Radical Submodules and Uniform Dimension of Modules

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Abstract

We investigate the relations between a radical submodule \( N \) of a module \( M \) being a finite intersection of prime submodules of \( M \) and the factor module \( M/N \) having finite uniform dimension. It is proved that if \( N \) is a radical submodule of a module \( M \) over a ring \( R \) such that \( M/N \) has finite uniform dimension, then \( N \) is a finite intersection of prime submodules. The converse is false in general but is true if the ring \( R \) is fully left bounded left Goldie and the module \( M \) is finitely generated. It is further proved that, in general, if a submodule \( N \) of a module \( M \) is a finite intersection of prime submodules, then the module \( M/N \) can have an infinite number of minimal prime submodules.

1. Introduction

Throughout this note all rings are associative with identity and all modules are unital left modules. Let \( R \) be a ring and let \( M \) be an \( R \)-module. A submodule \( K \) of \( M \) is called \textit{prime} if \( K \neq M \) and whenever \( r \in R \) and \( L \) is a submodule of \( M \) such that \( rL \subseteq K \) then \( rM \subseteq K \) or \( L \subseteq K \). In this case, the ideal \( P = \{ r \in R : rM \subseteq K \} \) is a prime ideal of \( R \) and we call \( K \) a \textit{P-prime} submodule of \( M \). For more information about prime submodules of \( M \) see, for example, [3]–[8] and [10]. A submodule \( N \) of a module \( M \) is called a \textit{radical} submodule if \( N \) is an intersection of prime submodules of \( M \). Note that radical submodules are proper submodules of \( M \).

Given a submodule \( N \) of a module \( M \), a decomposition \( N = K_1 \cap \cdots \cap K_n \) in terms of submodules \( K_i (1 \leq i \leq n) \) of \( M \), where \( n \) is a positive integer, is called \textit{irredundant}
if \( N \neq K_1 \cap \cdots \cap K_{i-1} \cap K_{i+1} \cap \cdots \cap K_n \) for all \( 1 \leq i \leq n \). In [11], a submodule \( N \) of a module \( M \) is said to have a prime decomposition if \( N \) is the intersection of a finite collection of prime submodules of \( M \). Let \( N \) be a submodule of an \( R \)-module \( M \) such that \( N \) has a prime decomposition. Then \( N \) will be said to have a normal prime decomposition if there exists a positive integer \( n \), distinct prime ideals \( P_i(1 \leq i \leq n) \) of \( R \) and \( P_i \)-prime submodules \( K_i \) \((1 \leq i \leq n)\) of \( M \) such that \( N = K_1 \cap \cdots \cap K_n \) is an irredundant decomposition.

**Lemma 1.1** (See [11, Corollary 2, Theorem 3 and Lemma 14].) Let \( R \) be any ring and let \( N \) be a submodule of an \( R \)-module \( M \) such that \( N \) has a prime decomposition. Then \( N \) has a normal prime decomposition. Moreover, if \( N = K_1 \cap \cdots \cap K_n \) and \( N = L_1 \cap \cdots \cap L_k \) are normal prime decompositions of \( N \) where \( K_i \) is \( P_i \)-prime for some prime ideal \( P_i(1 \leq i \leq n) \) and \( L_j \) is \( Q_j \)-prime for some prime ideal \( Q_j(1 \leq j \leq k) \), then \( n = k \) and \( \{ P_i : 1 \leq i \leq n \} = \{ Q_j : 1 \leq j \leq k \} \).

In Lemma 1.1, the prime ideals \( P_i \) \((1 \leq i \leq n)\) are called the associated prime ideals of \( N \). Given submodules \( G, H \) of an \( R \)-module \( M \) we set (\( G : H \)) = \( \{ r \in R : rH \subseteq G \} \). Note that (\( G : H \)) is an ideal of \( R \). Moreover, (\( G : H \)) = \( R \) if and only if \( H \subseteq G \).

**Lemma 1.2** (See [11, Theorem 6].) Let \( R \) be any ring and let \( N \) be a submodule of an \( R \)-module \( M \) such that \( N \) has a prime decomposition. Then a prime ideal \( P \) of \( R \) is an associated prime ideal of \( N \) if and only if \( P = (N : L) \) for some submodule \( L \) of \( M \).

A module \( M \) has finite uniform dimension if \( M \) does not contain a direct sum of an infinite number of non-zero submodules. Also, a non-zero module \( M \) is uniform if \( X \cap Y \neq 0 \) for all non-zero submodules \( X \) and \( Y \) of \( M \).

**Lemma 1.3** (See [9, 2.2.7, 2.2.8, 2.2.9].) A non-zero \( R \)-module \( M \) has finite uniform dimension if and only if there exist a positive integer \( n \) and independent uniform submodules \( U_i \) \((1 \leq i \leq n)\) of \( M \) such that \( U_1 \oplus \cdots \oplus U_n \) is an essential submodule of \( M \). Moreover, if \( V_i(1 \leq i \leq k) \) are independent uniform submodules of \( M \) such that \( V_1 \oplus \cdots \oplus V_k \) is essential in \( M \) then \( n = k \).

In Lemma 1.3, the positive integer \( n \) is called the uniform (or Goldie) dimension of \( M \) and is denoted by \( u(M) \). Let \( N \) be a submodule of a module \( M \). By Zorn’s Lemma the collection of submodules \( L \) of \( M \) such that \( L \cap N = 0 \) has a maximal member and any
such is called a complement of \( N \) (in \( M \)). A submodule \( K \) of \( M \) is called a complement (in \( M \)) if there exists a submodule \( N \) of \( M \) such that \( K \) is a complement of \( N \).

