On the Extended Hecke Groups $\overline{H}(\lambda_q)$

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Abstract

Hecke groups $H(\lambda_q)$ have been studied extensively for many aspects in the literature, [5], [8]. The Hecke group $H(\lambda_3)$, the modular group $\text{PSL}(2, \mathbb{Z})$, has especially been of great interest in many fields of mathematics, for example number theory, automorphic function theory and group theory. In this paper we consider the extended Hecke groups $\overline{H}(\lambda_q)$ which are defined analogously with the extended modular group. We find the conjugacy classes of torsion elements in $\overline{H}(\lambda_q)$. Using this we give some results about the normal subgroups and Fuchsian subgroups of $\overline{H}(\lambda_q)$.

Key Words: Extended Hecke group, conjugacy class, extended modular group.

1. Introduction

Hecke introduced an infinite class of discrete groups $H(\lambda_q)$ of linear fractional transformations preserving the upper-half plane [2]. The Hecke group $H(\lambda)$ is the group generated by

$$x(z) = -\frac{1}{z} \quad \text{and} \quad u(z) = z + \lambda,$$

where $\lambda = \lambda_q = 2 \cos \pi/q$, $q \geq 3$ integer, or $\lambda \geq 2$. We consider the former case. These Hecke groups $H(\lambda_q)$ are Fuchsian groups of the first kind. Let

$$y = xu = \frac{-1}{z + \lambda_q}.$$
Then $H(\lambda_q)$ has a presentation

$$H(\lambda_q) = \langle x, y : x^2 = y^q = 1 \rangle.$$  

For $q = 3$, the resulting Hecke group $H(\lambda_3)$ is the modular group $\text{PSL}(2, \mathbb{Z})$. In a sense Hecke groups are generalizations of the modular group. Similar to the extended modular group case, the extended Hecke group, denoted by $\overline{H}(\lambda_q)$, is the group obtained by adding the reflection

$$r(z) = 1 / \overline{z}$$

to the generators of $H(\lambda_q)$, and the Hecke group $H(\lambda_q)$ is a subgroup of index 2 in $\overline{H}(\lambda_q)$.

The extended Hecke group $\overline{H}(\lambda_q)$ has the presentation

$$\overline{H}(\lambda_q) = \langle x, y, r : x^2 = y^q = r^2 = (xr)^2 = (yr)^2 = 1 \rangle.$$  

Again, for $q = 3$, we obtain the extended modular group studied in [3], [4].

In this paper, firstly we find the conjugacy classes of finite order elements of the extended Hecke groups $\overline{H}(\lambda_q)$. In studying normal subgroups (especially Fuchsian ones), it is important to find the conjugacy classes of torsion elements. Also we show that in $\overline{H}(\lambda_p)$, $p \geq 3$ prime number, every normal subgroup with torsion has finite index (in fact the index is 2, 4 or $2p$).

### 2. Conjugacy Classes

It is well known that the Hecke group $H(\lambda_q)$ is a free product of a cyclic group of order 2 and a cyclic group of order $q$, i.e. $H(\lambda_q) \cong \mathbb{Z}_2 * \mathbb{Z}_q$, [1]. The extended Hecke group $\overline{H}(\lambda_q)$ is also known to be a free product with amalgamation as $\overline{H}(\lambda_q) \cong D_2 * \mathbb{Z}_2 * D_q$, [7], [4].

One of the consequences of the amalgam decomposition is a classification of the conjugacy classes of torsion elements in $\overline{H}(\lambda_q)$.

The following lemmas are trivial due to the presentation of $\overline{H}(\lambda_q)$ given in (1.1.1).

**Lemma 2.1** In $\overline{H}(\lambda_q)$, we have $y^s r = ry^{q-s}$, $1 \leq s \leq q - 1$.

**Lemma 2.2** In $\overline{H}(\lambda_q)$, we have
(i) $y^n r, 1 \leq n \leq q - 1$, is conjugate to $r$ by $y^t r$ where $t = \frac{nk + n}{2}$ for some $k \in \mathbb{Z}$ satisfying the condition $t \in \mathbb{Z}$ (note that $t$ is not unique).

(ii) $y^s, 1 \leq s \leq \frac{q - 1}{2}$, is conjugate to $y^{q-s}$ by $r$.

Now we can give the following theorem for the extended Hecke group $\mathcal{H}(\lambda_p)$ where $p \geq 3$ prime number. Note that, by the definition, $\mathcal{H}(\lambda_q)$ contains reflections.

