....,x_k)$ is a spherical function of order v with respect to Q.

Lemma 1.4 *Let K be an arbitrary integral vector, and L be a special vector with respect to the matrix A of the form Q. Then the equalities*

$$\begin{aligned}\wp_{g+NK,h}(\tau;Pv,Q)&=(-1)^{\frac{h^tAK}{N}}\wp_{gh}(\tau;Pv,Q),\\ \wp_{g,h+2L}(\tau;Pv,Q)&=\wp_{gh}(\tau;Pv,Q)\end{aligned}$$

are satisfied.[3].

For $M=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\Gamma_0(N)$ denote

$$\upsilon(M)=\left(i^{\frac{1}{2}\eta(\gamma)(sgn\delta-1)}\right)^{k+2v}(sgn\delta)^v\left(i^{(\frac{|\delta|-1}{2})^2}\right)^{k+2v}\left(\frac{2\Delta(sgn\delta)\beta}{|\delta|}\right)\left(\frac{-1}{|\delta|}\right),$$

where $\eta(\gamma)=1$ for $\gamma\geqslant 0, \eta(\gamma)=-1$ for $\gamma<0$. By $\upsilon_0(M)$ we denote $\upsilon(M)$ for $v=0$.

Lemma 1.5 *Let $Q_s=Q_s(x)$ $(s=1,2,...,j)$ be an integral positive quadratic form with k variables, $P_v^{(s)}=P_v^{(s)}(x)$ the corresponding spherical functions, A_s is a matrix of the form $Q_s(x)$, Δ_s be the discriminant of the matrix A_s, and N_s the level of the form Q_s. Moreover let $g^{(s)}$ and $h^{(s)}$ be vectors with even components and B_s be arbitrary complex number. Then the function*

$$X(\tau;Q_s)=\sum_{s=1}^{j}B_s\wp_{g^{(s)}h^{(s)}}(\tau;P_v^{(s)},Q_s) \tag{1.11}$$

is an integral modular form of the type $\left(-(\frac{k}{2}+v),N,\upsilon_0(M)\right)$ iff the conditions

$$N_s|N,\quad N_s^2|Q_s(g^{(s)})\ \text{and}\ 4N_s|\frac{N}{N_s}Q_s(h^{(s)})$$

are satisfied and for all α *and* δ *satisfying the condition* $\alpha\delta \equiv 1\,(mod N)$ *we get*

$$\sum_{s=1}^{j} B_s \wp_{\alpha g^{(s)}, -h^{(s)}}(\tau; P_v^{(s)}, Q_s)(sgn\delta)^v \left(\frac{(-1)^{\frac{k-1}{2}}\Delta_s}{|\delta|}\right)$$

$$= \left(\frac{(-1)^{\frac{k-1}{2}+v}\Delta}{|\delta|}\right) \sum_{s=1}^{j} B_s \wp_{g^{(s)} h^{(s)}}(\tau; P_v^{(s)}, Q_s). \text{ [2]}$$

Lemma 1.6 *If all conditions of Lemma 1.5 are satisfied and* $v > 0$ *then* $X(\tau; Q_s)$ *defined in (1.11) is a cusp form of the type* $\left(-(\frac{k}{2}+v), N, \upsilon_0(M)\right)$ [2].

2. Formulas for the Fourier Coefficients of Cusp Form for Some Quadratic Fourms

In the present paper, we obtain the formulas for the Fourier coefficient of cusp form for the quadratic form

$$Q_p = p \sum_{1 \leq i \leq j \leq p-2} x_i x_j + p \sum_{1 \leq i \leq p-2} x_i x_{p-1} + \frac{p-1}{2} x_{p-1}^2 \tag{2.1}$$

with $p-1$ variables.

