Fuzzy $\beta$-Compactness and Fuzzy $\beta$-Closed Spaces

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Abstract

The concepts of $\beta$-compactness and $\beta$-closed spaces in the fuzzy setting are defined and investigated. Fuzzy filterbases are used to characterize these concepts. A comparison between these types and some different forms of compactness in fuzzy topology is established.

Key Words: Fuzzy topological spaces, fuzzy $\beta$-compactness, fuzzy $\beta$-closed spaces, fuzzy filterbases.

1. Introduction

Compactness occupies a very important place in fuzzy topology and so do some of its forms. In [1], Abd El-Monsef et al introduced the concepts of $\beta$-open sets and $\beta$-continuous functions in general topology and Fath Alla in [8] introduced these concepts in fuzzy setting. In [5], some interesting properties of fuzzy $\beta$-compactness are investigated. The purpose of this paper is devoted to introduce and study the concepts of $\beta$-compactness and $\beta$-closed spaces in fuzzy setting. These notions generalize basic classical results (see [1], [2], [3], [4], [7] and [11]). Using fuzzy filterbases, we characterize fuzzy $\beta$-compactness and fuzzy $\beta$-closed spaces. We also explore some expected basic properties of these concepts.

AMS Subject Classification Code: 54A40, 54D30 and 54A20
2. Preliminaries

Throughout this paper, $X$ and $Y$ mean fuzzy topological spaces (fts, for short). A fuzzy point $x_t$ in $X$ is a fuzzy set having support $x \in X$ and value $t \in (0, 1]$ [13]. The complement and the support of a fuzzy set $u$ denoted by $\bar{u}$ and $S(u)$, respectively. For two fuzzy sets $u$ and $v$, we shall write $u \lor v$ ($u \land v$) to mean that $u$ is quasi coincident (not quasi coincident) with $v$, i.e., there exists $x \in X$ such that $u(x) + v(x) > 1$ ($u(x) + v(x) \leq 1$) [13].

**Definition 2.1** A fuzzy set $u$ in a fts $X$ is said to be:
(a) semiopen fuzzy set if $u \subseteq \text{cl} \text{nt} u$ [4];
(b) preopen fuzzy set if $u \subseteq \text{int} \text{cl} u$ [15];
(c) $\beta$-open fuzzy set if $u \subseteq \text{cl} \text{nt} \text{nt} u$ [8], equivalently, if there exists a preopen fuzzy set $A$ such that $A \subseteq u \subseteq \overline{A}$.

It is obvious that each semiopen and preopen fuzzy set implies $\beta$-open.

**Definition 2.2** [4, 10]. Let $u$ be a fuzzy set in a fts $X$, the fuzzy pre-closure (resp. semi-closure, pre-interior and semi-interior) of $u$ denoted by $\text{pcl} u$ (resp. $\text{scl} u$, $\text{pint} u$ and $\text{sin} t u$) are defined as follows:
$\text{pcl} u (\text{scl} u) = \{A : u \subseteq A, A \text{ is preclosed (semiclosed)}\}$;
$\text{pint} u (\text{sin} t u) = \{A : u \subseteq A, A \text{ is preopen (semiopen)}\}$.

**Definition 2.3** Let $u$ be a fuzzy set in a fts $X$. The fuzzy $\beta$-closure ($\beta\text{cl}$) and $\beta$-interior ($\beta\text{int}$) of $u$ are defined as follows:
$\beta\text{cl} u = \{A : u \subseteq A, A \text{ is } \beta\text{-closed}\}$;
$\beta\text{int} u = \{A : u \supseteq A, A \text{ is } \beta\text{-open}\}$.

It is obvious that $\beta\text{cl} \bar{u} = (\beta\text{int} u)'$ and $\beta\text{int} \bar{u} = (\beta\text{cl} u)'$.

**Definition 2.4** A function $f : X \rightarrow Y$ is said to be fuzzy $\beta$-continuous [8] (resp. $M\beta$-continuous) if the inverse image of every open (resp. $\beta$-open) fuzzy set in $Y$ is $\beta$-open (resp. $\beta$-open) fuzzy set in $X$.

