Shape Operator $A_H$ for Slant Submanifolds in Generalized Complex Space Forms

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Abstract

In this article, we establish an inequality between the sectional curvature function $K$ and the shape operator $A_H$ at the mean curvature vector for slant submanifolds in generalized complex space forms. Also a sharp relationship between the $k$-Ricci curvature and the shape operator $A_H$ is proved.

Key Words: Shape operator, slant submanifolds, generalized complex space form, $k$-Ricci curvature.

1. Preliminaries

In the introduction of [2], B. Y. Chen recalls as one of the basic problems in submanifold theory:

“Find simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold”.

In the above mentioned paper, B. Y. Chen establishes a relationship between sectional curvature function $K$ and the shape operator $A_H$ for submanifolds in real space forms.

Also, in [3], B. Y. Chen proves a sharp inequality between the $k$-Ricci curvature and the shape operator $A_H$.

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In [6], we establish a relationship between the sectional curvature function $K$ and the shape operator $A_H$ and a sharp relationship between the $k$-Ricci curvature and the shape operator $A_H$, respectively, for slant submanifolds in complex space forms.

Let $\tilde{M}$ be an almost Hermitian manifold with almost complex structure $J$ and Riemannian metric $g$. One denotes by $\tilde{\nabla}$ the operator of covariant differentiation with respect to $g$ in $\tilde{M}$.

**Definition.** If the almost complex structure $J$ satisfies

$$(\tilde{\nabla}_X J)Y + (\tilde{\nabla}_Y J)X = 0,$$

for any vector fields $X$ and $Y$ on $\tilde{M}$, then the manifold $\tilde{M}$ is called a *nearly-Kaehler manifold* [5], [11].

**Remark.** The above condition is equivalent to

$$(\tilde{\nabla}_X J)X = 0, \quad \forall X \in \Gamma T\tilde{M}.$$

For an almost complex structure $J$ on the manifold $\tilde{M}$, the *Nijenhuis tensor field* is defined by

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y],$$

for any vector fields $X, Y$ tangent to $\tilde{M}$, where $[,]$ is the Lie bracket.

A necessary and sufficient condition for a nearly-Kaehler manifold to be Kaehler is the vanishing of the Nijenhuis tensor $N_J$.

Any 4-dimensional nearly-Kaehler manifold is a Kaehler manifold.

**Example.** Let $S^6$ be the 6-dimensional unit sphere defined as follows:

Let $E^7$ be the set of all purely imaginary Cayley numbers. Then $E^7$ is a 7-dimensional subspace of the Cayley algebra $C$.

Let $\{1, e_0, e_1, ..., e_6\}$ be a basis of the Cayley algebra, $1$ being the unit element of $C$.

If $X = \sum_{i=0}^{6} x^i e_i$ and $Y = \sum_{i=0}^{6} y^i e_i$ are two elements of $E^7$, one defines the *scalar product* in $E^7$ by

$$<X, Y> = \sum_{i=0}^{6} x^i y^i,$$
and the vector product by
\[ X \times Y = \sum_{i \neq j} x^i y^j e_i \wedge e_j, \]
* being the multiplication operation of C.

Consider the 6-dimensional unit sphere \( S^6 \) in \( E^7 \):
\[ S^6 = \{ X \in E^7 \mid \langle X, X \rangle = 1 \}. \]

The scalar product in \( E^7 \) induces the natural metric tensor field \( g \) on \( S^6 \).

The tangent space \( T_X S^6 \) at \( X \in S^6 \) can naturally be identified with the subspace of \( E^7 \) orthogonal to \( X \).

Define the endomorphism \( J_X \) on \( T_X S^6 \) by
\[ J_X Y = X \times Y, \text{ for } Y \in T_X S^6. \]

It is easy to see that
\[ g(J_X Y, J_X Z) = g(Y, Z), \quad Y, Z \in T_X S^6. \]

The correspondence \( X \mapsto J_X \) defines a tensor field \( J \) such that \( J^2 = -I \).

Consequently, \( S^6 \) admits an almost Hermitian structure \((J, g)\).

This structure is a non-Kählerian nearly-Kählerian structure (its Betti numbers of even order are 0).

We will consider a class of almost Hermitian manifolds, called RK-manifolds, which contains nearly-Kähler manifolds.

