Spacelike Normal Curves in Minkowski Space $\mathbb{E}_1^3$

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Dedicated to Professor Dr. H. Hilmi Hacisalihoğlu

Abstract

In the Euclidean space $\mathbb{E}^3$, it is well known that normal curves, i.e., curves with position vector always lying in their normal plane, are spherical curves [3]. Necessary and sufficient conditions for a curve to be a spherical curve in Euclidean 3-space are given in [10] and [11].

In this paper, we give some characterizations of spacelike normals curves with spacelike, timelike or null principal normal in the Minkowski 3-space $\mathbb{E}_1^3$.

Key words and phrases: Normal Curves, Position Vector and Minkowski Space.

1. Introduction

In the Euclidean space $\mathbb{E}^3$, it is well-known that to each unit speed curve $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}^3$ with at least four countinuous derivatives, one can associate three mutually ortogonal unit vector fields $T$, $N$ and $B$, called respectively the tangent, the principal normal and the binormal vector fields. At each point $\alpha(s)$ of curve $\alpha$, the planes spanned by $\{T, N\}$, $\{T, B\}$ and $\{N, B\}$ are known respectively as the osculating plane, the rectifying plane and the normal plane. The curves $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}^3$ for which the position vector $\alpha$ always lie in their rectifying plane, are for simplicity called rectifying curves, (see [3]). Similarly, the curves for which the position vector $\alpha$ always lie in their osculating plane, are for simplicity called osculating curves; and finally, the curves for which the position vector...
always lie in their normal plane, are for simplicity called normal curves. By definition, for a normal curve, the position vector $\alpha(s)$ satisfies

$$\alpha(s) = \lambda(s)N(s) + \mu(s)B(s),$$

for some differentiable functions $\lambda$ and $\mu$ of $s \in I \subseteq \mathbb{R}$.

Characterization of rectifying curves is given in [3] and these curves are studied in Minkowski space $\mathbb{E}^3_1$ in [5]. In this paper, we characterize spacelike normal curves, lying fully in the Minkowski space $\mathbb{E}^3_1$.

2. Preliminaries

The Minkowski 3-space $\mathbb{E}^3_1$ is the Euclidean 3-space $\mathbb{E}^3$ provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where $(x_1, x_2, x_3)$ is a rectangular coordinate system of $\mathbb{E}^3$. 

Since $g$ is an indefinite metric, recall that a vector $v \in \mathbb{E}^3_1$ can have one of three Lorentzian causal characters: it can be spacelike if $g(v, v) > 0$ or $v = 0$, timelike if $g(v, v) < 0$ and null (lightlike) if $g(v, v) = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in $\mathbb{E}^3_1$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null (lightlike). Denote by $\{T, N, B\}$ the moving Frenet frame along the curve $\alpha(s)$ in the space $\mathbb{E}^3_1$. For an arbitrary curve $\alpha(s)$ in the space $\mathbb{E}^3_1$, the following Frenet formulae are given in [4, 9].

If $\alpha$ is a spacelike curve with a spacelike or timelike principal normal $N$, then the Frenet formulae read

$$\begin{bmatrix}
T' \\
N' \\
B'
\end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -ck_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

(1)

where $g(T, T) = 1, g(N, N) = \epsilon = \pm 1, g(B, B) = -\epsilon, g(T, N) = 0, g(T, B) = 0, g(N, B) = 0$.

If $\alpha$ is a spacelike curve with a null (lightlike) principal normal $N$, the Frenet formulae
are
\[
\begin{pmatrix}
T' \\
N' \\
B'
\end{pmatrix} = \begin{pmatrix}
0 & k_1 & 0 \\
0 & k_2 & 0 \\
-k_1 & 0 & -k_2
\end{pmatrix} \begin{pmatrix}
T \\
N \\
B
\end{pmatrix},
\]
where \( g(T, T) = 1, g(N, N) = 0, g(B, B) = 0, g(T, N) = 0, g(T, B) = 0, g(N, B) = 1 \). In this case, \( k_1 \) can take only two values: \( k_1 = 0 \) when \( \alpha \) is a straight line; \( k_1 = 1 \) in all other cases.