**Lemma 2.5** Let $f : X \rightarrow Y$ be a function, then the following are equivalent:
(a) $f$ is fuzzy $M\beta$-continuous.
(b) $f(\beta\text{cl} u) \leq \beta\text{cl} f(u)$, for every fuzzy set $u$ in $X$. 
Proof. (a)⇒(b): Let $u$ be a fuzzy set of $X$, then $\beta\text{c}l\ f(u)$ is $\beta$-closed. By (a) $f^{-1}(\beta\text{c}l\ f(u))$ is $\beta$-closed and so $f^{-1}(\beta\text{c}l\ f(u)) = \beta\text{c}l\ f^{-1}(\beta\text{c}l\ f(u))$. Since $u \leq f^{-1}(f(u))$, we have $\beta\text{c}l\ u \leq \beta\text{c}l\ f^{-1}(f(u)) \leq \beta\text{c}l\ f^{-1}(\beta\text{c}l\ f(u)) = f^{-1}(\beta\text{c}l\ f(u))$. Hence $f(\beta\text{c}l\ u) \leq \beta\text{c}l\ f(u)$.

(b)⇒(a): Let $v$ be a $\beta$-closed fuzzy set in $Y$. By (b) if $u = f^{-1}(v)$, then $\beta\text{c}l\ f^{-1}(v) \leq f^{-1}(\beta\text{c}l\ f(f^{-1}(v))) \leq f^{-1}(\beta\text{c}l\ v) = f^{-1}(v)$. Since $f^{-1}(v) \leq \beta\text{c}l\ f^{-1}(v)$, then $f^{-1}(v) = \beta\text{c}l\ f^{-1}(v)$. Hence $f^{-1}(v)$ is $\beta$-closed fuzzy set in $X$. Hence $f$ is fuzzy $M_\beta$-continuous.

Lemma 2.6 Let $f : X \to Y$ be a function, then the following are equivalent:

(a) $f$ is fuzzy $\beta$-continuous.

(b) $f(\beta\text{c}l\ u) \leq \text{c}l\ f(u)$, for every fuzzy set $u$ in $X$.

Proof. Obvious

Theorem 2.7 [16]. If $f : X \to Y$ is fuzzy open function, then $f^{-1}(\text{c}l\ (u)) \leq \text{c}l\ (f^{-1}(u))$, for every fuzzy set $u$ in $Y$.

Definition 2.8 [9]. A collection of fuzzy subsets $\xi$ of a fts $X$ is said to form a fuzzy filterbases iff for every finite collection $\{A_j : j = 1, ..., n\}$, $\cap_{j=1}^{n} A_j \neq 0_X$.

Definition 2.9 [9]. A collection $\mu$ of fuzzy sets in a fts $X$ is said to be cover of a fuzzy set $u$ of $X$ iff $\forall A \in \mu (A(x) = 1_X$, for every $x \in S(u)$. A fuzzy cover $\mu$ of a fuzzy set $u$ in a fts $X$ is said to have a finite subcover iff there exists a finite subcollection $\eta = \{A_1, ..., A_n\}$ of $\mu$ such that $\forall (A_j) (x) \geq u(x)$, for every $x \in S(u)$.

Definition 2.10 A fts $X$ is said to be strongly compact [14] (resp. semicompact [12]) iff every preopen (resp. semiopen) cover of $X$ has a finite subcover.

Definition 2.11 A fts $X$ is said to be almost compact [7] (resp. S-closed [6], s-closed [11], P-closed [17]) iff every open (resp. semiopen, semiopen, preopen) cover of $X$ has a finite subcollection whose closures (resp. closures, semi-closures, pre-closures) cover $X$. 

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3. Fuzzy \( \beta \)-Compact Space

**Definition 3.1** [5]. A fts \( X \) is said to be fuzzy \( \beta \)-compact iff for every family \( \mu \) of \( \beta \)-open fuzzy sets such that \( \forall A \in \mu \) there is a finite subfamily \( \eta \subseteq \mu \) such that \( \forall A \in \eta \) \( A = 1_X \).

**Definition 3.2** A fuzzy set \( u \) in a fts \( X \) is said to be fuzzy \( \beta \)-compact relative to \( X \) iff for every family \( \mu \) of \( \beta \)-open fuzzy sets such that \( \forall A \in \mu \) \( A \geq u(x) \) there is a finite subfamily \( \eta \subseteq \mu \) such that \( \forall A \in \eta \) \( A \geq u(x) \) for every \( x \in S(u) \).

**Remark 3.3** Since each of semiopen and preopen fuzzy set implies \( \beta \)-open, it is clear that every fuzzy \( \beta \)-compact space implies each of fuzzy strongly compact space and fuzzy semicompact space. But the converse need not be true.