**Lemma 1.4** (See [2, 1.10 and 5.10].) Let \( L, N \) be submodules of a module \( M \) with \( L \cap N = 0 \). Then there exists a complement \( K \) of \( N \) such that \( L \subseteq K \). Moreover, if \( M \) has finite uniform dimension then \( u(M) = u(N) + u(K) = u(M/K) + u(K) \).

We shall require the following result later. Its proof is included for completeness.

**Lemma 1.5** Given a positive integer \( n \), a module \( M \) has uniform dimension \( n \) if and only if there exist submodules \( L_i \) (\( 1 \leq i \leq n \)) such that

(a) \( M/L_i \) is a uniform module for all \( 1 \leq i \leq n \),

(b) \( 0 = L_1 \cap \cdots \cap L_n \), and

(c) \( 0 \neq L_1 \cap \cdots \cap L_{i-1} \cap L_{i+1} \cap \cdots \cap L_n \) for all \( 1 \leq i \leq n \).

Note that in Lemma 1.5, (b) and (c) can be restated thus: \( 0 = L_1 \cap \cdots \cap L_n \) is an irredundant decomposition.

**Proof.** Suppose first that \( M \) has uniform dimension \( n \). By Lemma 1.3, there exist independent uniform submodules \( U_i \) (\( 1 \leq i \leq n \)) of \( M \) such that \( U_1 \oplus \cdots \oplus U_n \) is an essential submodule of \( M \). For each \( 1 \leq i \leq n \), let \( K_i \) be a complement of \( U_i \) in \( M \) such that \( U_1 \oplus \cdots \oplus U_{i-1} \oplus U_{i+1} \oplus \cdots \oplus U_n \subseteq K_i \) (Lemma 1.4). By Lemma 1.4, \( M/K_i \) is a uniform module for each \( 1 \leq i \leq n \). Suppose that \( K_1 \cap \cdots \cap K_n \neq 0 \). Then \( (K_1 \cap \cdots \cap K_n) \cap (U_1 \oplus \cdots \oplus U_n) \neq 0 \). Let \( 0 \neq x = U_1 + \cdots + U_n \) where \( x \in K_1 \cap \cdots \cap K_n \) and \( u_i \in U_i \) (\( 1 \leq i \leq n \)). Then \( u_1 = x - u_2 - \cdots - u_n \in K_1 \cap U_1 = 0 \), so that \( u_1 = 0 \). Similarly, \( u_i = 0 \) (\( 2 \leq i \leq n \)), and hence \( x = 0 \), a contradiction. Therefore \( 0 = K_1 \cap \cdots \cap K_n \). Moreover, for each \( 1 \leq i \leq n \), \( 0 \neq U_i \subseteq K_1 \cap \cdots \cap K_{i-1} \cap K_{i+1} \cap \cdots \cap K_n \).

Conversely, suppose that \( M \) contains submodules \( L_i \) (\( 1 \leq i \leq n \)) satisfying (a), (b) and (c). Define a mapping \( \phi : M \rightarrow (M/L_1) \oplus \cdots \oplus (M/L_n) \) by \( \phi(m) = (m+L_1, \ldots, m+L_n) \) for all \( m \in M \). By (b), \( \phi \) is a monomorphism. Let \( 1 \leq i \leq n \). By (c) there exists \( 0 \neq m_i \in L_1 \cap \cdots \cap L_{i-1} \cap L_{i+1} \cap \cdots \cap L_n \) and hence \( m_i \not\in L_i \) by (b). It follows that \( 0 \neq (0, \ldots, 0, m_i + L_i, 0, \ldots, 0) = \phi(m_i) \in \phi(M) \). Hence \( \phi(M) \cap (0 \oplus \cdots \oplus 0 \oplus (M/L_i) \oplus 0 \oplus \cdots \oplus 0) \neq 0 \) for all \( 1 \leq i \leq n \). Hence \( \phi(M) \) is an essential submodule of
Before proceeding we make two comments about Lemma 1.5. Firstly, note that a non-zero module $M$ has finite uniform dimension if and only if the zero submodule is the intersection of a finite collection of irreducible submodules. Recall that a submodule $N$ of $M$ is called irreducible if the factor module $M/N$ is uniform. The second comment is that condition (a) in Lemma 1.5 is crucial because if $K$ and $L$ are non-zero submodules of a module $M$ such that $K \cap L = 0$ and $M/K$ and $M/L$ both have finite uniform dimension then $u(M) \leq u(M/K) + u(M/L)$ but it is not necessarily the case that $u(M) = u(M/K) + u(M/L)$. A simple example can be given to illustrate this fact. Let $\mathbb{Z}$ denote the ring of rational integers and $\mathbb{Q}$ the field of rational numbers. Let $M$ denote the $\mathbb{Z}$-module $\mathbb{Q}$ so that $u(M) = 2$. Let $K = \{(q,q) : q \in \mathbb{Q}\}$. Then $M = K \oplus (\mathbb{Q} \oplus 0)$ so that $u(M/K) = 1$. Let $n$ be any positive integer and let $\pi$ be any collection of $n$ distinct primes in $\mathbb{Z}$. Let $X$ denote the submodule $\sum_{p \in \pi} \sum_{k=1}^{\infty} \mathbb{Z}(1/p^k)$ of $\mathbb{Q}$. Note that $X$ consists of all $s/t$ in $\mathbb{Q}$ such that $s, t \in \mathbb{Z}, t \neq 0$ and $t$ is not divisible by any prime $p$ in $\pi$. Note that $\mathbb{Q}/X \cong (\mathbb{Q}/\mathbb{Z})/(X/\mathbb{Z})$ so that $u(\mathbb{Q}/X) = n$. Let $L$ denote the submodule $0 \oplus X$ of $M$. Then $K$ and $L$, are non-zero submodules of $M$ such that $K \cap L = 0$, $u(M/K) = 1$, $u(M/L) = n + 1$ and $u(M) = 2$, so that $u(M) \neq u(M/K) + u(M/L)$.