Let \( m \) be a fixed point in \( \mathbb{E}^3_1 \) and \( r > 0 \) be a constant. The pseudo-Riemannian sphere is defined by
\[
S^2_1(m, r) = \{ u \in \mathbb{E}^3_1 : g(u - m, u - m) = r^2 \};
\]
the pseudo-Riemannian hyperbolical space is defined by
\[
H^2_0(m, r) = \{ u \in \mathbb{E}^3_1 : g(u - m, u - m) = -r^2 \};
\]
the pseudo-Riemannian lightlike cone (quadric cone) is defined by
\[
C(m) = \{ u \in \mathbb{E}^3_1 : g(u - m, u - m) = 0 \}.
\]

3. The spacelike normal curves in \( \mathbb{E}^3_1 \)

In this section, we give some characterization theorems for spacelike normal curves.

**Theorem 3.1** Let \( \alpha = \alpha(s) \) be a unit speed spacelike normal curve in \( \mathbb{E}^3_1 \) with spacelike or timelike principal normal \( N \) and with curvatures \( k_1(s) > 0, k_2(s) \neq 0 \) for each \( s \in I \subset \mathbb{R} \). Then the following statements hold:

(i) The curvatures \( k_1(s) \) and \( k_2(s) \) satisfy the following equality
\[
\frac{1}{k_1(s)} = c_1 \cosh(\int k_2(s)ds) + c_2 \sinh(\int k_2(s)ds), \quad c_1, c_2 \in \mathbb{R};
\]

(ii) The principal normal and binormal component of the position vector of the curve are given respectively by
\[
g(\alpha(s), N) = a_1 \cosh(\int k_2(s)ds) + a_2 \sinh(\int k_2(s)ds)
\]
\[
g(\alpha(s), B) = a_1 \sinh(\int k_2(s)ds) + a_2 \cosh(\int k_2(s)ds), \quad a_1, a_2 \in \mathbb{R};
\]

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(iii) If the position vector of the curve is null vector, then \( \alpha \) lies on pseudo-Riemannian lightlike cone \( C(m) \) and the curvatures \( k_1(s) \) and \( k_2(s) \) satisfy

\[
\frac{1}{k_1(s)} = c_1[\cosh(\int k_2(s)ds) \pm \sinh(\int k_2(s)ds)].
\]

Conversely if \( \alpha(s) \) is a unit speed spacelike curve in \( E^3_1 \) with spacelike or timelike principal normal \( N \), the curvatures \( k_1(s) > 0, k_2(s) \neq 0 \) for each \( s \in I \subset \mathbb{R} \) and one of the statements (i), (ii) and (iii) hold, then \( \alpha \) is a normal curve or congruent to a normal curve.

**Proof.** Let us first suppose that \( \alpha(s) \) is a unit speed spacelike normal curve in \( E^3_1 \) with spacelike or timelike principal normal \( N \), where \( s \) is pseudo arclength parameter. Then by definition we have

\[
\alpha(s) = \lambda(s)N(s) + \mu(s)B(s).
\]

Differentiating this with respect to \( s \) and using the corresponding Frenet equations (1), we find

\[
\epsilon \lambda k_1 = -1, \quad \lambda' + \mu k_2 = 0, \quad \mu' + \lambda k_2 = 0.
\]

From the first and second equation in (3), we get

\[
\lambda = -\frac{\epsilon}{k_1}, \quad \mu = \frac{\epsilon}{k_2} \left( \frac{1}{k_1} \right)'.
\]

Thus

\[
\alpha(s) = -\frac{\epsilon}{k_1}N + \frac{\epsilon}{k_2} \left( \frac{1}{k_1} \right) ' B.
\]

Further, from the third equation in (3) and using (4), we find the following differential equation

\[
\left[ \frac{1}{k_2} \left( \frac{1}{k_1} \right)' \right]' - \frac{k_2}{k_1} = 0.
\]

Putting \( y(s) = \frac{1}{k_1} \) and \( p(s) = \frac{1}{k_2} \), equation (6) can be written as

\[
(p(s)y'(s))' - \frac{y(s)p(s)}{p(s)} = 0.
\]
If we change variables in the above equation as $t = \int \frac{1}{p(s)} ds$, then we get

$$\frac{d^2 y}{dt^2} = y = 0.$$  

The solution of the previous differential equation is

$$y = c_1 \cosh(t) + c_2 \sinh(t),$$

where $c_1, c_2 \in \mathbb{R}$. Therefore,

$$\frac{1}{k_1(s)} = c_1 \cosh\left(\int k_2(s) ds\right) + c_2 \sinh\left(\int k_2(s) ds\right).$$  \hspace{1cm} (7)