**Definition** [10]. A RK-manifold \((\tilde{M}, J, g)\) is an almost Hermitian manifold for which the curvature tensor \( \tilde{R} \) is invariant by \( J \), i.e.
\[ \tilde{R}(JX, JY, JZ, JW) = \tilde{R}(X, Y, Z, W), \]
for any \( X, Y, Z, W \in \Gamma T\tilde{M} \).

An almost Hermitian manifold \( \tilde{M} \) is of pointwise constant type if, for any \( p \in \tilde{M} \) and \( X \in T_p\tilde{M} \), we have
\[ \lambda(X, Y) = \lambda(X, Z), \]
where
\[ \lambda(X, Y) = \tilde{R}(X, Y, JX, JY) - \tilde{R}(X, Y, X, Y) \]
and \( Y \) and \( Z \) are unit tangent vectors on \( \tilde{M} \) at \( p \), orthogonal to \( X \) and \( JX \), i.e.
\[ g(Z, Z) = g(Y, Y) = 1, \]
\[ g(X, Y) = g(JX, Y) = g(X, Z) = g(JX, Z) = 0. \]

The manifold \( \tilde{M} \) is said to be of \textit{constant type} if for any unit \( X, Y \in \Gamma T\tilde{M} \) with \( g(X, Y) = g(JX, Y) = 0 \), \( \lambda(X, Y) \) is a constant function.

Recall the following result [10].

**Theorem.** Let \( \tilde{M} \) be a RK-manifold. Then \( \tilde{M} \) is of pointwise constant type if and only if there exists a function \( \alpha \) on \( \tilde{M} \) such that
\[ \lambda(X, Y) = \alpha [g(X, X)g(Y, Y) - (g(X, Y))^2 - (g(X, JY))^2], \]
for any \( X, Y \in \Gamma T\tilde{M} \).

Moreover, \( \tilde{M} \) is of constant type if and only if the above equality holds good for a constant \( \alpha \).

In this case, \( \alpha \) is the \textit{constant type} of \( \tilde{M} \).

**Definition.** A \textit{generalized complex space form} is a RK-manifold of constant holomorphic sectional curvature and of constant type.

We will denote a generalized complex space form by \( \tilde{M}(c, \alpha) \), where \( c \) is the constant holomorphic sectional curvature and \( \alpha \) the constant type, respectively.

Each complex space form is a generalized complex space form. The converse statement is not true. The sphere \( S^6 \) endowed with the standard nearly-Kaehler structure is an example of generalized complex space form which is not a complex space form.

Let \( \tilde{M}(c, \alpha) \) be a generalized complex space form of constant holomorphic sectional curvature \( c \) and of constant type \( \alpha \). Then the curvature tensor \( \tilde{R} \) of \( \tilde{M}(c, \alpha) \) has the following expression [10]:
\[ \tilde{R}(X, Y)Z = \frac{c + 3\alpha}{4} [g(Y, Z)X - g(X, Z)Y] + \]
(1.1)
Let $M$ be an $n$-dimensional submanifold of an $2m$-dimensional generalized complex space form $\tilde{M}(c, \alpha)$. We denote by $K(\pi)$ the sectional curvature of $M$ associated with a plane section $\pi \subset T_pM$, $p \in M$. Let $\nabla$ and $h$ be the Levi-Civita connection of $M$ and the second fundamental form, respectively.

Then the equation of Gauss is given by

$$\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + \frac{1}{4}[g(X, JZ)JY - g(Y, JZ)JX + 2g(X, JY)JZ].$$

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for any vectors $X, Y, Z, W$ tangent to $M$, where $R$ is the Riemann curvature tensor of $M$.

We denote by $H$ the mean curvature vector at $p \in M$, i.e.

$$H(p) = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i),$$

where $\{e_1, ..., e_{2m}\}$ is an orthonormal basis of the tangent space $T_p\tilde{M}(c, \alpha)$, such that $\{e_1, ..., e_n\}$ are tangent to $M$.

Also, we set

$$h'_{ij} = g(h(e_i, e_j), e_r), \quad i, j = 1, ..., n; \quad r = n + 1, ..., 2m,$$

and

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).$$

For any $p \in M$ and for any $X \in T_pM$, we put $JX = PX + FX$, where $PX \in T_pM, FX \in T^+_pM$.

We put

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(Pe_i, e_j).$$

Suppose $L$ is a $k$-plane section of $T_pM$ and $X$ a unit vector in $L$. We choose an orthonormal basis $\{e_1, ..., e_k\}$ of $L$ such that $e_1 = X$.