**Theorem 3.4** A fts \( X \) is \( \beta \)-compact iff for every collection \( \{ A_j : j \in J \} \) of \( \beta \)-closed fuzzy sets of \( X \) having the finite intersection property, \( \bigwedge_{j \in J} A_j \neq 0_X \).

**Proof.** Let \( \{ A_j : j \in J \} \) be a collection of \( \beta \)-closed fuzzy sets with the finite intersection property. Suppose that \( \bigwedge_{j \in J} A_j = 0_X \). Then \( \bigvee_{j \in J} A_j = 1_X \). Since \( \{ A_j : j \in J \} \) is a collection of \( \beta \)-open fuzzy sets cover of \( X \), then from the \( \beta \)-compactness of \( X \) it follows that there exists a finite subset \( F \subseteq J \) such that \( \bigvee_{j \in F} A_j = 1_X \). Then \( \bigwedge_{j \in F} A_j = 0_X \) which gives a contradiction and therefore \( \bigwedge_{j \in J} A_j \neq 0_X \).

Conversely, let \( \{ A_j : j \in J \} \) be a collection of \( \beta \)-open fuzzy sets cover of \( X \). Suppose that for every finite subset \( F \subseteq J \), we have \( \bigvee_{j \in F} A_j \neq 1_X \). Then \( \bigwedge_{j \in F} A_j \neq 0_X \). Hence \( \{ A_j : j \in J \} \) satisfies the finite intersection property. Then from the hypothesis we have \( \bigwedge_{j \in J} A_j \neq 0_X \) which implies \( \bigwedge_{j \in F} A_j \neq 1_X \) and this contradicting that \( \{ A_j : j \in J \} \) is a \( \beta \)-open cover of \( X \). Thus \( X \) is fuzzy \( \beta \)-compact.

Now, we give some results of fuzzy \( \beta \)-compactness in terms of fuzzy filterbases.

**Theorem 3.5** A fts \( X \) is fuzzy \( \beta \)-compact iff every filterbases \( \xi \) in \( X \), \( \bigwedge_{G \in \xi} \beta cl G \neq 0_X \).
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**Theorem 3.6** A fuzzy set $u$ in a fts $X$ is fuzzy $\beta$–compact relative to $X$ iff for every filterbases $\xi$ such that every finite of members of $\xi$ is quasi coincident with $u$, $(\bigwedge_{G \in \xi} \beta cl G) \cap u \neq 0_X$.

**Proof.** Let $u$ not be fuzzy $\beta$-compact relative to $X$, then there exists a $\beta$-open fuzzy set $\mu$ cover of $u$ such that $\mu$ has no finite subcover. Then for every finite subcollection $\{A_1, \ldots, A_n\}$ of $\mu$, there exists $x \in X$ such that $A_j(x) < 1$ for every $j = 1, \ldots, n$. Then $\hat{A}_j(x) > 0$, so that $\bigwedge_{j=1}^n \hat{A}_j(x) \neq 0_X$. Thus $\{\hat{A}_j : A_j \in \mu\}$ forms a filterbases in $X$. Since $\mu$ is $\beta$-open fuzzy set cover of $X$, then $(\bigvee_{A_j \in \mu} A_j)(x) = 1_X$ for every $x \in X$ and hence $\bigwedge_{A_j \in \mu} \beta cl \hat{A}_j(x) = \bigwedge_{A_j \in \mu} \hat{A}_j(x) = 0_X$, which is a contradiction. Then every $\beta$-open fuzzy set cover of $X$ has a finite subcover and hence $X$ is fuzzy $\beta$-compact.

Conversely, suppose there exists a filterbases $\xi$ such that $\bigwedge_{G \in \xi} \beta cl G = 0_X$, so that $(\bigvee_{G \in \xi} (\beta cl G))(x) = 1_X$ for every $x \in X$ and hence $\mu = \{(\beta cl G) : G \in \xi\}$ is a $\beta$-open fuzzy set cover of $X$. Since $X$ is fuzzy $\beta$-compact, then $\mu$ has a finite subcover. Then $(\bigvee_{j=1}^n (\beta cl g_j))(x) = 1_X$ and hence $(\bigvee_{j=1}^n g_j')(x) = 1_X$, so that $\bigwedge_{j=1}^n g_j = 0_X$ which is a contradiction, since the $g_j$ are members of filterbases $\xi$. Therefore $\bigwedge_{G \in \xi} \beta cl G \neq 0_X$ for every filterbases $\xi$.