We complete this section with two results about prime submodules.

**Lemma 1.6** (See [8, Proposition 1.4(ii)].) Let $N$ be a $P$-prime submodule of an $R$-module $M$, for some prime ideal $P$ of $R$, and let $K$ be a proper submodule of $M$ containing $N$ such that $K/N$ is a complement in $M/N$. Then $K$ is a $P$-prime submodule of $M$.

In what follows we shall be particularly interested in irreducible prime submodules of a module $M$, i.e. prime submodules $K$ of $M$ such that $M/K$ is a uniform module. For example, in the $\mathbb{Z}$-module $\mathbb{Q}$, the zero submodule of $\mathbb{Q}$ is an irreducible prime submodule. Lemma 1.6 has the following consequence.

**Corollary 1.7** Let $N$ be a $P$-prime submodule of an $R$-module $M$, for some prime ideal $P$ of $R$, and let $L$ be a non-zero submodule of $M$ such that $N \cap L = 0$. Let $K$ be a complement of $L$ in $M$ such that $N \subseteq K$. Then $K$ is a $P$-prime submodule of $M$. Moreover, if $L$ is a uniform module then $K$ is an irreducible $P$-prime submodule of $M$. 

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Proof. Note that $K \cap L = 0$ and $L \neq 0$ together imply $K \neq M$. It is easy to check that $K/N$ is a complement of $(L + N)/N$ in $M/N$. By Lemma 1.6, $K$ is a $P$-prime submodule of $M$. Now suppose that $L$ is uniform. By Lemma 1.4, $u(M/K) = u(L) = 1$, i.e. $K$ is an irreducible prime submodule of $M$. □

2. Modules with finite uniform dimension

In this section we shall prove that any radical submodule $N$ of a module $M$ such that the factor module $M/N$ has finite uniform dimension has a prime decomposition and we shall investigate the associated prime ideals of $N$.

Let $U$ be a uniform $R$-module. Let $P = \{ r \in R : rV = 0 \text{ for some non-zero submodule } V \text{ of } U \}$. Then $P$ is an ideal of $R$. Following [1] we shall call $P$ the assassinator of $U$. It can easily be checked that if $PW = 0$ for some non-zero submodule $W$ of $U$ then $P$ is a prime ideal of $R$.

Lemma 2.1 Let $U$ be a uniform submodule of an $R$-module $M$ and let $P$ be the assassinator of $U$. Suppose that $PM \cap U = 0$. Then there exists an irreducible $P$-prime submodule $K$ of $M$ such that $K \cap U = 0$.

Proof. Note that $PU = 0$ so that $P$ is a prime ideal of $R$. Let $K$ be a complement of $U$ in $M$ such that $PM \subseteq K$ (Lemma 1.4). Let $r \in R$ and let $L$ be a submodule of $M$ containing $K$ such that $rL \subseteq K$. Then $r(L \cap U) \subseteq K \cap U = 0$. Either $L \cap U = 0$ in which case $L = K$ or $L \cap U \neq 0$ in which case $r \in P$ because $P$ is the assassinator of $U$. It follows that $K$ is a $P$-prime submodule of $M$. By Lemma 1.4, $M/K$ is a uniform module and hence $K$ is an irreducible $P$-prime submodule of $M$. □

Lemma 2.2 Let $M$ be an $R$-module such that the zero submodule of $M$ is a radical submodule. Let $U$ be a uniform submodule of $M$ with assassinator $P$. Then $PM \cap U = 0$.

Proof. Let $A$ be a finitely generated left ideal of $R$ such that $A \subseteq P$. There exists a non-zero submodule $V$ of $U$ such that $AV = 0$. There exist prime submodules $K_\lambda$ ($\lambda \in \Lambda$) of $M$ such that $0 = \cap_{\lambda \in \Lambda} K_\lambda$. Let $\lambda \in \Lambda$. If $V \nsubseteq K_\lambda$ then $AV = 0 \subseteq K_\lambda$ gives $AM \subseteq K_\lambda$. Hence $AM \cap V \subseteq K_\lambda$. Thus $AM \cap V \subseteq \cap_{\lambda \in \Lambda} K_\lambda = 0$. Next $(AM \cap U) \cap V = AM \cap V = 0$,
so that $AM \cap U = 0$ because $U$ is uniform. Clearly it follows that $PM \cap U = 0$. 

**Theorem 2.3** Let $R$ be any ring and let $M$ be a non-zero $R$-module such that the zero submodule of $M$ is a radical submodule. Then the following statements are equivalent.

(i) The zero submodule of $M$ is a finite intersection of irreducible prime submodules of $M$.

(ii) $M$ has finite uniform dimension.

Moreover, in this case if $0 = K_1 \cap \cdots \cap K_n$ is any irredundant decomposition, where $K_i$ is an irreducible prime submodule of $M$ for each $1 \leq i \leq n$, then $n = u(M)$.