Define the Ricci curvature $Ric_L$ of $L$ at $X$ by

$$Ric_L(X) = K_{12} + K_{13} + ... + K_{1k},$$

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where $K_{ij}$ denotes the sectional curvature of the 2-plane section spanned by $e_i, e_j$. We simply called such a curvature a $k$-Ricci curvature.

The scalar curvature $\tau$ of the $k$-plane section $L$ is given by

$$\tau(L) = \sum_{1 \leq i < j \leq k} K_{ij}.$$  \hfill (1.8)

For each integer $k$, $2 \leq k \leq n$, the Riemannian invariant $\Theta_k$ on an $n$-dimensional Riemannian manifold $M$ is defined by

$$\Theta_k(p) = \frac{1}{k - 1} \inf_{L,X} \text{Ric}_L(X), \quad p \in M,$$  \hfill (1.9)

where $L$ runs over all $k$-plane sections in $T_pM$ and $X$ runs over all unit vectors in $L$.

Recall that for a submanifold $M$ in a Riemannian manifold, the relative null space of $M$ at a point $p \in M$ is defined by

$$N(p) = \{ X \in T_pM | h(X, Y) = 0, \forall Y \in T_pM \}.$$  \hfill (1.10)

### 2. Sectional curvature and shape operator

The notion of a slant submanifold of an almost Hermitian manifold was introduced by B. Y. Chen [1].

**Definition.** A submanifold $M$ of an almost Hermitian manifold $\widetilde{M}$ is said to be a slant submanifold if for any $p \in M$ and any nonzero vector $X \in T_pM$, the angle between $JX$ and the tangent space $T_pM$ is constant ($= \theta$).

We prove an inequality for an $n$-dimensional slant submanifold $M$ into a $2m$-dimensional generalized complex space form $\widetilde{M}(c, \alpha)$ of constant holomorphic sectional curvature $c$ and of constant type $\alpha$.

**Theorem 2.1.** Let $x : M \to \widetilde{M}(c, \alpha)$ be an isometric immersion of an $n$-dimensional $\theta$-slant submanifold into a $2m$-dimensional generalized complex space form $\widetilde{M}(c, \alpha)$ of constant holomorphic sectional curvature $c > \alpha > 0$. If there exists a point $p \in M$ and a number $b > \frac{c + 3\alpha}{4} + 3\frac{\alpha - \alpha}{2n} \cos^2 \theta$ such that $K \geq b$ at $p$, then the shape operator at the
mean curvature vector satisfies

\[ A_H > \frac{n-1}{n} [b - \frac{c + 3\alpha}{4} - 3 \frac{\alpha}{4(n-1)} \cos^2 \theta] I_n, \text{ at } p, \]  

(2.1)

where \( I_n \) is the identity map.

**Proof.** Let \( p \in M \) and a number \( b > \frac{c+3\alpha}{4} + \frac{3\alpha}{2\mu} \cos^2 \theta \) such that \( K \geq b \) at \( p \). We choose an orthonormal basis \( \{e_1, ..., e_n, e_{n+1}, ..., e_{2m}\} \) at \( p \) such that \( e_{n+1} \) is parallel to the mean curvature vector \( H \) and \( e_1, ..., e_n \) diagonalize the shape operator \( A_{n+1} \).

Then we have

\[ A_{n+1} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}, \]  

(2.2)

\[ A_r = (h_{ij}^r), i, j = 1, ..., n, r = n + 2, ..., 2m, \text{ trace } A_r = \sum_{i=1}^{n} h_{ii}^r = 0. \]  

(2.3)

For \( i \neq j \), we denote by

\[ u_{ij} = a_i a_j. \]  

(2.4)

From Gauss equation for \( X = Z = e_i, Y = W = e_j \), we get

\[ u_{ij} \geq b - \frac{c + 3\alpha}{4} - 3 \frac{\alpha}{4} g^2(e_i, J e_j) - \sum_{r=n+2}^{2m} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2]. \]  

(2.5)

We prove that \( u_{ij} \) have the following properties:

1. For any fixed \( i \in \{1, ..., n\} \), we have

\[ \sum_{i \neq j} u_{ij} \geq (n-1)(b - \frac{c + 3\alpha}{4}) - 3 \frac{\alpha}{4} \cos^2 \theta > 0. \]

2. \( u_{ij} \neq 0 \), for \( i \neq j \).