Theorem 2.3 Let $p \geq 3$ be prime number. There are exactly $\frac{p+5}{2}$ conjugacy classes of torsion elements in $\mathcal{H}(\lambda_p)$, three for those of order 2 and $\frac{p - 1}{2}$ for those of order $p$. In particular, any elliptic transformation of order 2 is conjugate to $x: z \to -1/z$ and any reflection is conjugate to one of $r: z \to 1/z$ or $x r: z \to -z$ while any elliptic transformation of order $p$ is conjugate to one of $y: z \to -1/(z + \lambda_p)$, $y^2, \ldots$, or $y^{\frac{p+1}{2}}$.

Proof. We consider the following presentation of $\mathcal{H}(\lambda_p)$ given in (1.1.1):

\[ \mathcal{H}(\lambda_p) = \langle x, y, r : x^2 = y^p = r^2 = (xr)^2 = (yr)^2 = 1 \rangle. \]

The decomposition of $\mathcal{H}(\lambda_p) \cong D_2 *_{\mathbb{Z}_2} D_p$ arose by letting

\[ G_1 = \langle x, r : x^2 = r^2 = (xr)^2 = 1 \rangle \cong D_2 \]

and

\[ G_2 = \langle y, r : y^p = r^2 = (yr)^2 = 1 \rangle \cong D_p. \]

From a theorem of Kurosh, see [6], we know that any element of finite order in a generalized free product $A *_{\mathcal{H}} B$ is conjugate to an element of finite order in one of the factors. Now let $g$ be any element of finite order in $\mathcal{H}(\lambda_p)$. Then $g$ must be conjugate to an element of finite order in one of the factors $G_1$ and $G_2$. Therefore to find the conjugacy classes of finite order elements in $\mathcal{H}(\lambda_p)$, we must find the conjugacy classes in these factors.

$G_1$ has one conjugacy class of elliptic elements of order 2 with representative $x$, two conjugacy classes of reflections with representatives $r$ and $xr$.

Similarly, in $G_2$, there are $p$ classes of reflections with representatives $r, y^1, \ldots, y^{p-1}$ and $p - 1$ classes of elliptic elements of order $p$ with representatives $y, y^2, \ldots, y^{p-1}$.

By Lemma 2.2 (i), $y^n r$ is conjugate to $r$, $1 \leq n \leq p - 1$, then we have one conjugacy
class of reflections with representative $r$. Also by Lemma 2.2 (ii), $y^r$ is conjugate to $y^{p-s}$ and so we have $\frac{p-1}{2}$ conjugacy classes of elliptic elements of order $p$ with representatives $y, y^2, \ldots, y^{\frac{p-1}{2}}$.

Therefore, in $\mathcal{H}(\lambda_p)$ as a whole there are: one class of elliptic elements of order 2 with representative $x$; two classes of reflections with representatives $r, xr$; and, $\frac{p-1}{2}$ classes of elliptic elements of order $p$ with representatives $y, y^2, \ldots, y^{\frac{p-1}{2}}$.

The situation is more complex for any odd $q$ which is not prime.

For odd and composite values of $q$, there are $\frac{\varphi(q)}{2}$ conjugacy classes of elliptic elements of order $q$ with representatives $y, y^{r_1}, \ldots, y^{\varphi(q)/2}$ where $(r_i, q) = 1, 1 \leq i \leq \frac{\varphi(q)}{2}$ and $\varphi(q)$ is the Euler function. There are: one conjugacy class of elliptic elements of order 2 with representative $x$; two conjugacy classes of reflections with representatives $r, xr$; and, in total $\frac{p-1}{2} - \frac{\varphi(q)}{2}$ conjugacy classes of elliptic elements of order $t_i$ where $t_i \mid q$.

For even $q$, firstly let us start with $q = 4$ and $q = 6$.

Theorem 2.4 (i) In $\mathcal{H}(\sqrt{2})$, there are two conjugacy classes of elliptic elements of order 2 with representatives $x, y^2$, two conjugacy classes of reflections with representatives $r, xr$ and one class of elliptic elements of order 4 with representative $y$.

(ii) In $\mathcal{H}(\sqrt{3})$, there are two conjugacy classes of elliptic elements of order 2 with representatives $x, y^3$, two conjugacy classes of reflections with representatives $r, xr$ and one class of elliptic elements of order 3 with representative $y^2$ and one class of elliptic elements of order 6 with representative $y$.

