On $\delta$-I-Continuous Functions

S. Yüksel, A. Açıkgöz and T. Noiri

Abstract

In this paper, we introduce a new class of functions called $\delta$-I-continuous functions. We obtain several characterizations and some of their properties. Also, we investigate its relationship with other types of functions.

Key words and phrases: $\delta$-I-cluster point, R-I-open set, $\delta$-I-continuous, strongly $\theta$-I-continuous, almost-I-continuous, SI-R space, AI-R space.

1. Introduction

Throughout this paper $\text{Cl}(A)$ and $\text{Int}(A)$ denote the closure and the interior of $A$, respectively. Let $(X, \tau)$ be a topological space and let $I$ an ideal of subsets of $X$. An ideal is defined as a nonempty collection $I$ of subsets of $X$ satisfying the following two conditions: (1) If $A \in I$ and $B \subseteq I$, then $B \in I$; (2) If $A \in I$ and $B \subseteq I$, then $A \cup B \in I$. An ideal topological space is a topological space $(X, \tau)$ with an ideal $I$ on $X$ and is denoted by $(X, \tau, I)$. For a subset $A \subseteq X$, $A^*(I) = \{x \in X \mid U \cap A \notin I \text{ for each neighborhood } U \text{ of } x\}$ is called the local function of $A$ with respect to $I$ and $\tau$ [4]. We simply write $A^*$ instead of $A^*(I)$ to be brief. $X^*$ is often a proper subset of $X$. The hypothesis $X = X^*[1]$ is equivalent to the hypothesis $\tau \cap I = \emptyset$ [5]. For every ideal topological space $(X, \tau, I)$, there exists a topology $\tau^*(I)$, finer than $\tau$, generated by $\beta(I, \tau) = \{U \setminus I : U \in \tau \text{ and } I \subseteq I\}$, but in general $\beta(I, \tau)$ is not always a topology [2]. Additionally, $\text{Cl}^*(A) = \text{A}(\text{A}^*)$ defines a Kuratowski closure operator for $\tau^*(I)$.

AMS Mathematics Subject Classification: 54C10.
In this paper, we introduce the notions of $\delta$-$I$-open sets and $\delta$-$I$-continuous functions in ideal topological spaces. We obtain several characterizations and some properties of $\delta$-$I$-continuous functions. Also, we investigate the relationships with other related functions.

2. $\delta$-$I$-open sets

In this section, we introduce $\delta$-$I$-open sets and the $\delta$-$I$-closure of a set in an ideal topological space and investigate their basic properties. It turns out that they have similar properties with $\delta$-open sets and the $\delta$-closure due to Veličko [6].

Definition 2.1 A subset $A$ of an ideal topological space $(X, \tau, I)$ is said to be an $R$-$I$-open (resp. regular open) set if $\text{Int}(\text{Cl}^*(A)) = A$ (resp. $\text{Int}(\text{Cl}(A)) = A$). We call a subset $A$ of $X$ $R$-$I$-closed if its complement is $R$-$I$-open.

Definition 2.2 Let $(X, \tau, I)$ be an ideal topological space, $S$ a subset of $X$ and $x$ a point of $X$.

1. $x$ is called a $\delta$-$I$-cluster point of $S$ if $S \setminus \text{Int}(\text{Cl}^*(U)) \neq \emptyset$ for each open neighborhood $x$.

2. The family of all $\delta$-$I$-cluster points of $S$ is called the $\delta$-$I$-closure of $S$ and is denoted by $[S]_{\delta-I}$ and

3. A subset $S$ is said to be $\delta$-$I$-closed if $[S]_{\delta-I} = S$. The complement of a $\delta$-$I$-closed set of $X$ is said to be $\delta$-$I$-open.

Lemma 2.1 Let $A$ and $B$ be subsets of an ideal topological space $(X, \tau, I)$. Then, the following properties hold:

1. $\text{Int}(\text{Cl}^*(A))$ is $R$-$I$-open;

2. If $A$ and $B$ are $R$-$I$-open, then $A \cap B$ is $R$-$I$-open;

3. If $A$ is regular open, then it is $R$-$I$-open;

4. If $A$ is $R$-$I$-open, then it is $\delta$-$I$-open and

5. Every $\delta$-$I$-open set is the union of a family of $R$-$I$-open sets.

Proof. (1) Let $A$ be a subset of $X$ and $V = \text{Int}(\text{Cl}^*(A))$. Then, we have $\text{Int}(\text{Cl}^*(V)) = \text{Int}(\text{Cl}^*(\text{Int}(\text{Cl}^*(A)))) \subset \text{Int}(\text{Cl}^*(\text{Cl}^*(A))) = \text{Int}(\text{Cl}^*(A)) = V$ and also $V = \text{Int}(V) \subset \text{Int}(\text{Cl}^*(V))$. Therefore, we obtain $\text{Int}(\text{Cl}^*(V)) = V$. 

40
(2) Let $A$ and $B$ be R-I-open. Then, we have $A \cap B = \text{Int}(Cl^*(A)) \cap \text{Int}(Cl^*(B)) = \text{Int}(Cl^*(A) \cap Cl^*(B)) \supset \text{Int}(Cl^*(A \cap B)) \supset \text{Int}(A \cap B) = A \cap B$. Therefore, we obtain $A \cap B = \text{Int}(Cl^*(A \cap B))$. This shows that $A \cap B$ is R-I-open.

(3) Let $A$ be regular open. Since $\tau^* \supset \tau$, we have $A = \text{Int}(Cl^*(A)) \subset \text{Int}(Cl(A)) = A$ and hence $A$ is R-I-open.