3. For distinct \( i, j, k \in \{1, ..., n\}, a_{ij}^2 = \frac{u_{ij} u_{ik}}{u_{jk}}. \)
4. We denote by \( S_k = \{ B \subset \{1, \ldots, n\}; |B| = k \} \) and for any \( B \in S_k \) we denote by \( \overline{B} = \{1, \ldots, n\} \setminus B \). Then, for a fixed \( k, 1 \leq k \leq \left\lceil \frac{n}{2} \right\rceil \) and each \( B \in S_k \), we have
\[
\sum_{j \in B} \sum_{t \in \overline{B}} u_{jt} > 0.
\]

5. For distinct \( i, j \in \{1, \ldots, n\}, u_{ij} > 0 \).

1. From (2.3), (2.4) and (2.5), we have:
\[
\sum_{j \neq i} u_{ij} \geq (n-1)(b - \frac{c + 3\alpha}{4}) - 3\frac{c - \alpha}{4} \|Pe_i\|^2 - \sum_{r=n+1}^{2m} [h_{ri}^r(\sum_{j \neq i} h_{rj}^r) - \sum_{j \neq i} (h_{ij}^r)^2] =
\]
\[
= (n-1)(b - \frac{c + 3\alpha}{4}) - 3\frac{c - \alpha}{4} \cos^2 \theta - \sum_{r=n+1}^{2m} [h_{ri}^r(-h_{ri}^r) - \sum_{j \neq i} (h_{ij}^r)^2] =
\]
\[
= (n-1)(b - \frac{c + 3\alpha}{4}) - 3\frac{c - \alpha}{4} \cos^2 \theta + \sum_{r=n+1}^{2m} \sum_{j=1}^{n} (h_{ij}^r)^2 \geq
\]
\[
\geq (n-1)(b - \frac{c + 3\alpha}{4}) - 3\frac{c - \alpha}{4} \cos^2 \theta > 0.
\]

2. If \( u_{ij} = 0, \) for \( i \neq j, \) then \( a_i = 0 \) or \( a_j = 0. \) \( a_i = 0 \) implies that \( u_{it} = a_i a_t = 0, \forall t \in \{1, \ldots, n\}, t \neq i. \)

It follows that
\[
\sum_{j \neq i} u_{ij} = 0,
\]
in contradiction with 1.

3. \[
\frac{u_{ij} u_{ik}}{u_{jk}} = \frac{a_i a_j a_k}{a_j a_k} = a_i^2.
\]

4. Since we can change the order of \( e_1, \ldots, e_n, \) we may assume \( B = \{1, \ldots, k\} \) and \( \overline{B} = \{k+1, \ldots, n\}. \) Then
\[
\sum_{j \in B} \sum_{t \in \overline{B}} u_{jt} = k(n-k)(b - \frac{c + 3\alpha}{4}) - 3\frac{c - \alpha}{4} \sum_{j=1}^{k} \sum_{t=k+1}^{n} g^2(Je_j, e_t) -
\]
\[
- \sum_{r=n+1}^{2m} \sum_{j=1}^{k} \sum_{t=k+1}^{n} [h_{ij}^r h_{it}^r - (h_{ij}^r)^2] \geq
\]

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\[
\geq k(n-k)(b-c+3\alpha/4) - 3k\frac{c-\alpha}{4}\cos^2\theta + \\
+ \sum_{r=n+2}^{2m} \left[ \sum_{j=1}^{k} \sum_{t=k+1}^{n} (h_{jt})^2 + \sum_{j=1}^{k} (h_{jj})^2 \right] \\
\geq k(n-k)(b-c+3\alpha/4) - 3k\frac{c-\alpha}{4}\cos^2\theta > 0.
\]

5. Assume \( u_{1n} < 0 \). From 3, we get \( u_{1i}u_{in} < 0 \), for \( 1 < i < n \).
Without loss of generality, we may assume
\[
\begin{cases}
  u_{12}, \ldots, u_{1n}, u_{(l+1)n}, \ldots, u_{(n-1)n} > 0, \\
  u_{1(l+1)}, \ldots, u_{1n}, u_{2n}, \ldots, u_{ln} < 0,
\end{cases}
\]
for some \( \left\lfloor \frac{n+1}{2} \right\rfloor \leq l \leq n-1 \).
If \( l = n-1 \), then \( u_{1n} + u_{2n} + \ldots + u_{(n-1)n} < 0 \), which contradicts to 1. Thus, \( l < n-1 \).
From 3, we get
\[
a_n^2 = \frac{u_{in}u_{tn}}{u_{it}} > 0, \tag{2.6}
\]
where \( 2 \leq i \leq l, l+1 \leq t \leq n-1 \). By (2.6) and (2.7), we obtain \( u_{it} < 0 \), which implies
\[
\sum_{i=1}^{l} \sum_{t=1}^{n} u_{it} = \sum_{i=2}^{l} \sum_{t=l+1}^{n} u_{it} + \sum_{i=1}^{l} u_{in} + \sum_{t=l+1}^{n} u_{1t} < 0.
\]
This contradicts to 4.