Theorem 2.1 *Let* Q_p *be the quadratic form defined in (2.1). Then the discriminant of* Q_p *is*

$$\begin{cases} -3 & \text{if } p = 3 \\ \frac{p^{p-2}}{2^{p-1}} & \text{if } p > 3. \end{cases}$$

Proof. For $p = 3$ we obtain the form

$$Q_3 = 3x_1^2 + 3x_1x_2 + x_2^2$$

which is a binary quadratic form. The discriminant of Q_3 is -3.

We know that for $p \geqslant 3$ the discriminant of Q_p is the determinant of the matrix A_p which corresponds to Q_p. The matrix A_p is

$$A_p = \begin{pmatrix} p & \frac{p}{2} & \frac{p}{2} & \cdots\cdots\cdots & \frac{p}{2} \\ \frac{p}{2} & p & \frac{p}{2} & \cdots\cdots\cdots & \frac{p}{2} \\ . & . & . & \cdots\cdots\cdots & . \\ . & . & . & \cdots\cdots\cdots & . \\ \frac{p}{2} & \frac{p}{2} & \frac{p}{2} & \cdots\cdots\cdots & \frac{p-1}{2} \end{pmatrix}_{(p-1)\times(p-1)} .$$

We want to find the determinant of A_p. To get this, using row operations, in the first step we obtain

$$p(-1)^{1+1}\begin{vmatrix} p & \frac{p}{2} & \frac{p}{2} & \cdots\cdots & \frac{p}{2} \\ \frac{p}{2} & p & \frac{p}{2} & \cdots\cdots & \frac{p}{2} \\ . & . & . & \cdots\cdots & . \\ . & . & . & \cdots\cdots & . \\ \frac{p}{2} & \frac{p}{2} & \frac{p}{2} & \cdots\cdots & \frac{p-1}{2} \end{vmatrix} + \frac{p}{2}(-1)^{1+2}\begin{vmatrix} \frac{p}{2} & \frac{p}{2} & \frac{p}{2} & \cdots\cdots & \frac{p}{2} \\ \frac{p}{2} & p & \frac{p}{2} & \cdots\cdots & \frac{p}{2} \\ . & . & . & \cdots\cdots & . \\ . & . & . & \cdots\cdots & . \\ \frac{p}{2} & \frac{p}{2} & \frac{p}{2} & \cdots\cdots & \frac{p-1}{2} \end{vmatrix}$$

$$+\frac{p}{2}(-1)^{1+3}\begin{vmatrix} \frac{p}{2} & p & \frac{p}{2} & \cdots\cdots & \frac{p}{2} \\ \frac{p}{2} & \frac{p}{2} & \frac{p}{2} & \cdots\cdots & \frac{p}{2} \\ . & . & . & \cdots\cdots & . \\ . & . & . & \cdots\cdots & . \\ \frac{p}{2} & \frac{p}{2} & \frac{p}{2} & \cdots\cdots & \frac{p-1}{2} \end{vmatrix} + \frac{p}{2}(-1)^{1+4}\begin{vmatrix} \frac{p}{2} & p & \frac{p}{2} & \cdots\cdots & \frac{p}{2} \\ \frac{p}{2} & \frac{p}{2} & p & \cdots\cdots & \frac{p}{2} \\ . & . & . & \cdots\cdots & . \\ . & . & . & \cdots\cdots & . \\ \frac{p}{2} & \frac{p}{2} & \frac{p}{2} & \cdots\cdots & \frac{p-1}{2} \end{vmatrix}$$

$$+\ldots+\frac{p}{2}(-1)^{1+(p-2)}\begin{vmatrix} \frac{p}{2} & p & \frac{p}{2} & \cdots\cdots & \frac{p}{2} \\ \frac{p}{2} & \frac{p}{2} & p & \cdots\cdots & \frac{p}{2} \\ . & . & . & \cdots\cdots & . \\ . & . & . & \cdots\cdots & . \\ \frac{p}{2} & \frac{p}{2} & \frac{p}{2} & \cdots\cdots & \frac{p-1}{2} \end{vmatrix}$$

$$+\frac{p}{2}(-1)^{1+(p-1)}\begin{vmatrix} \frac{p}{2} & p & \frac{p}{2} & \cdots\cdots & \frac{p}{2} \\ \frac{p}{2} & \frac{p}{2} & p & \cdots\cdots & \frac{p}{2} \\ . & . & . & \cdots\cdots & . \\ . & . & . & \cdots\cdots & . \\ \frac{p}{2} & \frac{p}{2} & \frac{p}{2} & \cdots\cdots & \frac{p}{2} \end{vmatrix}.$$