**Theorem 3.6** A fuzzy set $u$ in a fts $X$ is fuzzy $\beta$–compact relative to $X$ iff for every filterbases $\xi$ such that every finite of members of $\xi$ is quasi coincident with $u$, $(\bigwedge_{G \in \xi} \beta cl G) \cap u \neq 0_X$.

**Proof.** Let $u$ not be fuzzy $\beta$-compact relative to $X$, then there exists a $\beta$-open fuzzy set $\mu$ cover of $u$ such that $\mu$ has no finite subcover. Then $(\bigvee_{A_j \in \eta} A_j)(x) < u(x)$ for some $x \in S(u)$, so that $(\bigwedge_{A_j \in \eta} \hat{A}_j)(x) > \hat{u}(x) \geq 0$ and hence $\xi = \{\hat{A}_j : A_j \in \mu\}$ forms a filterbases and $\bigwedge_{A_j \in \eta} \hat{A}_j = u$. By hypothesis $(\bigwedge_{A_j \in \eta} \beta cl \hat{A}_j) \cap u \neq 0_X$ and hence $(\bigwedge_{A_j \in \eta} \hat{A}_j) \cap u \neq 0_X$.

Then for some $x \in S(u)$, $(\bigwedge_{A_j \in \mu} \hat{A}_j)(x) > 0_X$, that is $(\bigvee_{A_j \in \mu} \hat{A}_j)(x) < 1_X$, which is a contradiction. Hence $u$ is fuzzy $\beta$-compact relative to $X$.

Conversely, suppose that there exists a filterbases $\xi$ such that every finite of members of $\xi$ is quasi coincident with $u$ and $(\bigwedge_{G \in \xi} \beta cl G) \cap u \neq 0_X$. Then for every $x \in S(u)$, $(\bigwedge_{G \in \xi} \beta cl G)(x) = 0_X$ and hence $(\bigvee_{G \in \xi} (\beta cl G))(x) = 1_X$ for every $x \in S(u)$. Thus $\mu = \{(\beta cl G) : G \in \xi\}$ is $\beta$-open fuzzy set cover of $u$. Since $u$ is fuzzy $\beta$-compact
relative to $X$, then there exists a finite subcover, say $\{ (\beta cl G_1)’, \ldots, (\beta cl G_n)’ \}$, such that $(\bigwedge_{j=1}^{n} (\beta cl G_j)) (x) \geq u(x)$ for every $x \in S(u)$. Hence $(\bigwedge_{j=1}^{n} (\beta cl G_j)) (x) \leq \tilde{u}(x)$ for every $x \in S(u)$, so that $\bigwedge_{j=1}^{n} (\beta cl G_j) \not\subset u$, which is a contradiction. Therefore for every filterbases $\xi$ such that every finite of members of $\xi$ is quasi coincident with $u$, $(\bigwedge_{G \in \xi} \beta cl G) \not\subset u \neq 0_X$. 

Theorem 3.7 Every $\beta$-closed fuzzy subset of a fuzzy $\beta$-compact space is fuzzy $\beta$-compact relative to $X$.

Proof. Let $\xi$ be a fuzzy filterbases in $X$ such that $u \not\subset \liminf \{ G : G \in \lambda \}$ holds for every finite subcollection $\lambda$ of $\xi$ and a $\beta$-closed fuzzy set $u$. Consider $\xi = \{ u \} \cup \xi$. For any finite subcollection $\lambda^*$ of $\xi^*$, if $u \not\subset \lambda^*$, then $\bigwedge_{G \in \xi} \beta cl G \neq 0_X$. If $u \subset \lambda^*$ and since $u \not\subset \{ G : G \in \lambda^* - u \}$, then $\bigwedge_{G \in \xi} \beta cl G \neq 0_X$. Hence $\lambda^*$ is a fuzzy filterbases in $X$. Since $X$ is fuzzy $\beta$-compact, then $\bigwedge_{G \in \xi} \beta cl G \neq 0_X$, so that $(\bigwedge_{G \in \xi} \beta cl G) \not\subset u = (\bigwedge_{G \in \xi} \beta cl G) \not\subset u \neq 0_X$. Hence by Theorem 3.6, we have $u$ is fuzzy $\beta$-compact relative to $X$. 

Theorem 3.8 If a function $f : X \rightarrow Y$ is fuzzy $M\beta$-continuous and $u$ is fuzzy $\beta$-compact relative to $X$, then so is $f(u)$.