Now, we return to the proof of Theorem 2.1.
From 5, it follows that \( a_1, \ldots, a_n \) have the same sign. Assume \( a_j > 0, \forall j \in \{1, \ldots, n\} \).
Then
\[
\sum_{j \neq i} u_{ij} = a_i(a_1 + \ldots + a_n) - a_i^2 \geq (n-1)(b-c+3\alpha/4) - 3\frac{c-\alpha}{4}\cos^2\theta.
\]
From the above relation and from (2.2), we have
\[
a_i n \|H\| \geq (n-1)(b-c+3\alpha/4) - 3\frac{c-\alpha}{4}\cos^2\theta + a_i^2 >
\]

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This equation implies
\[ a_i \|H\| > \frac{n-1}{n} \left[b - \frac{c + 3\alpha}{4} - 3 \frac{c - \alpha}{4(n-1)} \cos^2 \theta \right], \]
and consequently (2.1).

In particular, for \( \alpha = 0 \), we refine Theorem 3.1 from [6].

For totally real submanifolds, we have the following

**Corollary 2.2.** Let \( x : M \to \overline{M}(c, \alpha) \) be an isometric immersion of an \( n \)-dimensional totally real submanifold into an \( 2m \)-dimensional generalized complex space form \( \overline{M}(c, \alpha) \). If there exists a point \( p \in M \) and a number \( b > \frac{c + 3\alpha}{4} \) such that \( K \geq b \) at \( p \), then the shape operator at the mean curvature vector satisfies
\[ A_H > \frac{n-1}{n} \left[b - \frac{c + 3\alpha}{4} - 3 \frac{c - \alpha}{4(n-1)} \cos^2 \theta \right] I_n, \text{ at } p, \]
where \( I_n \) is the identity map.

3. \( k \)-Ricci curvature and shape operator

We prove an inequality for a slant submanifold \( M \) of a \( 2m \)-dimensional generalized complex space form \( \overline{M}(c, \alpha) \) of constant holomorphic sectional curvature \( c \) and of constant type \( \alpha \).

**Theorem 3.1.** Let \( x : M \to \overline{M}(c, \alpha) \) be an isometric immersion of an \( n \)-dimensional \( \theta \)-slant submanifold \( M \) into a \( 2m \)-dimensional generalized complex space form \( \overline{M}(c, \alpha) \). Then, for any integer \( k, 2 \leq k \leq n \), and any point \( p \in M \), we have:

i) If \( \Theta_k(p) \neq \frac{c + 3\alpha}{4} + 3 \frac{c - \alpha}{4(n-1)} \cos^2 \theta \), then the shape operator at the mean curvature satisfies
\[ A_H > \frac{n-1}{n} [\Theta_k(p) - \frac{c + 3\alpha}{4} - 3 \frac{c - \alpha}{4(n-1)} \cos^2 \theta] I_n, \text{ at } p, \] (3.1)
where \( I_n \) denotes the identity map of \( T_pM \).
\[ ii) If \Theta_k(p) = \frac{c + 3\alpha}{4} + 3\frac{c - \alpha}{4(n-1)} \cos^2 \theta, \text{ then } A_H \geq 0 \text{ at } p. \]

\[ iii) A \text{ unit vector } X \in T_pM \text{ satisfies} \]

\[ A_H X = \frac{n - 1}{n} [\Theta_k(p) - \frac{c + 3\alpha}{4} - 3\frac{c - \alpha}{4(n-1)} \cos^2 \theta] X \]

(3.2)

if and only if \( \Theta_k(p) = \frac{c + 3\alpha}{4} + 3\frac{c - \alpha}{4(n-1)} \cos^2 \theta \) and \( X \in N(p) \).