If we continue in the same way we obtain

$$\begin{vmatrix} p & \frac{p}{2} & \frac{p}{2} & \cdots\cdots & \frac{p}{2} \\ \frac{p}{2} & p & \frac{p}{2} & \cdots\cdots & \frac{p}{2} \\ . & . & . & \cdots\cdots & . \\ . & . & . & \cdots\cdots & . \\ \frac{p}{2} & \frac{p}{2} & \frac{p}{2} & \cdots\cdots & \frac{p-1}{2} \end{vmatrix} = \frac{p^{p-3}}{2^{p-3}},$$

$$\begin{vmatrix} \frac{p}{2} & \frac{p}{2} & \frac{p}{2} & \cdots\cdots & \frac{p}{2} \\ \frac{p}{2} & p & \frac{p}{2} & \cdots\cdots & \frac{p}{2} \\ . & . & . & \cdots\cdots & . \\ . & . & . & \cdots\cdots & . \\ \frac{p}{2} & \frac{p}{2} & \frac{p}{2} & \cdots\cdots & \frac{p-1}{2} \end{vmatrix} = -\frac{p^{p-3}}{2^{p-2}},$$

$$\begin{vmatrix} \frac{p}{2} & p & \frac{p}{2} & \cdots\cdots & \frac{p}{2} \\ \frac{p}{2} & \frac{p}{2} & \frac{p}{2} & \cdots\cdots & \frac{p}{2} \\ . & . & . & \cdots\cdots & . \\ . & . & . & \cdots\cdots & . \\ \frac{p}{2} & \frac{p}{2} & \frac{p}{2} & \cdots\cdots & \frac{p-1}{2} \end{vmatrix} = \frac{p^{p-3}}{2^{p-2}},$$

$$\begin{vmatrix} \frac{p}{2} & p & \frac{p}{2} & \cdots\cdots & \frac{p}{2} \\ \frac{p}{2} & \frac{p}{2} & p & \cdots\cdots & \frac{p}{2} \\ . & . & . & \cdots\cdots & . \\ . & . & . & \cdots\cdots & . \\ \frac{p}{2} & \frac{p}{2} & \frac{p}{2} & \cdots\cdots & \frac{p-1}{2} \end{vmatrix} = -\frac{p^{p-3}}{2^{p-2}},$$

$$\vdots$$

$$\begin{vmatrix} \frac{p}{2} & p & \frac{p}{2} & \cdots\cdots & \frac{p}{2} \\ \frac{p}{2} & \frac{p}{2} & p & \cdots\cdots & \frac{p}{2} \\ . & . & . & \cdots\cdots & . \\ . & . & . & \cdots\cdots & . \\ \frac{p}{2} & \frac{p}{2} & \frac{p}{2} & \cdots\cdots & \frac{p-1}{2} \end{vmatrix} = \frac{p^{p-3}}{2^{p-2}},$$

$$\begin{vmatrix} \frac{p}{2} & p & \frac{p}{2} & \cdots\cdots & \frac{p}{2} \\ \frac{p}{2} & \frac{p}{2} & p & \cdots\cdots & \frac{p}{2} \\ . & . & . & \cdots\cdots & . \\ . & . & . & \cdots\cdots & . \\ \frac{p}{2} & \frac{p}{2} & \frac{p}{2} & \cdots\cdots & \frac{p}{2} \end{vmatrix} = \frac{p^{p-2}}{2^{p-2}},$$