(4) Let $A$ be any R-I-open set. For each $x \in A$, $(X-A) \cap A = \emptyset$ and $A$ is R-I-open. Hence $x \notin [X - A]_{\delta - I}$ for each $x \in A$. This shows that $x \notin (X-A)$ implies $x \notin [X - A]_{\delta - I}$. Therefore, we have $[X - A]_{\delta - I} \subset (X-A)$. Since in general, $S \subset [S]_{\delta - I}$ for any subset $S$ of $X$, $[X - A]_{\delta - I} = (X-A)$ and hence $A$ is $\delta$-I-open.

(5) Let $A$ be a $\delta$-I-open set. Then $(X-A)$ is $\delta$-I-closed and hence $(X-A) = [X - A]_{\delta - I}$. For each $x \in A$, $x \notin [X - A]_{\delta - I}$ and there exists an open neighborhood $V_x$ such that $\text{Int}(Cl^*(V_x)) \cap (X-A) = \emptyset$. Therefore, we have $x \in V_x \subset \text{Int}(Cl^*(V_x)) \subset A$ and hence $A = \bigcup \{\text{Int}(Cl^*(V_x)) \mid x \in A\}$. By (1), $\text{Int}(Cl^*(V_x))$ is R-I-open for each $x \in A$. □

Lemma 2.2 Let $A$ and $B$ be subsets of an ideal topological space $(X, \tau, I)$. Then, the following properties hold:

1. $A \subset [A]_{\delta - I}$;
2. If $A \subset B$, then $[A]_{\delta - I} \subset [B]_{\delta - I}$;
3. $[A]_{\delta - I} = \cap \{F \subset X \mid A \subset F$ and $F$ is $\delta$-I-closed\};
4. If $A$ is a $\delta$-I-closed set of $X$ for each $\alpha \in \Delta$, then $\cap \{A_\alpha \mid \alpha \in \Delta\}$ is $\delta$-I-closed;
5. $[A]_{\delta - I}$ is $\delta$-I-closed.

Proof. (1) For any $x \in A$ and any open neighborhood $V$ of $x$, we have $\emptyset \neq A \cap V \subset A \cap \text{Int}(Cl^*(V))$ and hence $x \in [A]_{\delta - I}$. This shows that $A \subset [A]_{\delta - I}$.

(2) Suppose that $x \notin [B]_{\delta - I}$. There exists an open neighborhood $V$ of $x$ such that $\emptyset = \text{Int}(Cl^*(V)) \cap B$; hence $\text{Int}(Cl^*(V)) \cap A = \emptyset$. Therefore, we have $x \notin [A]_{\delta - I}$.

(3) Suppose that $x \notin [A]_{\delta - I}$. For any open neighborhood $V$ of $x$ and any $\delta$-I-closed set $F$ containing $A$, we have $\emptyset \neq A \cap \text{Int}(Cl^*(V)) \subset F \cap \text{Int}(Cl^*(V))$ and hence $x \notin [F]_{\delta - I} = F$. This shows that $x \notin \cap \{F \subset X \mid A \subset F$ and $F$ is $\delta$-I-closed\}. Conversely, suppose that $x \notin [A]_{\delta - I}$. There exists an open neighborhood $V$ of $x$ such that $\text{Int}(Cl^*(V)) \cap A = \emptyset$. By Lemma 2.1, $X - \text{Int}(Cl^*(V))$ is a $\delta$-I-closed set which contains $A$ and does not contain $x$. Therefore, we obtain $x \notin \cap \{F \subset X \mid A \subset F$ and $F$ is $\delta$-I-closed\}. This completes the proof. □
(4) For each $\alpha \in \Delta$, $\bigcap_{\alpha \in \Delta} A_\alpha \cap \delta - \{A\} = A_\alpha$ and hence $\bigcap_{\alpha \in \Delta} A_\alpha = A_\alpha - I \bigcap \{A\}$. By (1), we obtain $\bigcap_{\alpha \in \Delta} A_\alpha = A_\alpha - \{A\}$. This shows that $\bigcap_{\alpha \in \Delta} A_\alpha$ is $\delta$-I-closed.

(5) This follows immediately from (3) and (4).

A point $x$ of a topological space $(X, \tau)$ is called a $\delta$-cluster point of a subset $S$ of $X$ if $\text{Int}(\text{Cl}(V)) \cap \delta - \{S\} = \emptyset$; for every open set $V$ containing $x$. The set of all $\delta$-cluster points of $S$ is called the $\delta$-closure of $S$ and is denoted by $\text{Cl}_\delta(S)$. If $\text{Cl}_\delta(S) = S$, then $S$ is said to be $\delta$-closed [6]. The complement of a $\delta$-closed set is said to be $\delta$-open. It is well-known that the family of regular open sets of $(X, \tau)$ is a basis for a topology which is weaker than $\tau$. This topology is called the semi-regularization of $\tau$ and is denoted by $\tau_S$. Actually, $\tau_S$ is the same as the family of $\delta$-open sets of $(X, \tau)$.

**Theorem 2.1** Let $(X, \tau, I)$ be an ideal topological space and $\tau_{\delta - I} = \{A \subset X \mid A$ is a $\delta$-I-open set of $(X, \tau, I)\}$. Then $\tau_{\delta - I}$ is a topology such that $\tau_S \subset \tau_{\delta - I} \subset \tau$.

**Proof.** By Lemma 2.1, we obtain $\tau_S \subset \tau_{\delta - I} \subset \tau$. Next, we show that $\tau_{\delta - I}$ is a topology.

(1) It is obvious that $\emptyset$, $X \in \tau_{\delta - I}$.