\[ iv) A_H = \frac{n - 1}{n} [\Theta_k(p) - \frac{c + 3\alpha}{4} - 3\frac{c - \alpha}{4(n-1)} \cos^2 \theta] I_n \text{ at } p \text{ if and only if } p \text{ is a totally geodesic point.} \]

**Proof.**

i) Let \( \{e_1, \ldots, e_n\} \) be an orthonormal basis of \( T_pM \). Denote by \( L_{i_1 \ldots i_k} \) the \( k \)-plane section spanned by \( e_{i_1}, \ldots, e_{i_k} \). It is easily seen by the definitions

\[ \tau(L_{i_1 \ldots i_k}) = \frac{1}{2} \sum_{i \in \{i_1, \ldots, i_k\}} \text{Ric}_{L_{i_1 \ldots i_k}}(e_i), \]  

(3.3)

\[ \tau(p) = \frac{1}{C_n^{n-2}} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \tau(L_{i_1 \ldots i_k}). \]  

(3.4)

Combining (3.3) and (3.4), we find

\[ \tau(p) \geq \frac{n(n-1)}{2} \Theta_k(p). \]  

(3.5)

From the equation of Gauss for \( X = Z = e_i, Y = W = e_j \), by summing, we obtain

\[ n^2 \|H\|^2 = 2\tau + \|h\|^2 - \frac{c + 3\alpha}{4} n(n-1) - 3\frac{c - \alpha}{4} \|P\|^2. \]  

(3.6)

We choose an orthonormal basis \( \{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2m}\} \) at \( p \) such that \( e_{n+1} \) is parallel to the mean curvature vector \( H(p) \) and \( e_1, \ldots, e_n \) diagonalize the shape operator \( A_{n+1} \).

Then we have the relations (2.2) and (2.3).

From (3.6), we get

\[ n^2 \|H\|^2 = 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 - \]

\[ - \frac{c + 3\alpha}{4} n(n-1) - 3\frac{c - \alpha}{4} \|P\|^2. \]  

(3.7)
On the other hand, since
\[ 0 \leq \sum_{i<j} (a_i - a_j)^2 = (n-1) \sum_i a_i^2 - 2 \sum_{i<j} a_i a_j, \]
we obtain
\[ n^2 \|H\|^2 = \left( \sum_{i=1}^n a_i \right)^2 = \sum_{i=1}^n a_i^2 + 2 \sum_{i<j} a_i a_j \leq n \sum_{i=1}^n a_i^2, \] (3.8)
which implies
\[ \sum_{i=1}^n a_i^2 \geq n \|H\|^2. \]

We have from (3.7)
\[ n^2 \|H\|^2 \geq 2 \tau + n \|H\|^2 - \frac{c+3\alpha}{4}n(n-1) - 3\frac{c-\alpha}{4}\|P\|^2, \] (3.9)
or, equivalently,
\[ \|H\|^2 \geq \frac{2\tau}{n(n-1)} - \frac{c+3\alpha}{4} - 3\frac{c-\alpha}{4n(n-1)}\|P\|^2. \] (3.10)

Since $M$ is a slant submanifold, from (3.5) and (3.10), we obtain
\[ \|H\|^2(p) \geq \Theta_k(p) - \frac{c+3\alpha}{4} - 3\frac{c-\alpha}{4n(n-1)}\|P\|^2 = \Theta_k(p) - \frac{c+3\alpha}{4} - 3\frac{c-\alpha}{4(n-1)} \cos^2 \theta. \] (3.11)

This shows that $H(p) = 0$ may occurs only when $\Theta_k(p) \leq \frac{c+3\alpha}{4} + 3\frac{c-\alpha}{4(n-1)} \cos^2 \theta$. Consequently, if $H(p) = 0$, statements i) and ii) hold automatically. Therefore, without loss of generality, we may assume $H(p) \neq 0$.