i.e. the determinant of the first matrix is $\frac{p^{p-3}}{2^{p-3}}$, the determinant of the second, third, ...,$(p-2)-$th matrix are same and is $\pm\frac{p^{p-3}}{2^{p-2}}$, and the determinant of the last ($(p-1)-$th) matrix is $\frac{p^{p-2}}{2^{p-2}}$. Hence

$$\begin{aligned} \det(A_p) &= p\left(\frac{p^{p-3}}{2^{p-3}}\right) - \frac{p}{2}\left(-\frac{p^{p-3}}{2^{p-2}}\right) + \frac{p}{2}\left(\frac{p^{p-3}}{2^{p-2}}\right) - \frac{p}{2}\left(-\frac{p^{p-3}}{2^{p-2}}\right) + \ldots\ldots - \frac{p}{2}\left(\frac{p^{p-2}}{2^{p-2}}\right) \\ &= \frac{p^{p-2}}{2^{p-3}} + (p-3)\frac{p^{p-2}}{2^{p-1}} - \frac{p^{p-1}}{2^{p-1}} \\ &= \frac{2^2p^{p-2} + (p-3)p^{p-2} - p^{p-1}}{2^{p-1}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{4p^{p-2}+(p-4)p^{p-2}+p^{p-2}-p^{p-1}}{2^{p-1}} \\
&= \frac{p^{p-2}(4+p-4)+p^{p-2}-p^{p-1}}{2^{p-1}} \\
&= \frac{p^{p-1}+p^{p-2}-p^{p-1}}{2^{p-1}} \\
&= \frac{p^{p-2}}{2^{p-1}}.
\end{aligned}
$$

Therefore the discriminant of Q_p is $\frac{p^{p-2}}{2^{p-1}}$. □

Now we obtain the formulas for the Fourier coefficients of cusp form for the quadratic form Q_p for $p = 3, 5$ and 7.

Let $r(n; Q)$ denote the number of representations of positive integer n by the quadratic form Q in k variables. Then it is well known that $r(n; Q)$ can be represented as

$$r(n; Q) = \rho(n; Q) + \vartheta(n; Q),$$

where $\rho(n; Q)$ is the singular series and $\vartheta(n; Q)$ is the Fourier coefficient of cusp form.

Theorem 2.2 *For the quadratic form Q_3 the equality*

$$r(n; Q_3) = \rho(n; Q_3) + \vartheta(n; Q_3)$$

is satisfied, where $r(n; Q_3)$ denote the number of representations of positive integer n by the quadratic form Q_3 , $\rho(n; Q_3)$ is the singular series and

$$\vartheta(n; Q_3) = \begin{cases} 12 & \text{for} \quad n = 1, \\ 18 & \text{for} \quad n = 2, \\ 30 & \text{for} \quad n = 3, \\ 48 & \text{for} \quad n = 4, \\ 36 & \text{for} \quad n = 5. \end{cases}$$

Proof. For the quadratic form Q_3, we get from (1.7) that $\alpha = -6$ for $\rho_2 = -\frac{1}{24}$. Therefore from (1.6) we obtain

$$\begin{aligned} E(\tau; Q_3) &= 1 + \sum_{n=1}^{\infty} (\alpha\sigma_{k-1}(n)z^n + \beta\sigma_{k-1}(n)z^{qn}) \\ &= 1 + \sum_{n=1}^{\infty} \rho(n;\, Q_3)z^n \\ &= 1 - 6\left(z + 3z^2 + 4z^3 + 7z^4 + 6z^5 + ...\right). \end{aligned} \tag{2.2}$$

Now consider the equation

$$Q_3(x_1, x_2) = n$$

for positive integer n.