Proof. Let $\{ A_j : j \in J \}$ be a $\beta$-open fuzzy set cover of $S(f(u))$. For $x \in S(u)$, $f(x) \in f(S(u)) = S(f(u))$. Since $f$ is fuzzy $M\beta$-continuous, then $\{ f^{-1}(A_j) : j \in J \}$ is $\beta$-open fuzzy set cover of $S(u)$. Since $u$ is fuzzy $\beta$-compact relative to $X$, there is a finite subfamily $\{ f^{-1}(A_j) : j = 1, \ldots, n \}$ such that $S(u) \subseteq \bigvee_{j=1}^{n} f^{-1}(A_j)$ which implies $S(u) \subseteq f^{-1}(\bigvee_{j=1}^{n} A_j)$ and then $S(f(u)) = f(S(u)) \subseteq ff^{-1}(\bigvee_{j=1}^{n} A_j) \subseteq \bigvee_{j=1}^{n} A_j$. Therefore $f(u)$ is fuzzy $\beta$-compact relative to $Y$. 

Lemma 3.9 If $f : X \rightarrow Y$ is fuzzy open and fuzzy continuous function, then $f$ is fuzzy $M\beta$-continuous.
Let \( v \) be an \( \beta \)-open fuzzy set in \( Y \); then \( v \subseteq cl int cl v \). So \( f^{-1}(v) \subseteq f^{-1}(cl int cl v) \subseteq cl (f^{-1}(int cl v)) \). Since \( f \) is fuzzy continuous, then \( f^{-1}(int cl v) = int (f^{-1}(cl v)) \). Also by Theorem 2.7, \( f^{-1}(int cl v) = int (f^{-1}(cl v)) \subseteq int (f^{-1}(cl v)) \). Thus \( f^{-1}(v) \subseteq cl (f^{-1}(int cl v)) \subseteq cl int cl (f^{-1}(cl v)) \). Hence the result.

**Corollary 3.10** Let \( f : X \rightarrow Y \) be fuzzy open and fuzzy continuous function and \( X \) is fuzzy \( \beta \)-compact, then \( f(X) \) is fuzzy \( \beta \)-compact.

**Proof.** It follows directly from Lemma 3.9 and Theorem 3.8.

**Definition 3.11** A function \( f : X \rightarrow Y \) is said to be fuzzy \( M \beta \)-open iff the image of every \( \beta \)-open fuzzy set in \( X \) is \( \beta \)-open in \( Y \).

**Theorem 3.12** Let \( f : X \rightarrow Y \) be a fuzzy \( M \beta \)-open bijective function and \( Y \) is fuzzy \( \beta \)-compact, then \( X \) is fuzzy \( \beta \)-compact.

**Proof.** Let \( \{A_j : j \in J\} \) be a collection of \( \beta \)-open fuzzy set cover of \( X \), then \( \{f(A_j) : j \in J\} \) is \( \beta \)-open fuzzy set covering of \( Y \). Since \( Y \) is fuzzy \( \beta \)-compact, there is a finite subset \( F \subseteq J \) such that \( \{f(A_j) : j \in F\} \) is an cover of \( Y \). But \( 1_X = f^{-1}(1_Y) = f^{-1} f(\bigvee_{j \in F} A_j) = \bigvee_{j \in F} A_j \) and therefore \( X \) is fuzzy \( \beta \)-compact.

4. **Fuzzy \( \beta \)-Closed Spaces**

**Definition 4.1** A fuzzy set \( u \) in a fts \( X \) is said to be a \( \beta q \)-nbd of a fuzzy point \( x_t \) in \( X \) if there exists a \( \beta \)-open fuzzy set \( A \leq u \) such that \( x_t \in \beta cl u \).

**Theorem 4.2** Let \( x_t \) be a fuzzy point in a fts \( X \) and \( u \) be any fuzzy set of \( X \), then \( x_t \in \beta cl u \) iff for every \( \beta q \)-nbd \( H \) of \( x_t \), \( H \sim u \).

**Proof.** Let \( x_t \in \beta cl u \) and there exists a \( \beta q \)-nbd \( H \) of \( x_t \), \( H \sim u \). Then there exists a \( \beta \)-open fuzzy set \( A \leq H \) in \( X \) such that \( x_t \in \beta cl A \), which implies \( A \sim u \) and hence \( u \leq A \). Since \( A \) is \( \beta \)-closed fuzzy set, then \( \beta cl u \leq A \). Since \( x_t \notin A \), then \( x_t \in \beta cl u \), which is a contradiction.
Conversely, let $x_t \notin \beta \text{cl } u = \bigwedge \{ A : A \text{ is } \beta\text{-closed in } X, A \supseteq u \}$. Then there exists a \( \beta\text{-closed fuzzy set } A \supseteq u \) such that $x_t \notin A$. Hence $x_t q q \hat{A} = H$, where $H$ is a $\beta$-open fuzzy set in $X$ and $H \sim q u$. Then there exists a $\beta q - nbd H$ of $x_t$ with $H \sim q u$. Hence the result. □

**Definition 4.3** A fits $X$ is said to be $\beta$-closed iff for every family $\mu$ of $\beta$-open fuzzy set such that $\bigvee A = 1_X$ there is a finite subfamily $\eta \subseteq \mu$ such that $( \bigvee_{A \in \eta} A)(x) = 1_X$, for every $x \in X$.