(2) Let $V_\alpha \in \tau_{\delta - I}$ for each $\alpha \in \Delta$. Then $X - V_\alpha$ is $\delta$-I-closed for each $\alpha \in \Delta$. By Lemma 2.2, $\bigcap_{\alpha \in \Delta} (X - V_\alpha)$ is $\delta$-I-closed and $\bigcap_{\alpha \in \Delta} (X - V_\alpha) = X \bigcup_{\alpha \in \Delta} V_\alpha$. Hence $\bigcup_{\alpha \in \Delta} V_\alpha$ is $\delta$-I-open.

(3) Let $A, B \in \tau_{\delta - I}$. By Lemma 2.1, $A = \bigcup_{\alpha \in \Delta_1} A_\alpha$ and $B = \bigcup_{\beta \in \Delta_2} B_\beta$, where $A_\alpha$ and $B_\beta$ are R-I-open sets for each $\alpha \in \Delta_1$ and $\beta \in \Delta_2$. Thus $A \cap B = \bigcup\{A_\alpha \cap B_\beta \mid \alpha \in \Delta_1, \beta \in \Delta_2\}$. Since $A_\alpha \cap B_\beta$ is R-I-open, $A \cap B$ is a $\delta$-I-open set by Lemma 2.1.

**Example 2.1** Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$ and $I = \{\emptyset, \{a\}\}$. Then $A = \{a, c\}$ is a $\delta$-I-open set which is not R-I-open. Since $\{a\}$ and $\{c\}$ are regular open sets, $A$ is a $\delta$-open set and hence $\delta$-I-open. But $A$ is not R-I-open. Because $A^* = \{b, c, d\}$ and $\text{Cl}^*(A) = A \cup A^* = X$. Therefore, we have $\text{Int}(\text{Cl}^*(A)) = X \neq A$.

For some special ideals, we have the following properties.

**Proposition 2.1** Let $(X, \tau, I)$ be an ideal topological space.

(1) If $I = \{\emptyset\}$ or the ideal $N$ of nowhere dense sets of $(X, \tau)$, then $\tau_{\delta - I} = \tau_S$.

(2) If $I = P(X)$, then $\tau_{\delta - I} = \tau$. 

42
Proof. (1) Let I = {∅}, then $S^* = \text{Cl}(S)$ for every subset S of X. Let A be R-I-open. Then $A = \text{Int}(\text{Cl}^*(A)) = (A \cup A^*) = \text{Int}(\text{Cl}(A))$ and hence A is regular open. Therefore, every $\delta$-I-open set is $\delta$-open and we obtain $\tau_{\delta-I} \subset \tau_S$. By Theorem 2.1, we obtain $\tau_{\delta-I} = \tau_S$.

Next, let $I = N$. It is well-known that $S^* = \text{Cl}(\text{Int}(\text{Cl}(S)))$ for every subset S of X. Let A be any R-I-open set. Then since A is open, $A = \text{Int}(\text{Cl}^*(A)) = \text{Int}(A \cup A^*) = \text{Int}(A \cup \text{Cl}(\text{Int}(\text{Cl}(A)))) = \text{Int}(\text{Cl}(\text{Int}(\text{Cl}(A)))) = \text{Int}(\text{Cl}(A))$. Hence A is regular open. Similarly to the case of $I = \{∅\}$, we obtain $\tau_{\delta-I} = \tau_S$.

(2) Let $I = P(X)$. Then $S^* = ∅$ for every subset S of X. Now, let A be any open set of X. Then $A = \text{Int}(A) = \text{Int}(A \cup A^*) = \text{Int}(\text{Cl}^*(A))$ and hence A is R-I-open. By Theorem 2.1, we obtain $\tau_{\delta-I} = \tau$.

3. $\delta$-I-continuous functions

Definition 3.1 A function $f:(X,\tau,I) → (Y,\Phi,J)$ is said to be $\delta$-I-continuous if for each $x \in X$ and each open neighborhood V of f(x), there exists an open neighborhood U of x such that $f(\text{Int}(\text{Cl}^*(U))) \subset \text{Int}(\text{Cl}^*(V))$.

Theorem 3.1 For a function $f:(X,\tau,I) → (Y,\Phi,J)$, the following properties are equivalent:

(1) f is $\delta$-I-continuous;

(2) For each $x \in X$ and each R-I-open set V containing f(x), there exists an R-I-open set containing x such that $f(U) \subset V$;

(3) $f([A]_{\delta-I}) \subset [f(A)]_{\delta-I}$ for every $A \subset X$;

(4) $[f^{-1}(B)]_{\delta-I} \subset f^{-1}([B]_{\delta-I})$ for every $B \subset Y$;

(5) For every $\delta$-I-closed set F of Y, $f^{-1}(F)$ is $\delta$-I-closed in X;

(6) For every $\delta$-I-open set V of Y, $f^{-1}(V)$ is $\delta$-I-open in X;

(7) For every R-I-open set V of Y, $f^{-1}(V)$ is $\delta$-I-open in X;

(8) For every R-I-closed set F of Y, $f^{-1}(F)$ is $\delta$-I-closed in X.

Proof. (1)⇒(2): This follows immediately from Definition 3.1.