**Proof.** (ii) $\Rightarrow$ (i) Suppose that $M$ has finite uniform dimension. Let $U_1$ be any uniform submodule of $M$ and let $P_1$ be the assassinator of $U_1$. By Lemma 2.2, $P_1 M \cap U_1 = 0$ and by Lemma 2.1 there exists an irreducible $P_1$-prime submodule $K_1$ of $M$ such that $K_1 \cap U_1 = 0$. If $u(M) = 1$ then $K_1 = 0$ and the result is proved.

Suppose that $u(M) \geq 2$. Let $U_2$ be any uniform submodule of $M$ such that $U_1 \cap U_2 = 0$. If $K_1 \cap (U_1 \oplus U_2) = 0$ then set $K_2 = M$. Suppose that $K_1 \cap (U_1 \oplus U_2) \neq 0$. Note that $K_1 \cap (U_1 \oplus U_2)$ embeds in $U_2$ (because $K_1 \cap U_1 = 0$) and hence $K_1 \cap (U_1 \oplus U_2)$ is a uniform submodule of $M$. Let $P_2$ be the assassinator of $K_1 \cap (U_1 \oplus U_2)$. As above, by Lemmas 2.2 and 2.1 there exists an irreducible $P_2$-prime submodule $K_2$ of $M$ such that $K_2 \cap \{K_1 \cap (U_1 \oplus U_2)\} = 0$ and hence $(K_1 \cap K_2) \cap (U_1 \oplus U_2) = 0$. If $u(M) = 2$ then $U_1 \oplus U_2$ is essential in $M$ and hence $K_1 \cap K_2 = 0$ so that again the result is true because $K_2 \cap K_2 = 0$ is an irreducible prime submodule.

Suppose that $u(M) \geq 3$. Let $U_3$ be any uniform submodule of $M$ such that $(U_1 \oplus U_2) \cap U_3 = 0$. By the above argument there exists a submodule $K_3$ of $M$ such that $(K_1 \cap K_2 \cap K_3) \cap (U_1 \oplus U_2 \oplus U_3) = 0$ and either $K_3 = M$ or $K_3$ is an irreducible prime submodule of $M$. Repeat this process to obtain a sequence $U_i (i \geq 1)$ of independent uniform submodules and a sequence $K_i (i \geq 1)$ of submodules such that $K_1$ is an irreducible prime submodule and for each $i \geq 2$ the submodule $K_i = M$ or $K_i$ is irreducible prime satisfying

$$(K_1 \cap \cdots \cap K_s) \cap (U_1 \oplus \cdots \oplus U_s) = 0$$

for each positive integer $s$. Let $n = u(M) \geq 1$. Then $U_1 \oplus \cdots \oplus U_n$ is an essential submodule of $M$ and hence $K_1 \cap \cdots \cap K_n = 0$.

$\square$
Corollary 2.4 Let $N$ be a radical submodule of an $R$-module $M$. Then $N$ is a finite intersection of irreducible prime submodules of $M$ if and only if $M/N$ has finite uniform dimension. In this case, $N$ has a prime decomposition.

Proof. By Theorem 2.3. \qed

In certain circumstances, every radical submodule of a module $M$ is an intersection of irreducible prime submodules. In order to prove this we begin with the following lemma.

Lemma 2.5 Let $P$ be a prime ideal of a ring $R$ and let $M$ be an $R$-module such that $0$ is a $P$-prime submodule of $M$ and every non-zero submodule contains a uniform submodule of $M$. Then the zero submodule is an intersection of irreducible $P$-prime submodules of $M$.

Proof. By Zorn’s Lemma $M$ contains a maximal independent collection of uniform submodules $U_\lambda (\lambda \in \Lambda)$ and by hypothesis $\oplus_{\lambda \in \Lambda} U_\lambda$ is an essential submodule of $M$. Let $\mu \in \Lambda$ and let $L_\mu = \oplus_{\lambda \neq \mu} U_\lambda$. Note that $L_\mu$ is a submodule of $M$ such that $L_\mu \cap U_\mu = 0$. By Lemma 1.4 there exists a complement $K_\mu$ of $U_\mu$ in $M$ such that $L_\mu \subseteq K_\mu$. Now Lemma 1.6 gives that $K_\mu$ is $P$-prime. It is easy to check that $(\cap_{\lambda \in \Lambda} K_\lambda) \cap (\oplus_{\lambda \in \Lambda} U_\lambda) = 0$ and hence $\cap_{\lambda \in \Lambda} K_\lambda = 0$ where $K_\lambda$ is a $P$-prime submodule of $M$ for each $\lambda \in \Lambda$. \qed

We shall say that a (non-zero) $R$-module $M$ has many uniforms if for every prime submodule $K$ of $M$ and for each element $m \in M \setminus K$, the submodule $(Rm+K)/K$ contains a uniform submodule.

Theorem 2.6 Let $M$ be an $R$-module with many uniforms. Then, for any prime ideal $P$ of $R$, every $P$-prime submodule of $M$ is an intersection of irreducible $P$-prime submodules of $M$. Moreover, every radical submodule of $M$ is an intersection of irreducible prime submodules of $M$.