For any even $q > 6$, we can only say that there are $\frac{\varphi(q)}{2}$ conjugacy classes of elliptic elements of order $q$ with representatives $y, y^{r_1}, \ldots, y^{\varphi(q)/2}$ where $(r_i, q) = 1$. Also, there are two conjugacy classes of elliptic elements of order 2 with representatives $x, y^{\frac{\varphi(q)}{2}}$ and two conjugacy classes of reflections with representatives $r, xr$. Furthermore, there are totally $\frac{q}{2} - \frac{\varphi(q)}{2} - 1$ conjugacy classes of elliptic elements of order $t_i$ where $t_i \mid q$ and $t_i \neq 2$.

Here we mention briefly conjugacy classes of torsion elements in Hecke groups $H(\lambda_q)$. Note that most of these results can be obtained by use of the notion of fundamental region of Hecke groups. Similar to the extended Hecke group case, for any even $q$, we can only say that there are $\varphi(q)$ conjugacy classes of elliptic elements of order $q$ with

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representative $y, y^2, \ldots, y^{\varphi(q)}$ where $(r_i, q) = 1$, two classes of elliptic elements of order 2 with representatives $x, y$. Furthermore, there are in total $q - \varphi(q) - 2$ conjugacy classes of elliptic elements of order $t_i$ where $t_i \mid q, t_i \neq 2$. If $q$ is prime, then there are only $q$ conjugacy classes of torsion elements in $H(\lambda_q)$, one for those of order 2 and $q - 1$ for those of order $q$. In particular, any elliptic transformation of order 2 is conjugate to $x : z \to -1/z$ and any elliptic transformation of order $q$ is conjugate to one of $y : z \to -1/(z + \lambda_p)$, $y^2, \ldots, y^{q-1}$. For odd and composite values of $q$, there are $\varphi(q)$ conjugacy classes of elliptic elements of order $q$ with representatives $y, y^2, \ldots, y^{\varphi(q)}$ where $(r_i, q) = 1, 1 \leq i \leq \varphi(q)$. There are one conjugacy classes of elliptic elements of order 2 with representative $x$ and totally $q - 1 - \varphi(q)$ conjugacy classes of elliptic elements of order $t_i$ where $t_i \mid q$.

Using Theorem 2.3 we get the following.

**Theorem 2.5** If $G$ is a normal subgroup of $\overline{H}(\lambda_p)$, $p$ prime, and $G$ has torsion, then the index $[\overline{H}(\lambda_p) : G]$ is finite.

**Proof.** Suppose $g \in \overline{H}(\lambda_p)$ and $g$ has finite order. Since $G \leq \overline{H}(\lambda_p)$, if $g \in G$ then $N(g) \subseteq G$ implies $[\overline{H}(\lambda_p) : G] \mid [\overline{H}(\lambda_p) : N(g)]$ where $N(g)$ is the normal closure of $g$ in $\overline{H}(\lambda_p)$.

Since $[\overline{H}(\lambda_p) : N(g)] = [\overline{H}(\lambda_p) : N(g^*)]$ where $g^*$ is any conjugate of $g$, we complete the argument by showing that $[\overline{H}(\lambda_p) : N(h)]$ is finite. Here $h$ is any of the class representatives of torsion elements listed in Theorem 2.3. Now $h = x, r, xr, y, y^2, \ldots, y^{\frac{p-1}{2}}$. The quotient group $\overline{H}(\lambda_p) / N(h)$ is the group obtained by adding the relation $h = 1$ to the relations of $\overline{H}(\lambda_p)$, [6].

(1) Suppose $h = x$. Then

$$\overline{H}(\lambda_p)/N(x) \cong < x, y, r : x^2 = y^p = r^2 = (xr)^2 = (yr)^2 = 1, x = 1 >$$

$$\cong < y, r : y^p = r^2 = (yr)^2 = 1 > \cong D_p.$$ 

Therefore $[\overline{H}(\lambda_p) : N(x)] = 2p$.

(2) Suppose $h = r$. Then we find

$$\overline{H}(\lambda_p)/N(r) \cong < x, y : x^2 = y^p = r^2 = (xr)^2 = (yr)^2 = 1, r = 1 >$$

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\( \cong x : x^2 = 1 \cong C_2 \)

since \( y^2 = y^p = 1 \). Therefore \( \overline{H}(\lambda_p) : N(r) \) = 2.