(2)⇒(3): Let $x \in X$ and $A \subset X$ such that $f(x) \in f([A]_{\delta-I})$. Suppose that $f(x) \notin [f(A)]_{\delta-I}$. Then there exists an R-I-open neighborhood V of f(x) such that $f(A) \cap V = \emptyset$. By (2), there exists an R-I-open neighborhood U of x such that $f(U) \subset V$. Since $f(A) \cap f(U) \subset f(A) \cap V = \emptyset$, $f(A \cap f(U) = \emptyset$. Hence, we get that $U \cap A \subset f^{-1}(f(U)) \cap f^{-1}(f(A)) = f^{-1}(f(U) \cap f(A))$.}

43
Let \( x \in X \) and \( V \) be an open set containing \( f(x) \). Now, set \( V_0 = \text{Int}(\text{Cl}^*(V)) \), then by Lemma 2.1 \( Y-V_0 \) is an R-I-closed set. By (8), \( f^{-1}(Y-V_0) = X-f^{-1}(V) \) is \( \delta \)-I-closed. Therefore, \( f^{-1}(V) \) is \( \delta \)-I-open.

\( 8) \Rightarrow (1): \) Let \( x \in X \) and \( V \) be an open set containing \( f(x) \). Now, set \( V_0 = \text{Int}(\text{Cl}^*(V)) \), then by Lemma 2.1 \( Y-V_0 \) is an R-I-closed set. By (8), \( f^{-1}(Y-V_0) = X-f^{-1}(V) \) is \( \delta \)-I-closed. Therefore, \( f^{-1}(V) \) is \( \delta \)-I-open.

\( 8) \Rightarrow (1): \) Let \( x \in X \) and \( V \) be an open set containing \( f(x) \). Now, set \( V_0 = \text{Int}(\text{Cl}^*(V)) \), then by Lemma 2.1 \( Y-V_0 \) is an R-I-closed set. By (8), \( f^{-1}(Y-V_0) = X-f^{-1}(V) \) is \( \delta \)-I-closed. Therefore, \( f^{-1}(V) \) is \( \delta \)-I-open.

\( 8) \Rightarrow (1): \) Let \( x \in X \) and \( V \) be an open set containing \( f(x) \). Now, set \( V_0 = \text{Int}(\text{Cl}^*(V)) \), then by Lemma 2.1 \( Y-V_0 \) is an R-I-closed set. By (8), \( f^{-1}(Y-V_0) = X-f^{-1}(V) \) is \( \delta \)-I-closed. Therefore, \( f^{-1}(V) \) is \( \delta \)-I-open.

\( 8) \Rightarrow (1): \) Let \( x \in X \) and \( V \) be an open set containing \( f(x) \). Now, set \( V_0 = \text{Int}(\text{Cl}^*(V)) \), then by Lemma 2.1 \( Y-V_0 \) is an R-I-closed set. By (8), \( f^{-1}(Y-V_0) = X-f^{-1}(V) \) is \( \delta \)-I-closed. Therefore, \( f^{-1}(V) \) is \( \delta \)-I-open.
Proposition 3.1 Let \((X, \tau, I)\) be an ideal topological space, \(A, X_o\) subsets of \(X\) such that \(A \subset X_o\) and \(X_o\) is open in \(X\).

1. If \(A\) is \(R-I\)-open in \((X, \tau, I)\), then \(A\) is \(R-I\)-open in \((X_o, \tau/X_o, I/X_o)\).

2. If \(A\) is \(\delta-I\)-open in \((X, \tau, I)\), then \(A\) is \(\delta-I\)-open in \((X_o, \tau/X_o, I/X_o)\).

Proof.

1. Let \(A\) be \(R-I\)-open in \((X, \tau, I)\). Then \(A = \text{Int}(\text{Cl}(A))\) and \(\text{Cl}(X_o A) = \text{Cl}(A) \cap X_o = (\text{Cl}(A) \cap X_o) \subset X_o\). Hence we have \(\text{Int}(\text{Cl}(A)) \cap X_o = (\text{Cl}(A) \cap X_o) \subset X_o\). Therefore, \(A\) is \(R-I\)-open in \((X_o, \tau/X_o, I/X_o)\).

2. Let \(A\) be \(\delta-I\)-open set of \((X, \tau, I)\). By Lemma 2.1, \(A = \bigcup_{\alpha \in \Delta} A_\alpha\), where \(A_\alpha\) is \(R-I\)-open set of \((X, \tau, I)\) for each \(\alpha \in \Delta\). By (1), \(A\) is \(R-I\)-open in \((X_o, \tau/X_o, I/X_o)\) for each \(\alpha \in \Delta\) and hence \(A\) is \(\delta-I\)-open in \((X_o, \tau/X_o, I/X_o)\).

Theorem 3.2 If \(f:(X, \tau, I) \rightarrow (Y, \Phi, J)\) is a \(\delta-I\)-continuous function and \(X_o\) is a \(\delta-I\)-open set of \((X, \tau, I)\), then the restriction \(f/X_o:(X_o, \tau/X_o, I/X_o) \rightarrow (Y, \Phi, J)\) is \(\delta-I\)-continuous.

Proof. Let \(V\) be any \(\delta-I\)-open set of \((Y, \Phi, J)\). Since \(f\) is \(\delta-I\)-continuous, \(f^{-1}(V)\) is \(\delta-I\)-open in \((X, \tau, I)\). Since \(X_o\) is \(\delta-I\)-open, by Theorem 2.1 \(X_o \cap f^{-1}(V)\) is \(\delta-I\)-open in \((X, \tau, I)\) and hence \(X_o \cap f^{-1}(V)\) is \(\delta-I\)-open in \((X_o, \tau/X_o, I/X_o)\) by Proposition 3.1. This shows that \((f/X_o)^{-1}(V)\) is \(\delta-I\)-open in \((X_o, \tau/X_o, I/X_o)\) and hence \(f/X_o\) is \(\delta-I\)-continuous.