Thus we have proved statement (i). Next, substituting (7) into (4) and (5), we get

$$\lambda = -\epsilon [c_1 \cosh\left(\int k_2(s) ds\right) + c_2 \sinh\left(\int k_2(s) ds\right)],$$

$$\mu = \epsilon [c_1 \sinh\left(\int k_2(s) ds\right) + c_2 \cosh\left(\int k_2(s) ds\right)],$$

and

$$\alpha = -\epsilon [c_1 \cosh\left(\int k_2(s) ds\right) + c_2 \sinh\left(\int k_2(s) ds\right)] N$$

$$+ \epsilon [c_1 \sinh\left(\int k_2(s) ds\right) + c_2 \cosh\left(\int k_2(s) ds\right)] B.$$  \hspace{1cm} (8)

Therefore, from (8) we easily find that

$$g(\alpha, \alpha) = \epsilon (c_1^2 - c_2^2),$$  \hspace{1cm} (9)

$$g(\alpha, N) = a_1 \cosh\left(\int k_2(s) ds\right) + a_2 \sinh\left(\int k_2(s) ds\right),$$  \hspace{1cm} (10)

$$g(\alpha, B) = a_1 \sinh\left(\int k_2(s) ds\right) + a_2 \cosh\left(\int k_2(s) ds\right),$$  \hspace{1cm} (11)

where $a_1 = -c_1 \in \mathbb{R}, a_2 = -c_2 \in \mathbb{R}$. Consequently, we have proved (ii).
Next, suppose that $\alpha$ is a normal curve with a null (lightlike) position vector. Then we have $g(\alpha, \alpha) = 0$. Substituting this into equation (9), we obtain $c_1^2 = c_2^2$. Then (7) becomes

$$\frac{1}{k_1(s)} = c_1 [\cosh(\int k_2(s)ds) \pm \sinh(\int k_2(s)ds)].$$  \hfill (12)

On the other hand, let us consider the vector

$$m = \alpha(s) + \frac{\epsilon}{k_1} N - \frac{\epsilon}{k_2} \left( \frac{1}{k_1} \right)' B.$$

Differentiating this with respect to $s$ and using corresponding Frenet equations (1), we find $m' = 0$, and therefore $m = \text{constant}$. Then $g(\alpha - m, \alpha - m) = 0$, which means that $\alpha$ lies on $C(m)$. Consequently, we have proved statement (iii).

Conversely, suppose that statement (i) holds. Then we have

$$\frac{1}{k_1(s)} = c_1 \cosh(\int k_2(s)ds) + c_2 \sinh(\int k_2(s)ds).$$

Differentiating this with respect to $s$, we get

$$\left[ \frac{1}{k_2} \left( \frac{1}{k_1} \right)' \right]' = \frac{k_2}{k_1}.$$

By applying Frenet equations (1), we obtain

$$\frac{d}{ds} \left[ \alpha(s) + \frac{\epsilon_1}{k_1} N - \frac{\epsilon_1}{k_2} \left( \frac{1}{k_1} \right)' B \right] = 0.$$

Consequently, $\alpha$ is congruent to a normal curve. Next, assume that statement (ii) holds. Then the equations (9) and (10) are satisfied. Differentiating (9) with respect to $s$ and using (10), we find $g(\alpha, T) = 0$, which means that $\alpha$ is normal curve. Finally, assume that statement (iii) holds. Then $\alpha$ lies on light cone $C(m)$ with vertex at $m$, $m = \text{constant}$ and curvatures $k_1(s)$ and $k_2(s)$ satisfy the equation (12). Hence we have

$$g(\alpha - m, \alpha - m) = 0.$$
Differentiating this four times with respect to $s$ and using Frenet equations (1), we get

$$\alpha(s) - m = -\frac{c}{k_1}N + \left(\frac{c}{k_2}\right)\frac{1}{k_1}B.$$  

This means that, up to a translation for vector $m$, curve $\alpha$ is congruent to a normal curve. Let us put $m = 0$. Then using (12) we easily find $g(\alpha, \alpha) = 0$, which proves the theorem. \hfill \Box

**Theorem 3.2** Let $\alpha = \alpha(s)$ be unit speed spacelike normal curve in $\mathbb{E}^3_1$ with curvatures $k_1(s) > 0$, $k_2(s) \neq 0$, non-null principal normal $N$ and non-null position vector. Then:

(i) The position vector $\alpha$ is spacelike if and only if the curve $\alpha$ lies on the pseudo-Riemannian sphere $S^2_1(m, r)$ and there holds

$$\frac{1}{k_1(s)} = \pm \sqrt{c^2 + cr^2} \cosh\left(\int k_2(s)ds\right) + c \sinh\left(\int k_2(s)ds\right), \quad c \in \mathbb{R}, \quad \epsilon = \pm 1; \quad (13)$$