Proof. Let $P$ be a prime ideal of $R$ and let $K$ be a $P$-prime submodule of $M$. Applying Lemma 2.5 to the module $M/K$ we see that $0 = \cap_{\lambda \in \Lambda} K_\lambda/K$ where $K_\lambda$ is a submodule containing $K$ such that $K_\lambda/K$ is an irreducible $P$-prime submodule of $M/K$ for each $\lambda \in \Lambda$. Clearly $K = \cap_{\lambda \in \Lambda} K_\lambda$ where $K_\lambda$ is an irreducible $P$-prime submodule of $M$ for
each \( \lambda \in \Lambda \). The last part is clear. \( \square \)

Note that if \( R \) is a left Noetherian ring then every non-zero left \( R \)-module has many uniforms. More generally, if a ring \( R \) has left Krull dimension then every non-zero left \( R \)-module has many uniforms by [9, 6.2.4 and 6.2.6]. A ring \( R \) is called \textit{left semi-artinian} if every non-zero cyclic left \( R \)-module contains a simple submodule. For example, right perfect rings are left semi-artinian. Clearly if \( R \) is a left semi-artinian ring then every non-zero left \( R \)-module has many uniforms. (For more information on left semi-artinian rings see [2, pp26-28].) In the next section we shall show that if \( R \) is any commutative ring, or more generally any ring satisfying a polynomial identity, then every non-zero \( R \)-module has many uniforms.

Next we give a characterization of the associated prime ideals of a radical submodule \( N \) in case \( M/N \) has finite uniform dimension (compare Lemma 1.2).

\textbf{Theorem 2.7} Let \( N \) be a radical submodule of an \( R \)-module \( M \) such that \( M/N \) has finite uniform dimension. Then \( P \) is an associated prime ideal of \( N \) if and only if \( P \) is the assassinator of a uniform submodule of the module \( M/N \).

\textbf{Proof.} Suppose first that \( L \) is a submodule of \( M \) containing \( N \) such that \( L/N \) is a uniform module. Let \( P \) be the assassinator of \( L/N \). By Lemma 2.2, \( P = (N:L) \) and by Lemma 1.2, \( P \) is an associated prime ideal of \( N \).

Conversely, suppose that \( P \) is an associated prime ideal of \( N \). Let \( N = K_1 \cap \cdots \cap K_n \) be a normal prime decomposition of \( N \) where \( K_i \) is a \( P_i \)-prime submodule of \( M \) for some prime ideal \( P_i \) for each \( 1 \leq i \leq n \) and \( n \) is a positive integer. Without loss of generality, we can suppose that \( P = P_1 \) (Lemma 1.1). If \( n = 1 \) then \( N = K_1 \) and so \( N \) is a \( P \)-prime submodule of \( M \). Let \( H \) be a submodule of \( M \) properly containing \( N \) such that \( H/N \) is a uniform module. Clearly \( P \) is the assassinator of \( H/N \).

Now suppose that \( n \geq 2 \). Since \( K_2 \cap \cdots \cap K_n \neq N \) it follows that there exists a submodule \( G \) of \( K_2 \cap \cdots \cap K_n \) properly containing \( N \) such that \( G/N \) is a uniform module. Note that \( PG \subseteq K_1 \cap \cdots \cap K_n = N \). On the other hand, let \( r \in R \) and let \( J \) be a submodule of \( G \) such that \( rJ \subseteq N \). Then \( rJ \subseteq K_1 \). Either \( J \subseteq K_1 \) in which case \( J \subseteq K_1 \cap \cdots \cap K_n = N \) or \( r \in P \). It follows that \( P \) is the assassinator of the uniform submodule \( G/N \) of \( M/N \). \( \square \)
Corollary 2.8 Let $N$ be a radical submodule of an $R$-module $M$ such that $M/N$ has finite uniform dimension. Then a prime ideal $P$ of $R$ is the assassinator of a uniform submodule of the module $M/N$ if and only if $P = (N : L)$ for some submodule $L$ of $M$.

Proof. By Lemma 1.2 and Theorem 2.5. \hfill \Box

3. Modules over fully bounded rings

We now consider when it is the case that every submodule $N$ of a module $M$ with $N$ having a prime decomposition has the property that the factor module $M/N$ has finite uniform dimension. Note that if $F$ is a field and $V$ an infinite dimensional vector space over $F$ then the zero subspace of $V$ is a prime submodule, but the $F$-module $V$ does not have finite uniform dimension. Because of this example we shall consider finitely generated modules. But even for finitely generated modules there are problems. In [1, Example 1.22] an example is given of a right Noetherian domain such that the left $R$-module $R$ does not have finite uniform dimension. Thus we shall also restrict the choice of the ring $R$.

A prime ring $R$ is left bounded if every essential left ideal contains a non-zero two-sided ideal. A general ring $R$ is a fully left bounded left Goldie ring (left FBG-ring for short) if, for each prime ideal $P$ of $R$, the prime ring $R/P$ is a left bounded left Goldie ring. Clearly commutative rings are (left) FBG-rings, as are rings with polynomial identity by [9, 13.6.6].

Let $R$ be a prime left Goldie ring. An element $c$ of $R$ is regular if $cr \neq 0$ and $rc \neq 0$ for every non-zero element $r$ of $R$. An $R$-module $M$ is called torsion-free if $cm \neq 0$ for every regular element $c$ of $R$ and non-zero element $m$ of $M$. On the other hand, $M$ is a torsion module if for each $m \in M$ there exists a regular element $c$ of $R$ such that $cm = 0$.

Lemma 3.1 (See [8, Lemma 2.6].) Let $P$ be a prime ideal of a ring $R$ such that $R/P$ is a left bounded left Goldie ring and let $K$ be a submodule of an $R$-module $M$. Then $K$ is a $P$-prime submodule of $M$ if and only if $P = (K : M)$ and the $(R/P)$-module $M/K$ is torsion-free.

Let $P$ be a prime ideal of a ring $R$. By a maximal $P$-prime submodule of an $R$-module $M$ we mean a $P$-prime submodule $K$ of $M$ such that $K$ is not properly contained
in any $P$-prime submodule of $M$. By a *maximal prime* submodule of $M$ we shall mean a submodule which is a maximal $Q$-prime submodule of $M$ for some prime ideal $Q$ of $R$. In [7], given a prime ideal $P$ of $R$, a submodule $L$ of a module $M$ is called $P$-maximal if $L$ is maximal in the collection of submodules $H$ of $M$ such that $P = (H : M)$.