Theorem 3.3 If \(f:(X, \tau, I) \rightarrow (Y, \Phi, J)\) and \(g:(Y, \Phi, J) \rightarrow (Z, \varphi, K)\) are \(\delta-I\)-continuous, then so is \(gof:(X, \tau, I) \rightarrow (Z, \varphi, K)\).

Proof. It follows immediately from Cor. 3.1.

Theorem 3.4 If \(f, g:(X, \tau, I) \rightarrow (Y, \Phi, J)\) are \(\delta-I\)-continuous functions and \(Y\) is a Hausdorff space, then \(A = \{x \in X : f(x) = g(x)\}\) is a \(\delta-I\)-closed set of \((X, \tau, I)\).

Proof. We prove that \(X-A\) is \(\delta-I\)-open set. Let \(x \in X-A\). Then, \(f(x) \neq g(x)\). Since \(Y\) is Hausdorff, there exist open sets \(V_1\) and \(V_2\) containing \(f(x)\) and \(g(x)\), respectively, such that \(V_1 \cap V_2 = \emptyset\). From here we have \(\text{Int}(\text{Cl}(V_1)) \cap \text{Int}(\text{Cl}(V_2)) = \emptyset\). Thus, we obtain that \(\text{Int}(\text{Cl}(V_1)) \cap \text{Int}(\text{Cl}(V_2)) = \emptyset\). Since \(f\) and \(g\) are \(\delta-I\)-continuous, there exists an open
neighborhood $U$ of $x$ such that $f(\text{Int}(\text{Cl}^*(U))) \subseteq \text{Int}(\text{Cl}^*(V_1))$ and $g(\text{Int}(\text{Cl}^*(U))) \subseteq \text{Int}(\text{Cl}^*(V_2))$. Hence we obtain that $\text{Int}(\text{Cl}^*(V_1)) \subseteq f^{-1}(\text{Int}(\text{Cl}^*(V_1)))$ and $\text{Int}(\text{Cl}^*(V_2)) \subseteq g^{-1}(\text{Int}(\text{Cl}^*(V_2)))$. From here we have $\text{Int}(\text{Cl}^*(V_1)) \cap g^{-1}(\text{Int}(\text{Cl}^*(V_2))) = \emptyset$. Suppose that $f^{-1}(\text{Int}(\text{Cl}^*(V_1))) \cap g^{-1}(\text{Int}(\text{Cl}^*(V_2))) \neq \emptyset$. Hence there exists a point $z$ such that $z \in f^{-1}(\text{Int}(\text{Cl}^*(V_1))) \cap g^{-1}(\text{Int}(\text{Cl}^*(V_2)))$. Thus, $f(z) \in \text{Int}(\text{Cl}^*(V_1))$, $g(z) \in \text{Int}(\text{Cl}^*(V_2))$, and $z \in A$. Since $z \in A$, $f(z) = g(z)$. Therefore, we have $f(z) \in \text{Int}(\text{Cl}^*(V_1)) \cap \text{Int}(\text{Cl}^*(V_2))$ and $g(z) \in \text{Int}(\text{Cl}^*(V_2)) \cap \text{Int}(\text{Cl}^*(V_1)) \neq \emptyset$. This is a contradiction to $\text{Int}(\text{Cl}^*(V_1)) \cap \text{Int}(\text{Cl}^*(V_2)) = \emptyset$. Hence we obtain that $f^{-1}(\text{Int}(\text{Cl}^*(V_1))) \cap g^{-1}(\text{Int}(\text{Cl}^*(V_2))) \neq \emptyset$. Thus $f^{-1}(\text{Int}(\text{Cl}^*(V_1))) \cap g^{-1}(\text{Int}(\text{Cl}^*(V_2))) \subseteq X-A$. Since $\text{Int}(\text{Cl}^*(U)) \subseteq f^{-1}(\text{Int}(\text{Cl}^*(V_1))) \cap g^{-1}(\text{Int}(\text{Cl}^*(V_2)))$, we have that there exists an open neighborhood of $x$ such that $x \in U \text{Int}(\text{Cl}^*(U)) \subseteq X-A$. Therefore, $X-A$ is a $\delta$-I-open set. This shows that $A$ is $\delta$-I-closed.

4. Comparisons

**Definition 4.1** A function $f:(X,\tau,I)\rightarrow (Y,\Phi,J)$ is said to be strongly $\theta$-I-continuous (resp. $\theta$-I-continuous, almost $I$-continuous) if for each $x \in X$ and each open neighborhood $V$ of $f(x)$, there exists an open neighborhood $U$ of $x$ such that $f(\text{Cl}^*(U)) \subseteq V$ (resp. $f(\text{Cl}^*(U)) \subseteq \text{Cl}^*(V)$, $f(U) \subseteq \text{Int}(\text{Cl}^*(V))$).

**Definition 4.2** A function $f:(X,\tau,I)\rightarrow (Y,\Phi,J)$ is said to be almost-I-open if for each $R$-I-open set $U$ of $X$, $f(U)$ is open in $Y$.

**Theorem 4.1** (1) If $f:(X,\tau,I)\rightarrow (Y,\Phi,J)$ is strongly $\theta$-I-continuous and $g:(Y,\Phi,J)\rightarrow (Z,\varphi,K)$ is almost $I$-continuous, then $gof:(X,\tau,I)\rightarrow (Z,\varphi,K)$ is $\delta$-I-continuous.