(3) Let \( h = xr \). Then

\[ \overline{H}(\lambda_p)/N(xr) \cong x, y : x^2 = y^p = r^2 = (xr)^2 = (yr)^2 = 1, \ xr = 1 > . \]

As \( xr = 1 \) we see that \( x = r \) since \( r^2 = 1 \). Therefore we get

\[ \overline{H}(\lambda_p)/N(xr) \cong x, y : x^2 = y^p = (xy)^2 = 1 > \cong D_p, \]

so \( |\overline{H}(\lambda_p) : N(xr)| = 2p \).

(4) Let \( h = y \). Then

\[ \overline{H}(\lambda_p)/N(y) \cong x, r : x^2 = r^2 = (xr)^2 = 1 > \cong D_2, \]

so \( |\overline{H}(\lambda_p) : N(y)| = 4 \).

(5) Similarly, if \( h = y^a, 2 \leq a \leq \frac{p-1}{2} \), then \( (a, p) = 1 \) and so we have

\[ \overline{H}(\lambda_p)/N(y^a) \cong x, r : x^2 = r^2 = (xr)^2 = 1 > \cong D_2. \]

Hence \( |\overline{H}(\lambda_p) : N(y^a)| = 4 \).

Thus in all cases, the index is finite. \( \square \)

We can restate this as in the following corollary.

**Corollary 2.6** If \( G < \overline{H}(\lambda_p) \) and \( G \) has an elliptic element or reflection then \( |\overline{H}(\lambda_p) : G| \) divides \( 4p \) (divides \( 2, 4 \), or \( 2p \), depending on elliptic element or reflection).

Since the index of Hecke group \( H(\lambda_q) \) in \( \overline{H}(\lambda_q) \) is 2 and since if \( G \) is normal in \( H(\lambda_q) \), then is also normal in \( \overline{H}(\lambda_q) \), we have

**Corollary 2.7** If \( G < H(\lambda_p) \), \( p \geq 3 \) prime number, and \( G \) has an elliptic element then the index \( |H(\lambda_p) : G| \) divides \( 2p \) (divides \( 2 \) or \( p \), depending on elliptic element).
3. Fuchsian Subgroups

It is well-known that $H(\lambda_q)$ is discontinuous in the upper half-plane and Fuchsian with the real line as a fixed circle.

If $C$ is a circle, we let $P(C)$ be the Fuchsian stabilizer of $C$ in $\overline{H}(\lambda_q)$, i.e. the subgroup of $\overline{P}(\lambda_q)$ which maps both $C$ and the interior of $C$ on itself and $P_N(C)$ the normal closure of $P(C)$ in $\overline{P}(\lambda_q)$.

Now we can give the following corollary for the extended Hecke group $\overline{H}(\lambda_p)$ for prime number $p \geq 3$.

**Corollary 3.1** If $P(C)$ contains any elliptic element in $\overline{P}(\lambda_p)$ then the index $|\overline{H}(\lambda_p) : P_N(C)|$ is finite. In particular, $|\overline{H}(\lambda_p) : P_N(C)|$ divides $2, 4, 2p$.

Let $L(C)$ be the general stabilizer of the circle $C$ in $\overline{H}(\lambda_p)$, i.e. the subgroup of $\overline{H}(\lambda_p)$ which maps $C$ on itself and let $L_N(C)$ be its normal closure. Then we have

**Theorem 3.2** If the circle $C$ is fixed by either an elliptic or parabolic element or a reflection in $\overline{H}(\lambda_p)$ then $|\overline{H}(\lambda_p) : L_N(C)|$ is finite (in fact $|\overline{H}(\lambda_p) : L_N(C)|$ divides $4p$).

**Proof.** If the circle $C$ is fixed by an elliptic element or reflection, then the result follows from Theorem 2.5. Now let $C$ is fixed by a parabolic map $t$. Any parabolic element in $\overline{H}(\lambda_p)$ is conjugate to a translation $t' : z \rightarrow z + \alpha$. So, if $v t v^{-1} = t'$, then $v(C)$ is a fixed circle of $t'$. The real line is fixed by $t'$. Further, the real line is fixed by $x : z \rightarrow -1/z$. Then $v^{-1} x v$ fixes $C$, so $L(C)$ contains an elliptic map. From Theorem 2.5, $|\overline{H}(\lambda_p) : L_N(C)|$ is finite and in fact it divides $4p$. \qed

**References**


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