**Remark 4.4** From the above definition and other types of fuzzy compactness, one can draw the following diagram:

\[
\begin{array}{ccc}
F\text{-semicompact} & \rightarrow & Fs\text{-closed} \\
\uparrow & & \uparrow \\
F\beta\text{-compact} & \rightarrow & F\beta\text{-closed} \\
\downarrow & & \downarrow \\
F\text{-strongly compact} & \rightarrow & FP\text{-closed}
\end{array}
\]

where $F =$fuzzy.

**Example 4.5** Let $X \neq 0_X$ be a set and $u_n(x) = 1 - \frac{1}{n}$ for every $x \in X$ and $n \in N^+$. The collection $\{ u_n : n \in N^+ \}$ is a base for a fuzzy topology on $X$. The collection $\{ u_n : n \in N^+ \}$ is obviously a $\beta$–open fuzzy set cover of $X$. On the other hand we have $\beta \text{cl } u = 1_X$ for every $n \geq 3$. Hence $X$ is $\beta$-closed but not fuzzy $\beta$-compact, (see [6]).

**Remark 4.6** Example 4.5 also shows that:

(i) Each of the concepts $Fs$–closed, $FS$–closed and $FP$–closed spaces does not imply $F\beta$-compact.

(ii) Since the collection $\{ u_n : n \in N^+ \}$ is also semiopen (resp. preopen) fuzzy sets cover of $X$, then $X$ is $F\beta$-closed space but not $F$–semicompact space (resp. $F$–strongly compact space).

**Example 4.7** Let $X = I = [0, 1]$ and consider the following fuzzy sets

\[
U_1(x) = \frac{1}{\sqrt{5}} , \quad U_2(x) = \frac{1}{\sqrt{3}} , \quad U_3(x) = \frac{1}{\sqrt{2}} , \quad \ldots \ldots \ldots , \forall x \in I.
\]
Let $\sigma = \{u_j : j \in N^+\} \cup \{0_X, 1_X\}$. It is clear that $\sigma$ is a fuzzy topology on $X$. Now, the collection $\{u_j : j \in N^+\}$ is a semiopen (resp. preopen) fuzzy set cover of $X$ but not has a finite subcover. So $X$ is not $F-$semicompact space (resp. $F-$strongly compact space). Since the semi-closure (resp. pre-closure) of every semiopen (resp. preopen) fuzzy set of $X$ is $1_X$, then $X$ is $F_s-$closed (resp. $FP_s-$closed).

Remark 4.8 Example 4.7 is also shows that each of the concepts $FS-$closed and $FP-$closed spaces does not imply each of $F-$semicompact and $F-$strongly compact spaces.

Remark 4.9 From Remark 3.3, Example 4.5, Remark 4.6, Example 4.7 and Remark 4.8, it is clear that:

(i) $FS-$closed and $FP-$closed spaces are independent notions.
(ii) $FS-$closed and $F-$strongly compact spaces are independent notions.
(iii) $FP-$closed and $F-$semicompact spaces are independent notions.
(iv) $F\beta-$compact, $F-$semicompact and $F-$strongly compact spaces are independent notions.

Theorem 4.10 A fits $X$ is $\beta-$closed iff for every fuzzy $\beta-$open filterbases $\xi$ in $X$, $\bigwedge_{G \in \xi} \beta cl G \neq 0_X$.

**Proof.** Let $\mu$ be a $\beta-$open fuzzy set cover of $X$ and let for every finite subfamily $\eta$ of $\mu$, $(\bigvee_{A \in \eta} \beta cl A)(x) < 1_X$ for some $x \in X$. Then $(\bigwedge_{A \in \mu} A)(x) > 0_X$ for some $x \in X$. Thus $\{(\beta cl A) : A \in \mu\} = \xi$ forms a fuzzy $\beta-$open filterbases in $X$. Since $\mu$ is a $\beta-$open fuzzy set cover of $X$, then $\bigwedge_{A \in \mu} A = 0_X$ which implies $\bigwedge_{A \in \mu} (\beta cl A) = 0_X$, which is a contradiction. Then every $\beta-$open fuzzy set $\mu$ cover of $X$ has a finite subfamily $\eta$ such that $(\bigvee_{A \in \mu} A)(x) = 1_X$ for every $x \in X$. Hence $X$ is $\beta-$closed.