**Lemma 3.2** Let $P$ be a prime ideal of a ring $R$. Consider the following statements about a submodule $K$ of an $R$-module $M$.

(i) $K$ is $P$-maximal;

(ii) $K$ is maximal $P$-prime;

(iii) $K$ is irreducible $P$-prime.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Moreover, if $R/P$ is a left bounded left Goldie ring then (iii) $\Rightarrow$ (ii). If in addition $M$ is finitely generated, then (ii) $\Rightarrow$ (i).

**Proof.** (i) $\Rightarrow$ (ii) Let $K$ be a $P$-maximal submodule of $M$. Note that $P = (K : M)$. Let $r \in R$ such that $rL \subseteq K$ for some submodule $L$ of $M$ properly containing $K$. Let $A = (L : M)$. Then $P \subseteq A$ because $K$ is $P$-maximal. Now $rAM \subseteq rL \subseteq K$, so that $rA \subseteq P$ and hence $r \in P$. It follows that $K$ is $P$-prime. Clearly $K$ is a maximal $P$-prime submodule of $M$.

(ii) $\Rightarrow$ (iii) Let $K$ be a maximal $P$-prime submodule of $M$. Let $L$ be any submodule of $M$ properly containing $K$. Let $H$ be a submodule of $M$ containing $K$ such that $H/K$ is a complement of $L/K$ in $M/K$. Since $L/K \neq 0$ it follows that $H/K \neq M/K$. By Lemma 1.6, $H$ is a $P$-prime submodule of $M$. Then $H = K$. It follows that $L/K$ is an essential submodule of $M/K$. Therefore $M/K$ is a uniform module and $K$ is an irreducible $P$-prime submodule of $M$.

Now suppose that $R/P$ is a left bounded left Goldie ring. Let $K$ be an irreducible $P$-prime submodule of $M$. Let $G$ be any submodule of $M$ properly containing $K$. Let $m \in M$. Since $G/K$ is an essential submodule of the $(R/P)$-module $M/K$ it follows that $E(m + G) = 0$ for some essential left ideal $E$ of the ring $R/P$. By [9, 2.3.5.] there exists a regular element $\sigma$ of $R/P$ such that $\sigma(m + G) = 0$. It follows that $M/G$ is a torsion $(R/P)$-module for every submodule $G$ properly containing $K$. By Lemma 3.1, $N$ is a maximal $P$-prime submodule of $M$. 

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Finally, suppose that $M$ is a finitely generated module (and $R/P$ is left bounded left Goldie). Let $K$ be an irreducible $P$-prime submodule of $M$ and let $G$ be any submodule of $M$ properly containing $N$. As before, $M/G$ is a torsion $(R/P)$-module. By hypothesis, there exists an ideal $A$ of $R$ properly containing $P$ such that $AM \subseteq G$. Thus $P \subseteq (G : M)$. It follows that $K$ is $P$-maximal.

Let $M$ be a finitely generated $R$-module. Then $g(M)$ will denote the least number of elements in a smallest generating set of $M$.

**Lemma 3.3** Let $R$ be a prime left Goldie ring and let $M$ be a finitely generated torsion-free $R$-module. Then $M$ has finite uniform dimension and $u(M) \leq g(M)u(R)$.

**Proof.** Suppose that $M \neq 0$ and $g(M) = k$, for some positive integer $k$. There exists an epimorphism $\phi : R^k \to M$. Let $K = \ker \phi$. Then $R^k/K$ is torsion-free so that $K$ is a complement submodule of $R^k$ by [2, 1.10]. By Lemma 1.4,

$$ku(R) = u(R^k) = u(K) + u((R^k)/K) \geq u(R^k/K) = u(M).$$

**Corollary 3.4** Let $P$ be a prime ideal of a ring $R$ such that the ring $R/P$ is left bounded left Goldie and let $K$ be a $P$-prime submodule of a finitely generated $R$-module $M$. Then the $R$-module $M/K$ has finite uniform dimension and $u(M/K) \leq g(M/K)u(R/P)$.

**Proof.** By Lemmas 3.1 and 3.3

**Theorem 3.5** Let $R$ be a left FBG-ring. Then the following statements are equivalent for a submodule $N$ of a finitely generated $R$-module $M$.

(i) $N$ is a radical submodule of $M$ and $M/N$ has finite uniform dimension.

(ii) $N$ is a finite intersection of maximal prime submodules of $M$.

(iii) $N$ has a prime decomposition.
Proof.  (i) $\Rightarrow$ (ii) By Corollary 2.4 and Lemma 3.2.

(ii) $\Rightarrow$ (iii) Clear.

(iii) $\Rightarrow$ (i) Suppose that $N$ has a prime decomposition. Then $N$ is a radical submodule of $M$. Let $N = K_1 \cap \cdots \cap K_n$ be a prime decomposition where $K_i$ is a $P_i$-prime submodule of $M$ for some prime ideal $P_i$ of $R$ for each $1 \leq i \leq n$. For each $1 \leq i \leq n$, the prime ring $R/P_i$ is left bounded left Goldie. By Corollary 3.4, the $R$-module $M/K_i$ has finite uniform dimension. Since $M/N$ embeds in $(M/K_1) \oplus \cdots \oplus (M/K_n)$ it follows that $M/N$ has finite uniform dimension. \qed

Theorem 3.6 Let $R$ be a left FBG-ring and let $M$ be a non-zero $R$-module. Then, for any prime ideal $P$ of $R$, every $P$-prime submodule of $M$ is an intersection of maximal $P$-prime submodules of $M$. Moreover, every radical submodule of $M$ is an intersection of maximal prime submodules of $M$.