(2) The following implications hold:

\[
\text{strongly } \theta - I - \text{ continuous } \Rightarrow \delta - I - \text{ continuous } \Rightarrow \text{ almost } - I - \text{ continuous.} \quad (4.1)
\]

**Proof.** (1) Let $x \in X$ and $W$ be any open set of $Z$ containing $(gof)(x)$. Since $g$ is almost-I-continuous, there exists an open neighborhood $V \subseteq Y$ of $f(x)$ such that $g(V) \subseteq \text{Int}(\text{Cl}^*(W))$. 

46
Since \( f \) is strongly \( \theta \)-continuous, there exists an open neighborhood \( U \subseteq X \) of \( x \) such that \( f(Cl^*(U)) \subseteq V \). Hence we have \( g(f(Cl^*(U))) \subseteq g(V) \) and \( g(f(Int(Cl^*(U)))) \subseteq g(V) \cap Int(Cl^*(W)) \). Thus, we obtain \( g(f(Int(Cl^*(U)))) \subseteq Int(Cl^*(W)) \). This shows that \( gof \) is \( \delta \)-continuous.

(2) Let \( f \) be strongly \( \theta \)-continuous. Let \( x \in X \) and \( V \) be any open neighborhood of \( f(x) \). Then, there exists an open neighborhood \( U \subseteq X \) such that \( f(Cl(U)) \subseteq V \). Hence we have \( g(f(Int(Cl(U)))) \subseteq g(V) \). Since \( f(U) = f(Cl(U)) \), \( U \subseteq Int(X) = X \). Thus, \( f \) is \( \delta \)-continuous. Let \( f \) be \( \delta \)-continuous. Now we prove that \( f \) is almost \( \theta \)-continuous. Then, for each \( x \in X \) and each open neighborhood \( V \) of \( f(x) \), there exists an open neighborhood \( U \subseteq X \) such that \( f(Int(Cl(U))) \subseteq Int(Cl(V)) \). Since \( U \subseteq Int(Cl(U)) \), \( f(U) \subseteq Int(Cl(V)) \). Thus, \( f \) is almost \( \theta \)-continuous.

**Remark 4.1** The following examples enable us to realize that none of these implications in Theorem 4.1 (2) is reversible.

**Example 4.1** Let \( X = \{a, b, c\} \), \( \tau = \emptyset, X, \{a\}, \{a, c\} \), \( I = \emptyset, X, \{a, b\} \) and \( J = \emptyset, \{b\}, \{c\}, \{b, c\} \). The identity function \( f: (X, \tau, I) \rightarrow (X, \Phi, J) \) is \( \delta \)-continuous but it is not strongly \( \theta \)-continuous.

(i) Let \( a \in X \) and \( V = \{a, b\} \subseteq \Phi \) such that \( f(a) \in V \). \( V^* = (\{a, b\})^* = \{a, b, c\} = X \), \( Cl^*(V) = V \cup V^* = X \) and \( Int(Cl^*(V)) = Int(X) = X \). Then, there exists an open \( U \subseteq \{a, c\} \subseteq X \) such that \( a \in U \). We have \( U^* = (\{a\})^* = \{a, b, c\}, Cl^*(U) = U \cup U^* = \{a, b, c\} \) and \( Int(Cl^*(U)) = \{a, c\} \). Since \( f(Int(Cl^*(U))) = f(\{a, c\}) = \{a, c\} \subseteq Int(Cl^*(V)) \). \( f \) is not \( \delta \)-continuous.

(ii) Let \( b \in X \) and \( V = \{a, b\} \subseteq \Phi \) such that \( f(b) \in V \). \( V^* = (\{a, b\})^* = \{a, b, c\} = X \), \( Cl^*(V) = V \cup V^* = X \) and \( Int(Cl^*(V)) = Int(X) = X \). Then, there exists an open \( U = X \) such that \( b \in U \). We have \( Cl^*(U) = Cl^*(X) = X \) and \( Int(Cl^*(U)) = Int(X) \). Since \( f(Int(Cl^*(U))) = f(X) = X \) and \( U \subseteq Int(Cl^*(V)) = X \).

(iii) Let \( x = a, b \) or \( c \) and \( V = X \subseteq \Phi \) such that \( f(x) \in V \). \( Cl^*(V) = V \cup V^* = X \) and \( Int(Cl^*(V)) = Int(X) = X \). Then, there exists an open \( U = X \) such that \( x \in U \). We have \( Cl^*(U) = Cl^*(X) = X \) and \( Int(Cl^*(U)) = Int(X) \). Since \( f(Int(Cl^*(U))) = f(X) = X \) and \( X \subseteq Int(Cl^*(V)) = X \). By (i), (ii) and (iii), \( f \) is \( \delta \)-continuous. On the other hand by (i), since \( f(Cl^*(U)) = f(Cl^*(\{a\})) = f(\{a, b, c\}) = \{a, b, c\} \) is not subset of \( V = \{a, b\} \), \( f \) is not strongly \( \theta \)-continuous.
Example 4.2 Let $X = \{a,b,c,d\}$, $\tau = \emptyset, X, \{a,c\}, \{d\}, \{a,b,c\}, \{a,c,d\}$, $I = \emptyset, \{d\}$ and $J = \emptyset, \{c\}, \{d\}, \{a,d\}$. The identity function $f : (X, \tau, I) \rightarrow (X, \tau, J)$ is almost $I$-continuous but it is not $\delta I$-continuous. (i) Let $x = a$ or $c \in X$ and $V = \{a,c\} \in \Phi = \tau$ such that $f(x) \in V$. $V^* = \{a,c\}^* = \{a,b,c\}$, $Cl^*(V) = \emptyset \cup V^* = \{a,b,c\}$ and $Int(Cl^*(V)) = \{a,c\}$. Then, there exists an open $U = \{a,c\} \subset X$ such that $x \in U$. We have $U^* = \{a,c\}^* = \{a,b,c\}$ and $Int(Cl^*(U)) = Int(\{a,b,c\}) = \{a,b,c\}$. Since $f(U) = f(\{a,c\}) = \{a,c\} \subset Int(Cl^*(V)) = \{a,c\}.$