Conversely, suppose there exists a fuzzy $\beta-$open filterbases $\xi$ in $X$ such that $\bigwedge_{G \in \xi} \beta cl G = 0_X$, so that $(\bigvee_{G \in \xi} (\beta cl G))(x) = 1_X$ for every $x \in X$ and hence $\mu = \{(\beta cl G) : G \in \xi\}$ is a $\beta-$open fuzzy set cover of $X$. Since $X$ is $\beta-$closed, then $\mu$ has a finite subfamily $\eta$ such that $(\bigvee_{G \in \eta} (\beta cl G))(x) = 1_X$ for every $x \in X$, and hence $\bigwedge_{G \in \eta} (\beta cl (\beta cl G))(x) = 0_X$. Thus $\bigwedge_{G \in \eta} G = 0_X$ which is a contradiction, since all the $G$ are members of filterbases. \qed
Definition 4.11 A fuzzy set $u$ in a fts $X$ is said to be $\beta$-closed relative to $X$ iff for every family $\mu$ of $\beta$-open fuzzy sets such that $\bigvee_{A \in \mu} A = u$, there is a finite subfamily $\eta \subseteq \mu$ such that $(\bigvee_{A \in \eta} \beta \text{cl} A)(x) \geq u(x)$ for every $x \in S(u)$.

Theorem 4.12 A fuzzy subset $u$ in a fts $X$ is $\beta$-closed relative to $X$ iff every fuzzy $\beta$-open filterbases $\xi$ in $X$, $(\bigwedge_{G \in \xi} \beta \text{cl} G) \wedge u = 0_X$, there exists a finite subfamily $\lambda$ of $\xi$ such that $(\bigwedge_{G \in \lambda} G) \wedge u = 0_X$.

Proof. Let $u$ be a $\beta$-closed relative to $X$, suppose $\xi$ is a fuzzy $\beta$-open filterbases in $X$ such that for every finite subfamily $\lambda$ of $\xi$, $(\bigwedge_{G \in \lambda} G) \wedge u = 0_X$. Then for every $x \in S(u)$, $(\bigwedge_{G \in \xi} \beta \text{cl} G)(x) = 0_X$ and hence $(\bigvee_{G \in \xi} \beta \text{cl} G')'(x) = 1_X$ for every $x \in S(u)$. Then $\mu = \{(\beta \text{cl} G)^{'} : G \in \xi\}$ is a $\beta$-open fuzzy set cover of $u$ and hence there exists a finite subfamily $\lambda \subseteq \xi$ such that $(\bigvee_{G \in \lambda} \beta \text{cl} (\beta \text{cl} G)^{'}) \geq u$, so that $(\bigwedge_{G \in \lambda} \beta \text{cl} (\beta \text{cl} G)^{'}) = \bigwedge_{G \in \lambda} \beta \text{int}(\beta \text{cl} G) \leq \hat{u}$ and hence $\bigwedge_{G \in \lambda} G \leq \hat{u}$. Then $\bigwedge_{G \in \lambda} G \wedge u$ which is a contradiction.