This equation

1. has six integral solutions $(-1,1), (-1,2), (0,-1), (0,1), (1,-2), (1,1)$ for $n = 1$,

2. has no integral solution for $n = 2$ and $n = 5$,

3. has six integral solutions $(-2,3), (-1,0), (-1,3), (1,-3), (1,0), (2,-3)$ for $n = 3$,

4. has six integral solutions $(-2,2), (-2,4), (0,-2), (0,2), (2,-4), (2,-2)$ for $n = 4$.

Therefore from (1.5) we obtain

$$\wp(\tau; Q_3) = 1 + 6z + 6z^3 + 6z^4 + ... \tag{2.3}$$

Using (2.2) and (2.3) we get

$$\begin{aligned} X(\tau; Q_3) &= \wp(\tau; Q_3) - E(\tau; Q_3) \\ &= 12z + 18z^2 + 30z^3 + 48z^4 + 36z^5 + ... \end{aligned} \tag{2.4}$$

is a cusp form of the type $(1,\, \Gamma_0(3),\, \chi)$. Therefore from (2.4) it is clear that

$$\vartheta(n; Q_3) = \begin{cases} 12 & \text{for} \quad n = 1, \\ 18 & \text{for} \quad n = 2, \\ 30 & \text{for} \quad n = 3, \\ 48 & \text{for} \quad n = 4, \\ 36 & \text{for} \quad n = 5. \end{cases}$$

□

Theorem 2.3 *For the quadratic form Q_5 the equality*

$$r(n;Q_5) = \rho(n;Q_5) + \vartheta(n;Q_5)$$

is satisfied, where $r(n;Q_5)$ denote the number of representations of positive integer n by the quadratic form Q_5 , $\rho(n;Q_5)$ is the singular series and

$$\vartheta(n;Q_5) = -\frac{1}{15881}\begin{cases} 61440 & for \quad n=1, \\ 394150 & for \quad n=2, \\ 1402700 & for \quad n=3, \\ 4485120 & for \quad n=4, \\ 7423820 & for \quad n=5. \end{cases}$$

Proof. For the quadratic form Q_5 we get from (1.7) that $\alpha = \frac{61440}{15881}$ for $\rho_4 = \frac{1}{240}$. Therefore from (1.6) we obtain

$$\begin{aligned} E(\tau;Q_5) &= 1+\sum_{n=1}^{\infty}(\alpha\sigma_{k-1}(n)z^n + \beta\sigma_{k-1}(n)z^{qn}) \\ &= 1+\sum_{n=1}^{\infty}\rho(n;\ Q_5)z^n \\ &= 1+\frac{61440}{15881}\left(z+9z^2+28z^3+73z^4+126z^5+......\right). \end{aligned} \tag{2.5}$$

Now consider the equation

$$Q_5(x_1,x_2,x_3,x_4) = n$$

for positive integer n.

This equation

1. has no integral solution for $n=1$ and $n=4$,

2. has ten integral solutions $(-1,-1,-1,4)$,$(-1,0,0,1)$,$(0,-1,0,1)$,$(0,0,-1,1)$, $(0,0,0,-1)$,$(0,0,0,1)$,$(0,0,1,-1)$,$(0,1,0,-1)$,$(1,0,0,-1)$,$(1,1,1,-4)$ for $n=2$,

3. has twenty integral solutions $(-1,-1,-1,3)$, $(-1,-1,0,2)$, $(-1,-1,0,3)$, $(-1,0,-1,2)$, $(-1,0,-1,3)$, $(-1,0,0,2)$, $(0,-1,-1,2)$, $(0,-1,-1,3)$, $(0,-1,0,2)$, $(0,0,-1,2)$, $(0,0,1,-2)$, $(0,1,0,-2)$, $(0,1,1,-3)$, $(0,1,1,-2)$, $(1,0,0,-2)$, $(1,0,1,-3)$, $(1,0,1,-2)$, $(1,1,0,-3)$, $(1,1,0,-2)$, $(1,1,1,-3)$ for $n=3$,