Proof.  We shall prove that $M$ has many uniforms. Let $Q$ be a prime ideal of $R$ and let $K$ be a $Q$-prime submodule of $M$. Let $m \in M \setminus K$. Note that the ring $R/Q$ is a left bounded left Goldie ring and the $(R/Q)$-module $M/K$ is torsion-free (see Lemma 3.1). Hence $(Rm + K)/K$ is a torsion-free cyclic $(R/Q)$-module. There exists a non-essential left ideal $\mathcal{T}$ of $R = R/Q$ such that $(Rm + K)/K \cong \mathcal{R}/\mathcal{T}$. Next there exists a uniform left ideal $\mathcal{U}$ of $\mathcal{R}$ such that $\mathcal{L} \cap \mathcal{U} = 0$, and hence $\mathcal{U}$ embeds in $(Rm + K)/K$. It follows that $M$ has many uniforms. By Theorem 2.6 and Lemma 3.2, every $P$-prime submodule is an intersection of maximal $P$-prime submodules of $M$, for each prime ideal $P$ of $R$. The last part is clear.

Next we shall examine the fully left bounded condition further. We begin with the following result. \qed

Lemma 3.7 Let $R$ be a prime ring such that every ideal is finitely generated as a left ideal and let $M$ be a finitely generated $R$-module such that the zero submodule $0 = K_1 \cap \cdots \cap K_n$ where $n$ is a positive integer and $K_i$ is a maximal 0-prime submodule of $M$ for each $1 \leq i \leq n$. Let $L$ be a submodule of $M$ such that $L \cap K_1 \cap \cdots \cap K_{i-1} \cap K_{i+1} \cap \cdots \cap K_n \not\subseteq K_i$ for each $1 \leq i \leq n$. Then there exists a non-zero ideal $A$ of $R$ such that $AM \subseteq L$.

Proof.  The result is proved by induction on $n$. Suppose that $n = 1$. Then $0$ is a maximal 0-prime submodule of $M$ and $L$ is a non-zero submodule of $M$. Let
H = \{m \in M : Bm \subseteq L \text{ for some non-zero ideal } B \text{ of } R\}. It is easy to check that H is a submodule of M. Let x \in M such that Cx \subseteq H for some non-zero ideal C of R. There exist a positive integer k and elements c_i \in C(1 \leq i \leq k) such that C = Rc_1 + \cdots + Rc_k. For each 1 \leq i \leq k there exists a non-zero ideal D_i of R such that D_ix \subseteq L. Let D = D_1 \cdots D_kC. Then D is a non-zero ideal of R such that Dx = D_1 \cdots D_kCkx = \sum_{i=1}^{k} D_1 \cdots D_kc_ix \subseteq L, and hence x \in H. It follows that if H \neq M then H is a 0-prime submodule of M. Because 0 is a maximal 0-prime submodule of M, we deduce that H = M. Now M is finitely generated and it easily follows that AM \subseteq L for some non-zero ideal A of R.

Now suppose that n \geq 2. Let K = K_1 \cap \cdots \cap K_{n-1}. Note that \{[(L \cap K_n) + K]/K]\cap [(K_1/K) \cap \cdots \cap (K_{i-1}/K) \cap (K_{i+1}/K) \cap \cdots \cap (K_{n-1}/K)] \notin K_i/K for all 1 \leq i \leq n - 1. By induction on n there exists a non-zero ideal A_1 of R such that A_1(M/K) \subseteq [(L \cap K_n) + K]/K, i.e. A_1M \subseteq (L \cap K_n) + K. On the other hand, L \cap K \notin K_n so that, by the case n = 1, there exists a non-zero ideal A_2 of R such that A_2(M/K_n) \subseteq [(L \cap K) + K_n]/K_n, i.e. A_2M \subseteq (L \cap K) + K_n. Let A = A_1A_2. Then A is a non-zero ideal of R and

\[AM \subseteq [(L \cap K_n) + K] \cap [(L \cap K) + K_n] \subseteq (L \cap K) + (L \cap K_n) \subseteq L,\]

because K \cap K_n = 0.

Corollary 3.8 Let R be a prime ring such that every ideal is finitely generated as a left ideal and let M be a finitely generated left R-module such that the zero submodule is the intersection of a finite collection of maximal 0-prime submodules. Let L be an essential submodule of M. Then there exists a non-zero ideal A of R such that AM \subseteq L.

Proof. There exist a positive integer n and maximal 0-prime submodules K_i(1 \leq i \leq n) such that 0 = K_1 \cap \cdots \cap K_n and 0 \neq K_1 \cap \cdots \cap K_{i-1} \cap K_{i+1} \cap \cdots \cap K_n for all 1 \leq i \leq n. Clearly L \cap K_1 \cap \cdots \cap K_{i-1} \cap K_{i+1} \cap \cdots \cap K_n \notin K_i for all 1 \leq i \leq n. The result follows by Lemma 3.6.

Theorem 3.9 The following statements are equivalent for a left Noetherian ring R.
(i) $R$ is fully left bounded.

(ii) Every radical submodule of every finitely generated $R$-module is a finite intersection of maximal prime submodules of $M$.

(iii) Every radical submodule of the $R$-module $R$ is a finite intersection of maximal prime submodules of the $R$-module $R$.