(ii) Let $x = a$, $c$ or $d \in X$ and $V = \{a,c,d\} \in \Phi = \tau$ such that $f(x) \in V$. $V^* = \{a,c,d\}^* = \{a,b,c,d\}$ and $Cl^*(V) = \emptyset \cup V^* = \{a,b,c,d\}$ and $Int(Cl^*(V)) = X$. Then, there exists an open $U = \{a,c,d\} \subset X$ such that $x \in U$. We have $U^* = \{a,c,d\}^* = \{a,b,c,d\} = X$ and $Int(Cl^*(U)) = Int(X) = X$. Since $f(U) = f(\{a,c,d\}) = \{a,c,d\} \subset Int(Cl^*(V)) = \{a,b,c,d\}$.

(iii) Let $x = a$, $b$ or $c \in X$ and $V = \{a,b,c\} \in \Phi = \tau$ such that $f(x) \in V$. $V^* = \{a,b,c\}^* = \{a,b,c\}$ and $Cl^*(V) = \emptyset \cup V^* = \{a,b,c\}$ and $Int(Cl^*(V)) = \{a,b,c\}$. Then, there exists an open $U = \{a,b,c\} \subset X$ such that $x \in U$. We have $U^* = \{a,b,c\}^* = \{a,b,c\}$ and $Int(Cl^*(U)) = \{a,b,c\} = \{a,b,c\}$. Since $f(U) = f(\{a,b,c\}) = \{a,b,c\} \subset Int(Cl^*(V)) = \{a,b,c\}$.

(iv) Let $d \in X$ and $V = \{d\} \in \Phi = \tau$ such that $f(d) \in V$. $V^* = \{d\}^* = \emptyset$ and $Cl^*(V) = \{d\}$. Then, there exists an open $U = \{d\} \subset X$ such that $d \in U$. We have $U^* = \{d\}^* = \emptyset$ and $Int(Cl^*(U)) = \{d\}$. Since $f(U) = f(\{d\}) = \{d\} \subset Int(Cl^*(V)) = \{d\}$. By (i), (ii), (iii) and (iv), f is almost $I$-continuous. On the other hand by (i), since $f(Int(Cl^*(U))) = f(\{a,b,c\}) = \{a,b,c\}$ is not subset of $Int(Cl^*(V))$ and $Int(Cl^*(V)) = \{a,c\}$, f is not $\delta I$-continuous.

Definition 4.3 An ideal topological space $(X, \tau, I)$ is said to be an SI-R space if for each $x \in X$ and each open neighborhood $V$ of $x$, there exists an open neighborhood $U$ of $x$ such that $x \in U \subset Int(Cl^*(U)) \subset V$.

Theorem 4.2 For a function $f : (X, \tau, I) \rightarrow (Y, \Phi, J)$, the following are true:

1. If $Y$ is an SI-R space and $f$ is $\delta I$-continuous, then $f$ is continuous.
2. If $X$ is an SI-R space and $f$ is almost $I$-continuous, then $f$ is $\delta I$-continuous.

Proof. (1) Let $Y$ be an SI-R space. Then, for each open neighborhood $V$ of $f(x)$, there exists an open neighborhood $V_0$ of $f(x)$ such that $f(x) \in V_0 \subset Int(Cl^*(V_0)) \subset V$. Since $f$ is $\delta I$-continuous, there exists an open neighborhood $U_0$ of $x$ such that $f(Int(Cl^*(U_0))) \subset Int(Cl^*(V_0))$. Since $U_0$ is an open set, $f(U_0) \subset f(Int(Cl^*(U_0))) \subset Int(Cl^*(V_0)) \subset V$. Thus, $f(U_0) \subset V$ and hence $f$ is continuous.
(2) Let \( x \in X \) and \( V \) be an open neighborhood of \( f(x) \). Since \( f \) is almost I-continuous, there exists an open neighborhood \( U \) of \( x \) such that \( f(U) \subseteq \text{Int}(\text{Cl}(V)). \) Since \( X \) is an SI-R space, there exists an open neighborhood \( U_1 \) of \( x \) such that \( \text{Int}(\text{Cl}(U_1)) \subseteq U. \) Thus \( f(\text{Int}(\text{Cl}(U_1))) \subseteq f(U) \subseteq \text{Int}(\text{Cl}(V)). \) Therefore \( f \) is \( \delta \)-I-continuous.

**Corollary 4.1** If \( (X, \tau, I) \) and \( (Y, \Phi, J) \) are SI-R spaces, then the following concepts on a function \( f:(X, \tau, I) \to (Y, \Phi, J) \): \( \delta \)-I-continuity, continuity and almost I-continuity are equivalent.