4. has twenty integral solutions $(-2,-1,-1,5)$, $(-1,-2,-1,5)$, $(-1,-1,-2,5)$, $(-1,-1,-1,5)$, $(-1,0,0,0)$, $(-1,0,1,0)$, $(-1,1,0,0)$, $(0,-1,0,0)$, $(0,-1,1,0)$,

$(0,0,-1,0),(0,0,1,0),(0,1,-1,0),(0,1,0,0),(1,-1,0,0),(1,0,-1,0),(1,0,0,0),$ $(1,1,1,-5),(1,1,2,-5),(1,2,1,-5),(2,1,1,-5)$ for $n=5$.

Therefore from (1.5) we obtain

$$\wp(\tau;Q_5)=1+10z^2+20z^3+20z^5+..... \tag{2.6}$$

Using (2.5) and (2.6) we get

$$\begin{aligned} X(\tau;Q_5) &= \wp(\tau;Q_5)-E(\tau;Q_5) \\ &= -\frac{1}{15881}\begin{pmatrix} 61440z+394150z^2+1402700z^3+ \\ 4485120z^4+7423820z^5+....... \end{pmatrix} \end{aligned} \tag{2.7}$$

is a cusp form of the type $(2,\,\Gamma_0(5),\,\chi)$. Therefore from (2.7) it is clear that

$$\vartheta(n;Q_5)=-\frac{1}{15881}\begin{cases} 61440 & \text{for} \quad n=1, \\ 394150 & \text{for} \quad n=2, \\ 1402700 & \text{for} \quad n=3, \\ 4485120 & \text{for} \quad n=4, \\ 7423820 & \text{for} \quad n=5. \end{cases}$$

□

Theorem 2.4 *For the quadratic form Q_7 the equality*

$$r(n;Q_7)=\rho(n;Q_7)+\vartheta(n;Q_7)$$

is satisfied, where $r(n;Q_7)$ denote the number of representations of positive integer n by the quadratic form Q_7 , $\rho(n;Q_7)$ is the singular series and

$$\vartheta(n;Q_7)=\frac{1}{4747561247799}\begin{cases} -132120576 & \textit{for} \quad n=1, \\ -4359979008 & \textit{for} \quad n=2, \\ 66433620048642 & \textit{for} \quad n=3, \\ -139651448832 & \textit{for} \quad n=4, \\ 198984563486982 & \textit{for} \quad n=5. \end{cases}$$

Proof. For the quadratic form Q_7, we get from (1.7) that $\alpha=\frac{132120576}{4747561247799}$ for $\rho_6=-\frac{1}{504}$. Therefore from (1.6) we obtain

$$\begin{aligned} E(\tau;Q_7) &= 1+\sum_{n=1}^{\infty}(\alpha\sigma_{k-1}(n)z^n+\beta\sigma_{k-1}(n)z^{qn}) \\ &= 1+\sum_{n=1}^{\infty}\rho(n;\,Q_7)z^n \\ &= 1+\frac{132120576}{4747561247799}\left(z+33z^2+244z^3+1057z^4+3126z^5+......\right). \end{aligned} \tag{2.8}$$

Now consider the equation

$$Q_7(x_1, x_2, x_3, x_4, x_5, x_6) = n$$

for positive integer n.

This equation

1. has no integral solution for $n = 1, 2$ and 4,

2. has fourteen integral solutions $(-1,-1,-1,-1,-1,6), (-1,0,0,0,0,1), (0,-1,0,0,0,1), (0,0,-1,0,0,1), (0,0,0,-1,0,1), (0,0,0,0,-1,1), (0,0,0,0,0,-1), (0,0,0,0,0,1), (0,0,0,0,1,-1), (0,0,0,1,0,-1), (0,0,1,0,0,-1), (0,1,0,0,0,-1), (1,0,0,0,0,-1), (1,1,1,1,1,-6)$ for $n = 3$,