(ii) The position vector $\alpha$ is timelike if and only if the curve $\alpha$ lies on the pseudo-hyperbolical space $\mathbb{H}^2_0(m, r)$ and there holds

$$\frac{1}{k_1(s)} = \pm \sqrt{c^2 - cr^2} \cosh\left(\int k_2(s)ds\right) + c \sinh\left(\int k_2(s)ds\right), \quad c \in \mathbb{R}, \quad \epsilon = \pm 1. \quad (14)$$

**Proof.** Let us first assume that the position vector $\alpha$ is spacelike. Then $g(\alpha, \alpha) = r^2$, $r \in \mathbb{R}^+$. Substituting this into (9), we get $c_1 = \pm \sqrt{c^2 + cr^2}$. By using the last equation and (7), we obtain that (13) holds. Next, let us consider the vector

$$m = \alpha + \left(\epsilon/k_1\right)N - \left(\epsilon/k_2\right)(1/k_1)'B.$$  

Differentiating this and using the corresponding Frenet equations, we get $m' = 0$. Consequently, $m = \text{constant}$. It follows that $g(\alpha - m, \alpha - m) = r^2$, which means that $\alpha$ lies on pseudo-Riemannian sphere $S^2_1(m, r)$ with center $m$ and of radius $r$. Conversely, assume that (13) holds and that $\alpha$ lies on $S^2_1(m, r)$. Then $g(\alpha - m, \alpha - m) = r^2$, where $r \in \mathbb{R}^+$. Differentiating this four times with respect to $s$ and using Frenet equations, we find

$$\alpha - m = -(\epsilon/k_1)N + (\epsilon/k_2)(1/k_1)'B.$$  

Therefore, up to a translation for a vector \( m \), \( \alpha \) is congruent to a normal curve. In particular, let us put \( m = 0 \). Then (13) implies that \( g(\alpha, \alpha) = r^2 \), which proves statement (i).

The proof of statement (ii) is analogous to the proof of statement (i). \( \square \)

**Remark.** The spacelike curves with a null principal normal \( N \), in the space \( \mathbb{E}_1^3 \) can have the first curvature \( k_1 = 0 \) or \( k_1 = 1 \) [7]. If \( k_1 = 0 \), then \( \alpha(s) \) is straight line. Therefore \( \alpha(s) \) is in direction of \( T(s) \) for each \( s \). For straight line we have \( N = B = 0 \), so we do not have normal plane \( \{N, B\} \). Therefore, if \( k_1 = 0 \) then \( \alpha(s) \) can not be normal curve.

**Theorem 3.3** Let \( \alpha(s) \) be unit speed spacelike normal curve in \( \mathbb{E}_1^3 \) with a null principal normal \( N \) and \( k_1 = 1 \). Then \( \alpha \) is normal curve if and only if the principal normal and binormal component of the position vector are, respectively, \( g(\alpha, N) = -1 \), \( g(\alpha, B) = c \), \( c \in \mathbb{R} \).

**Proof.** Let us first assume that \( \alpha(s) \) is normal curve. Then we have

\[
\alpha(s) = \lambda(s)N(s) + \mu(s)B(s). \tag{15}
\]

Differentiating this with respect to \( s \) and using Frenet equations (2), we get

\[
\lambda' + \lambda k_2 = 0 \quad \text{and} \quad \mu' - \mu k_2 = 0 \tag{16}
\]

We obtain from the third equation in (16) that \( k_2 = 0 \). Then the second equation in (16) implies \( \lambda' = 0 \). Thus \( \lambda = c \), \( c \in \mathbb{R} \) and therefore

\[
\alpha = cN - B. \tag{17}
\]

Finally, we obtain \( g(\alpha, N) = -1 \), \( g(\alpha, B) = c \).