**Definition 4.4** An ideal topological space \( (X, \tau, I) \) is said to be an AI-R space if for each \( R \)-I-closed set \( F \subseteq X \) and each \( x \notin F \), there exist disjoint open sets \( U \) and \( V \) in \( X \) such that \( x \in U \) and \( F \subseteq V. \)

**Theorem 4.3** An ideal topological space \( (X, \tau, I) \) is an AI-R space if and only if each \( x \in X \) and each \( R \)-I-open neighborhood \( V \) of \( x \), there exists an \( R \)-I-open neighborhood \( U \) of \( x \) such that \( x \in U \subseteq \text{Cl}(U) \subseteq V. \)

**Proof.** **Necessity.** Let \( x \in V \) and \( V \) be \( R \)-open. Then \( \{x\} \cap (X-V) = \emptyset \). Since \( X \) is an AI-R space, there exist open sets \( U_1 \) and \( U_2 \) containing \( x \) and \( (X-V) \), respectively, such that \( U_1 \cap U_2 = \emptyset \). Then \( \text{Cl}(U_1) \cap U_2 = \emptyset \) and hence \( \text{Cl}(U_1) \subseteq \text{Cl}(U_1) \subseteq (X-U_2) \subseteq V. \) Thus \( x \in U_1 \subseteq \text{Cl}(U_1) \subseteq \text{Cl}(U_1) \subseteq V \) and we obtain that \( U_1 \subseteq \text{Int}(\text{Cl}(U_1)) \subseteq \text{Cl}(U_1). \) Let \( \text{Int}(\text{Cl}(U_1)) = U. \) Thus, we have \( \text{Cl}(U) = \text{Cl}(\text{Int}(\text{Cl}(U_1))) \subseteq \text{Cl}(\text{Cl}(U_1)) = \text{Cl}(U_1) \subseteq \text{Cl}(U) \) and \( U_1 \subseteq U \subseteq \text{Cl}(U_1) \subseteq \text{Cl}(U_1) \subseteq V. \) Therefore, there exists an \( R \)-I-open set \( U \) such that \( x \in U \subseteq \text{Cl}(U) \subseteq V. \)

**Sufficiency.** Let \( x \in X \) and an \( R \)-I-closed set \( F \) such that \( x \notin F \). Then, \( X-F \) is an \( R \)-I-open neighborhood of \( x \). By hypothesis, there exists an \( R \)-I-open neighborhood \( V \) of \( x \) such that \( x \in V \subseteq \text{Cl}(V) \subseteq \text{Cl}(V) \subseteq X-F. \) From here we have \( F \subseteq X-\text{Cl}(V) \subseteq (X-\text{Cl}(V)), \) where \( X-\text{Cl}(V) \) is an open set. Moreover, we have that \( V \cap (X-\text{Cl}(V)) = \emptyset \) and \( V \) is open. Therefore, \( X \) is an AI-R space.

**Theorem 4.4** For a function \( f:(X, \tau, I) \to (Y, \Phi, J) \), the following are true:

1. If \( Y \) is an AI-R space and \( f \) is \( \theta \)-I-continuous, then \( f \) is \( \delta \)-I-continuous.
(2) If $X$ is an $AI-R$ space, $Y$ is an $SI-R$ space and $f$ is $\delta-I$-continuous, then $f$ is strongly $\theta-I$-continuous.

**Proof.** (1) Let $Y$ be an $AI-R$ space. Then, for each $x \in X$ and each $R-I$-open neighborhood $V$ of $f(x)$, there exists an $R-I$-open neighborhood $V_o$ of $f(x)$ such that $f(x) \in V_o \subseteq Cl^*(V_o) \subseteq V$. Since $f$ is $\theta-I$-continuous, there exists an open neighborhood $U_o$ of $x$ such that $f(Cl^*(U_o)) \subseteq Cl^*(V_o)$. Hence, we obtain that $f(Int(Cl^*(U_o))) \subseteq f(Cl^*(V_o)) \subseteq V$ and thus $f(Int(Cl^*(U_o))) \subseteq V$. By Theorem 3.1, $f$ is $\delta-I$-continuous.

(2) Let $X$ be an $AI-R$ space and $Y$ an $SI-R$ space. For each $x \in X$ and each open neighborhood $V$ of $f(x)$, there exists an open set $V_o$ such that $f(x) \in V_o \subseteq Int(Cl^*(V_o)) \subseteq V$ since $Y$ is an $SI-R$ space. Since $f$ is $\delta-I$-continuous, there exists an open set $U$ containing $x$ such that $f(Int(Cl^*(U))) \subseteq Cl^*(V_o)$. By Lemma 2.1, $Int(Cl^*(U))$ is $R-I$-open and since $X$ is $AI-R$, by Theorem 4.3 there exists an $R-I$-open set $U_o$ such that $x \in V_o \subseteq Cl^*(U_o) \subseteq Int(Cl^*(U))$. Every $R-I$-open set is open and hence $U_o$ is open. Moreover, we have $f(Cl^*(U_o)) \subseteq V$. This shows that $f$ is strongly $\theta-I$-continuous.

---

**Theorem 4.5** If a function $f:(X,\tau,I) \rightarrow (Y,\Phi,J)$ is $\theta-I$-continuous and almost-$I$-open, then it is $\delta-I$-continuous.

**Proof.** Let $x \in X$ and $V$ be an open neighborhood of $f(x)$. Since $f$ is $\theta-I$-continuous, there exists an open neighborhood of $x$ such that $f(Cl^*(U)) \subseteq Cl^*(V)$; therefore, $f(Int(Cl^*(U))) \subseteq Cl^*(V)$. Since $f$ is almost-$I$-open, we have $f(Int(Cl^*(U))) \subseteq Int(Cl^*(V))$. This shows that $f$ is $\delta-I$-continuous.

---

**References**


S. YÜKSEL, A. AÇIKGÖZ
Department of Mathematics,
University of Selçuk,
42079 Konya-TURKEY
e-mail: syuksel@selcuk.edu.tr.

T. NOIRI
Department of Mathematics,
Yatsushiro College of Technology,
Yatsushiro, Kumamoto,
866-8501 JAPAN
e-mail: noiri@as.yatsushiro-nct.ac.jp

Received 03.09.2003