From the equation of Gauss we get
\[ a_i a_j = K_{ij} - \frac{c+3\alpha}{4} - 3\frac{c-\alpha}{4} g^2(e_i, Je_j) - \sum_{r=n+2}^{2m} \left[ h_r^{ij} h_r^{ij} - (h_r^{ij})^2 \right]. \] (3.12)
By (3.12), we obtain
\[ a_1(a_{i_2} + \ldots + a_{i_k}) = Ric_{L_{i_1i_2}\ldots i_k}(e_1) - (k - 1) \frac{c + 3\alpha}{4} - \frac{c - \alpha}{4} \sum_{j=2}^{k} g^2(e_1, Je_{i_j}) - \sum_{r=n+2}^{2m} \sum_{j=2}^{k} [h^r_{i_{1j}} h^r_{i_j} - (h^r_{i_j})^2], \]
which yields
\[ a_1(a_2 + \ldots + a_n) = \frac{1}{C_{n-2}} \sum_{2 \leq i_2 < \ldots < i_k \leq n} Ric_{L_{i_1i_2}\ldots i_k}(e_1) - (n - 1) \frac{c + 3\alpha}{4} - \frac{c - \alpha}{4} \sum_{j=2}^{n} g^2(e_1, Je_j) + \sum_{r=n+2}^{2m} \sum_{j=1}^{n} (h^r_{i_j})^2. \]
We find
\[ a_1(a_2 + \ldots + a_n) \geq (n - 1)[\Theta_k(p) - \frac{c + 3\alpha}{4} - 3\frac{c - \alpha}{4(n - 1)} \cos^2 \theta]. \]
Then
\[ a_1(a_1 + a_2 + \ldots + a_n) = a_1^2 + a_1(a_2 + \ldots + a_n) \geq a_1^2 + (n - 1)[\Theta_k(p) - \frac{c + 3\alpha}{4} - 3\frac{c - \alpha}{4(n - 1)} \cos^2 \theta] \geq (n - 1)[\Theta_k(p) - \frac{c + 3\alpha}{4} - 3\frac{c - \alpha}{4(n - 1)} \cos^2 \theta]. \]
Since \( n \|H\| = a_1 + \ldots + a_n \), the above equation implies
\[ A_H \geq \frac{n - 1}{n} [\Theta_k(p) - \frac{c + 3\alpha}{4} - 3\frac{c - \alpha}{4(n - 1)} \cos^2 \theta] I_n. \]
The equality does not hold, because in our case \( H(p) \neq 0 \).

The assertion ii) is obvious.

iii) Let \( X \in T_p M \) a unit vector satisfying (3.2). By (3.16) and (3.14) one has \( a_1 = 0 \) and \( h^r_{i_j} = 0, \forall j \in \{1, \ldots, n\}, r \in \{n + 2, \ldots, 2m\} \), respectively. The above conditions imply \( \Theta_k(p) = \frac{c + 3\alpha}{4} + 3\frac{c - \alpha}{4(n - 1)} \cos^2 \theta \) and \( X \in N(p) \).

The converse is clear.
iv) The equality (3.2) holds for any $X \in T_pM$ if and only if $N(p) = T_pM$, i.e. $p$ is a totally geodesic point.

**Remark.** If we denote by $\lambda_i$ the eigenvalues of $A_H$, i.e. $\lambda_i = a_i \|H\|$, $i \in \{1, ..., n\}$, we obtain the following inequality for arbitrary submanifolds of generalized complex space forms:

$$\lambda_i \geq \frac{n-1}{n}[\Theta_k(p) - \frac{c+3\alpha}{4} - 3\frac{c-\alpha}{4(n-1)}]\|Pe_i\|^2.$$ 

In particular, for $\alpha = 0$, we obtain Theorem 4.1 from [6].

**Corollary 3.2.** Let $x : M \to \tilde{M}(c, \alpha)$ be an isometric immersion of an $n$-dimensional totally real submanifold $M$ into a generalized complex space form $\tilde{M}(c, \alpha)$. Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have:

i) If $\Theta_k(p) \neq \frac{c+3\alpha}{4}$, then the shape operator at the mean curvature vector satisfies

$$A_H > \frac{n-1}{n}[\Theta_k(p) - \frac{c+3\alpha}{4}]I_n, \text{ at } p,$$

where $I_n$ denotes the identity map of $T_pM$.

ii) If $\Theta_k(p) = \frac{c+3\alpha}{4}$, then $A_H \geq 0$ at $p$.

iii) A unit vector $X \in T_pM$ satisfies

$$A_H X = \frac{n-1}{n}[\Theta_k(p) - \frac{c+3\alpha}{4}]X$$

if and only if $\Theta_k(p) = \frac{c+3\alpha}{4}$ and $X \in N(p)$.

iv) $A_H = \frac{n-1}{n}[\Theta_k(p) - \frac{c+3\alpha}{4}]I_n$ at $p$ if and only if $p$ is a totally geodesic point.

**References**


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