3. has fortytwo integral solutions $(-1,-1,-1,-1,5), (-1,-1,0,-1,-1,5), (-1,0,-1,0,0,2), (-1,0,0,0,0,2), (0,-1,0,-1,0,2), (0,0,-1,-1,0,2), (0,0,0,-1,-1,2), (0,0,0,0,1,-2), (0,0,1,0,0,-2), (0,1,0,0,0,-2), (0,1,1,0,0,-2), (1,0,0,0,1,-2), (1,0,1,1,1,-5), (1,1,1,0,1,-5), (-1,-1,-1,-1,0,5), (-1,-1,0,0,0,2), (-1,0,0,-1,0,2), (0,-1,-1,-1,-1,5), (0,-1,0,0,-1,2), (0,0,-1,0,-1,2), (0,0,0,-1,0,2), (0,0,0,1,0,-2), (0,0,1,0,1,-2), (0,1,0,0,1,-2), (0,1,1,1,1,-5), (1,0,0,1,0,-2), (1,1,0,0,0,-2), (1,1,1,1,0,-5), (-1,-1,-1,0,-1,5), (-1,0,-1,-1,-1,5), (-1,0,0,0,-1,2), (0,-1,-1,0,0,2), (0,-1,0,0,0,2), (0,0,-1,0,0,2), (0,0,0,0,-1,2), (0,0,0,1,1,-2), (0,0,1,1,0,-2), (0,1,0,1,0,-2), (1,0,0,0,0,-2), (1,0,1,0,0,-2), (1,1,0,1,1,-5), (1,1,1,1,1,-5)$ for $n = 5$.

Therefore from (1.5) we obtain

$$\wp(\tau; Q_7) = 1 + 14z^3 + 42z^5 + ... \tag{2.9}$$

Using (2.8) and (2.9) we get

$$\begin{aligned} X(\tau; Q_7) &= \wp(\tau; Q_7) - E(\tau; Q_7) \\ &= \frac{1}{4747561247799} \begin{pmatrix} -132120576z - 4359979008z^2 \\ +66433620048642z^3 - 139651448832z^4 \\ +198984563486982z^5 + ... \end{pmatrix} \end{aligned} \tag{2.10}$$

is a cusp form of the type $(3, \Gamma_0(7), \chi)$. Therefore from (2.10) it is clear that

$$\vartheta(n; Q_7) = \frac{1}{4747561247799} \begin{cases} -132120576 & \text{for} \quad n = 1, \\ -4359979008 & \text{for} \quad n = 2, \\ 66433620048642 & \text{for} \quad n = 3, \\ -139651448832 & \text{for} \quad n = 4, \\ 198984563486982 & \text{for} \quad n = 5. \end{cases}$$

□

Theorem 2.5 *For the quadratic form Q_p we get*

$$ord\,(\wp(\tau;Q_p), i\infty, \Gamma_0(p)) = \frac{p-1}{2}$$

and

$$a_{\frac{p-1}{2}}(Q_p) = 2p$$

for $p = 3, 5$ and 7.

Proof. We know from (2.3), (2.6) and (2.9) that

$$\begin{aligned} \wp(\tau;Q_3) &= 1 + 6z + 6z^3 + 6z^4 + ... \\ \wp(\tau;Q_5) &= 1 + 10z^2 + 20z^3 + 20z^5 + ... \\ \wp(\tau;Q_7) &= 1 + 14z^3 + 42z^5 + ... \end{aligned} \tag{2.11}$$

Therefore

$$ord\,(\wp(\tau;Q_p), i\infty, \Gamma_0(p)) = \frac{p-1}{2}$$

by (1.1).

Using (2.11) it is clear that

$$\begin{aligned} a_1(Q_3) &= 6, \\ a_2(Q_5) &= 10, \\ a_3(Q_7) &= 14. \end{aligned}$$

Therefore

$$a_{\frac{p-1}{2}}(Q_p) = 2p. \qquad \square$$

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