Conversely, let $u$ not be a $\beta$-closed fuzzy set relative to $X$, then there exists a $\beta$-open fuzzy set $\mu$ cover of $u$ such that every finite subfamily $\eta \subseteq \mu$, $(\bigvee_{A \in \eta} \beta \text{cl} A)(x) \leq u(x)$ for some $x \in S(u)$ and hence $(\bigwedge_{A \in \eta} (\beta \text{cl} A)^{'})'(x) > \hat{u}(x) \geq 0$ for some $x \in S(u)$. Thus $\xi = \{(\beta \text{cl} A)^{'}, A \in \mu\}$ forms a fuzzy $\beta$-open filterbases in $X$. Let there exists a finite subfamily $\{(\beta \text{cl} A)^{'}, A \in \eta\}$ such that $(\bigwedge_{A \in \eta} (\beta \text{cl} A)^{'}) \wedge u = 0_X$. Then $u \leq (\bigvee_{A \in \eta} \beta \text{cl} A)$. So there exists a finite subfamily $\eta \subseteq \mu$ such that $(\bigvee_{A \in \eta} \beta \text{cl} A) \geq u$ which is a contradiction. Then for each finite subfamily $\lambda = \{(\beta \text{cl} A)^{'}, A \in \eta\}$ of $\xi$, we have $(\bigwedge_{A \in \eta} (\beta \text{cl} A)^{'})qu$. Hence by the given condition $(\bigwedge_{A \in \mu} (\beta \text{cl} (\beta \text{cl} A)^{'}) \wedge u \neq 0_X$, so there exists $x \in S(u)$ such that $(\bigwedge_{A \in \mu} (\beta \text{cl} (\beta \text{cl} A)^{'})'(x) > 0_X$. Then $(\bigvee_{A \in \mu} (\beta \text{cl} (\beta \text{cl} A)^{'})'(x) = (\bigvee_{A \in \mu} \beta \text{int}(\beta \text{cl} A))(x) < 1_X$, and hence $(\bigvee_{A \in \mu} A)(x) < 1_X$ which contradicts the fact that $\mu$ is a $\beta$-open fuzzy set cover of $u$. Therefore $u$ is fuzzy $\beta$-closed relative to $X$. \hfill \Box

Definition 4.13 A fuzzy set $u$ of $X$ is said to be fuzzy $\beta$-regular if it is both $\beta$-open
Proposition 4.14 If \( u \) is \( \beta \)-open fuzzy set in \( X \), then \( \beta \text{cl} \ u \) is \( \beta \)-regular.

Proof. Since \( \beta \text{cl} \ u \) is \( \beta \)-closed, we must show that \( \beta \text{cl} \ u \) is \( \beta \)-open. Since \( u \) is \( \beta \)-open in \( X \), \( v \leq u \leq \text{cl} \ v \) holds for some preopen fuzzy set \( v \) in \( X \). Therefore, we have \( v \leq \beta \text{cl} \ v \leq \beta \text{cl} \ u \leq \text{cl} \ v \), and hence \( \beta \text{cl} \ u \) is \( \beta \)-open.

Theorem 4.15 For a fts \( X \), the following are equivalent:

(a) \( X \) is \( \beta \)-closed space.

(b) Every cover of \( X \) by fuzzy \( \beta \)-regular sets has a finite subcover.

(c) For every collection \( \{ A_j : j \in J \} \) of fuzzy \( \beta \)-regular sets such that \( \bigwedge_{j \in J} A_j = 0_X \), there exists a finite subset \( F \subseteq J \) such that \( \bigwedge_{j \in F} A_j = 0_X \).

Proof. It is obvious from Proposition 4.14 and from the facts that, for every collection \( \{ A_j : j \in J \} \), \( \bigvee_{j \in J} A_j = \bigwedge_{j \in J} A_j \), \( \bigwedge_{j \in F} A_j = \bigvee_{j \in F} A_j \) and

\[ A \text{ is } \beta \text{-open fuzzy set iff } \tilde{A} \text{ is } \beta \text{-closed fuzzy set}. \]

Theorem 4.16 Let \( f : X \rightarrow Y \) be a fuzzy \( \beta \)-continuous surjection function. If \( X \) is \( \beta \)-closed space, then \( Y \) is almost compact.

Proof. Let \( \{ A_j : j \in J \} \) be an open fuzzy set cover of \( Y \). Then \( \{ f^{-1}(A_j) : j \in J \} \) is a \( \beta \)-open fuzzy set cover of \( X \). By hypothesis, there exists a finite subset \( F \subseteq J \) such that \( \bigvee_{j \in F} \beta \text{cl} f^{-1}(A_j) = 1_X \). From the surjectivity of \( f \) and by Lemma 2.6, \( 1_Y = f(1_X) = f(\bigvee_{j \in F} \beta \text{cl} f^{-1}(A_j)) \leq \bigvee_{j \in F} \text{cl}(f^{-1}(A_j)) = \bigvee_{j \in F} A_j \). Hence \( Y \) is almost compact.

Using Lemma 2.5, we have also the following theorem which can proved similarly to Theorem 4.16.

Theorem 4.17 If \( f : X \rightarrow Y \) is fuzzy \( M \beta \)-continuous surjection function and \( X \) is fuzzy \( \beta \)-closed space, then \( Y \) is so.
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Acknowledgement

It is a pleasure to thank the referees for their comments which resulted in an improved presentation of the paper.

References


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Received 14.04.2003