Conversely, let \( g(\alpha, N) = -1 \), \( g(\alpha, B) = c \). Then differentiating with respect to \( s \), we find \( k_2 = 0 \) and \( g(\alpha, T) = 0 \), which means that \( \alpha \) is normal curve. \( \square \)

**Theorem 3.4** Let \( \alpha(s) \) be unit speed spacelike normal curve in \( \mathbb{E}_1^3 \) with a null principal normal \( N \) and \( k_1 = 1 \). Then \( \alpha \) lies on pseudo-Riemannian sphere \( \mathbb{S}_1^2(m; r) \) if and only if \( \alpha \) is plane normal curve with the equation \( \alpha - m = -\frac{r^2}{2}N - B \).
Proof. Suppose that $\alpha$ lies on pseudo-Riemannian sphere $S^2_1(m, r)$. Then we have

$$g(\alpha - m, \alpha - m) = r^2, \quad r \in \mathbb{R}^+.$$ 

Differentiating this and applying Frenet formulae, we find

$$k_2 g(N, \alpha - m) = 0.$$

Thus $k_2 = 0$, and $\alpha$ is plane curve. We will prove that it is normal curve. Decompose the vector $\alpha - m$ by

$$\alpha - m = aT + bN + cB,$$

where $a = a(s), b = b(s), c = c(s)$ are arbitrary functions of $s$.

Then $g(\alpha - m, T) = 0 = a$, $g(\alpha - m, N) = c = -1$, $g(\alpha - m, B) = b$. Differentiating $g(\alpha - m, B) = b$, we get $b = b_0 = \text{constant}$. We obtain that

$$\alpha - m = b_0 N - B,$$

and since $g(\alpha - m, \alpha - m) = r^2$, we have $g(\alpha - m, \alpha - m) = -2b_0 = r^2$ and $b_0 = -\frac{r^2}{2}$.

Finally, $\alpha$ has the equation

$$\alpha - m = -\frac{r^2}{2} N - B,$$

and it is congruent to a normal curve.

Conversely, if $\alpha$ is plane normal curve with the equation $\alpha - m = -\frac{r^2}{2} N - B$ where $r \in \mathbb{R}^+$ and $m = (m_1, m_2, m_3) \in \mathbb{E}^3_1$, then we have $k_2 = 0$. Next, we get that $m = \alpha + \frac{r^2}{2} N + B$ which differentiating in $s$ gives $m' = 0$. Thus $m = \text{constant} \in \mathbb{E}^3_1$, (i.e. $m$ is constant vector). Therefore, $\alpha$ lies on $S^2_1(m, r)$. $\Box$

**Theorem 3.5** Let $\alpha(s)$ be unit speed spacelike normal curve in $\mathbb{E}^3_1$ with a null principal normal $N$ and $k_1 = 1$. Then $\alpha$ lies on pseudo-Riemannian hyperbolical space $H^2_0(m, r)$ if and only if $\alpha$ is plane normal curve with the equation $\alpha - m = -\frac{r^2}{2} N - B$, where $r \in \mathbb{R}^+$. 

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**Theorem 3.6** Let \( \alpha(s) \) be unit speed spacelike normal curve in \( \mathbb{E}^3_1 \) with a null principal normal \( N \) and \( k_1 = 1 \). Then \( \alpha \) lies on light cone \( C(m) \) with vertex at \( m \) if and only if \( \alpha \) is congruent to a normal curve with the equation \( \alpha(s) = -B(s) \).

**Proof.** Suppose that \( \alpha \) lies on light cone \( C(m) \) with vertex at point \( m \in \mathbb{E}^3_1 \). Then

\[
g(\alpha - m, \alpha - m) = 0.
\]

Differentiating the previous equation and using Frenet equations (2), we get \( g(\alpha - m, T) = 0 \), \( g(\alpha - m, N) = -1 \) and \( k_2 = 0 \). Next, decompose the vector \( \alpha - m \) by

\[
\alpha - m = aT + bN + cB,
\]

where \( a = a(s), b = b(s), c = c(s) \) are arbitrary functions of \( s \).

Then \( g(\alpha - m, T) = 0 = a, g(\alpha - m, N) = c = -1, g(\alpha - m, B) = b \). Differentiating \( g(\alpha - m, B) = b \), we get \( b = b_0 = \text{constant} \). It follows that

\[
\alpha - m = b_0 N - B.
\]

Since \( g(\alpha - m, \alpha - m) = 0 = -2b_0 \), we get \( b_0 = 0 \). Thus \( \alpha - m = -B \). Therefore, up to a translation for the vector \( m \), \( \alpha \) is congruent to a normal curve and \( \alpha = -B \).

Conversely, assume that \( \alpha \) is congruent to a normal curve with the equation \( \alpha = -B \). Differentiating this we get \( k_2 = 0 \). Let us consider the vector \( m = \alpha + B \). Taking the derivative of the last equation, we find \( m = \text{constant} \) and finally \( g(\alpha - m, \alpha - m) = 0 \), which means that \( \alpha \) lies on the light cone \( C(m) \). \( \square \)

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