(iv) Every prime ideal $P$ of $R$ is a finite intersection of maximal $P$-prime submodules of the $R$-module $R$.

Proof. (i) ⇒ (ii) By Theorem 3.5.

(ii) ⇒ (iii) Clear.

(iii) ⇒ (iv) Let $P$ be any prime ideal of $R$. By (iii) there exist a positive integer $n$, prime ideals $P_i(1 \leq i \leq n)$ and maximal $P_i$-prime submodules $K_i(1 \leq i \leq n)$ of $R$ such that $P = K_1 \cap \cdots \cap K_n$ and $P \neq K_1 \cap \cdots \cap K_{i-1} \cap K_{i+1} \cap \cdots \cap K_n$ for all $1 \leq i \leq n$. For each $1 \leq i \leq n$, $PR \subseteq K_i$ so that $P \subseteq (K_i : R) = P_i$. Suppose that $P \neq P_i$ for some $1 \leq i \leq n$. Then $P_i(K_1 \cap \cdots \cap K_{i-1} \cap K_{i+1} \cap \cdots \cap K_n) \subseteq P$, so that $K_1 \cap \cdots \cap K_{i-1} \cap K_{i+1} \cap \cdots \cap K_n = P$, a contradiction. Thus $P = P_i(1 \leq i \leq n)$. This proves (iv).

(iv) ⇒ (i) Let $Q$ be any prime ideal of $R$. Let $M$ denote the $R$-module $R/Q$. Then the $(R/Q)$-module $M$ satisfies the hypotheses of Corollary 3.8. Let $E$ be any left ideal of $R$ containing $Q$ such that $E/Q$ is an essential left ideal of $R/Q$. By Corollary 3.8 there exists an ideal $A$ of $R$ properly containing $Q$ such that $(A/Q)(R/Q) \subseteq E/Q$, i.e. $A \subseteq E$. Hence $R/Q$ is left bounded.

Finally, note that if $R$ is an arbitrary ring and $N$ is a radical submodule of an $R$-module $M$ such that the module $M/N$ has only a finite number of minimal prime submodules then $N$ has a prime decomposition (see [8, p.1059]). The converse is false. Consider the following result.

Theorem 3.10 Let $P$ and $Q$ be prime ideals of a ring $R$ such that $P \nsubseteq Q$ and $Q \nsubseteq P$ and let $N$ be the submodule $P \oplus Q$ of the $R$-module $R \oplus R$. Then $N = K \cap L$ where $K$ is the $P$-prime submodule $P \oplus R$ and $L$ is the $Q$-prime submodule $R \oplus Q$ of $M$. Moreover,
the minimal prime submodules of $M/N$ are $K/N, L/N$ and $BM/N$ where $P + Q \subseteq B$ and $B/(P + Q)$ is a minimal prime ideal of the ring $R/(P + Q)$.

**Proof.** The first part is clear. Let $G$ be a submodule of $M$ containing $N$ such that $G/N$ is a minimal prime submodule of $M/N$. Note that $G$ is a prime submodule of $M$. Now $P(R \oplus 0) \subseteq G$ gives $R \oplus 0 \subseteq G$ or $PM \subseteq G$. If $R \oplus 0 \subseteq G$ then $R \oplus Q \subseteq G$ and $(R \oplus Q)/N$ is a prime submodule of $M/N$ so that $G/N = (R \oplus Q)/N$. Suppose that $PM \subseteq G$. Next $Q(0 \oplus R) \subseteq G$ gives that $G/N = (P \oplus R)/N$ or $QM \subseteq G$. Suppose that $QM \subseteq G$. Then $(P + Q)M \subseteq G$. Because $P + Q$ is contained in the prime ideal $(G : M)$ there exists a prime ideal $B$ of $R$ such that $P + Q \subseteq B \subseteq (G : M)$ and $B/(P + Q)$ is a minimal prime ideal of the ring $R/(P + Q)$. Note that $BM/N$ is a prime submodule of $M/N$ such that $BM/N \subseteq G/N$. Then $G/N = BM/N$. \qed

Let $S$ be a commutative domain such that there exists a proper ideal $A$ of $S$ such that the ring $S/A$ has an infinite number of minimal prime ideals. Let $R$ denote the polynomial ring $S[X]$ where $X$ is the set of indeterminates $\{x_a : a \in A\}$. Let $P = \sum_{a \in A} Rx_a$ and let $Q = \sum_{a \in A} R(x_a - a)$. Then $P$ and $Q$ are prime ideals of $R$ because $R/P \cong R/Q \cong S$. Moreover, $P + Q = P + A$ and $R/(P + Q) \cong S/A$, so that the ring $R/(P + Q)$ contains an infinite number of minimal prime ideals. If $N$ is the submodule $P \oplus Q$ of the $R$-module $M = R \oplus R$ then $N$ has a prime decomposition but the $R$-module $M/N$ contains an infinite number of minimal prime submodules by Theorem 3.10.

To find a commutative domain $S$ and an ideal $A$ with the above properties we proceed as follows. Let $T$ be any commutative von Neumann regular ring which is not Artinian. Then every prime ideal of $T$ is maximal and $T$ contains an infinite number of (minimal) prime ideals. Let $S = \mathbb{Z}[X]$ denote the polynomial ring in the set $X = \{x_t : t \in T\}$ of indeterminates. Then $S$ is a commutative domain and there exists a ring epimorphism $\phi : S \to T$ such that $\phi(x_t) = t$ ($t \in T$). Let $A$ denote the kernel of $\phi$. Then $A$ is an ideal of $S$ such that $S/A \cong T$.

**References**


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