Groups Whose Proper Subgroups are Hypercentral of Length at Most $\leq \omega$

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Abstract

Groups, all proper subgroups of which are hypercentral of length at most $\omega$ and every proper subgroup of which is a $B_n$-group for a natural number $n$ depending on the subgroup, are studied in this article.

Key Words: hypercentral groups, locally nilpotent groups

1. Introduction

For $n \geq 0$, we denote by $B_n$ the class of groups in which every subnormal subgroup has defect at most $n$. $B_n$-groups are considered by many authors, both for special cases and in general. For results related to $B_1$-groups see [10], [4], [11], for $B_2$, $B_3$, $B_4$-groups see [5], [2] and for the general case see [6], [3]. It was shown in [14] that there exists a group $G$ that is a hypercentral group of length exactly $\omega + 1$ and all of its subgroups are subnormal. The split extension $G$ of a group of type $C_2^\omega$ by the inverting automorphism, is hypercentral of length $\omega + 1$ and every proper subgroup of $G$ is nilpotent. A group $G$ is locally graded if every non-trivial finitely generated subgroup of $G$ has a finite non-trivial image. We denote by $N_0$ class of groups in which every subgroup is subnormal.

The focus of this paper are those locally nilpotent groups whose every proper subgroup is a hypercentral of length at most $\omega$; and where every proper subgroup of these...
hypercentrals are $B_n$-groups in general, and prove that every such $B_n$-group is either soluble or a $N_0$-group.

2. Main Results

**Theorem 1** Let $G$ be a periodic hypercentral group and let every proper subgroup $H$ of $G$ be a $B_n$-group for some natural number $n$ depending on $H$. If $G$ is hypercentral of length at most $\omega$, then $G$ is nilpotent.

**Proof.** Suppose that $G$ is not nilpotent. Then $G$ is hypercentral of length $\omega$ and $G = \bigcup_{i=0}^{\infty} Z_i(G)$. For all $x \in G$, there exists $i \in N$ such that $x \in Z_i(G)$. Since $Z_i(G)$ is nilpotent for all natural numbers $i$, for all $x \in G$, $\langle x \rangle$ is a subnormal subgroup of $G$. Thus $G$ is a Baer group. Since $G$ is hypercentral, $G' < G$ and also $G'$ is nilpotent, by Lemma 6.1 of [6]. Since $G/G'$ is abelian, $G$ is soluble. Every proper subgroup of $G$ is nilpotent, again by Lemma 6.1 of [6]. If $G$ has no maximal subgroup, then every subgroups of $G$ are subnormal by Theorem 3.1.(ii) of [15]. Thus $G$ is nilpotent by Theorem 2.7 of [8]. If $G$ has a maximal subgroup, then there is a maximal subgroup $M$ such that $G = \langle x \rangle M$ for some $x \in G$. Since $G$ is Baer, $\langle x \rangle M$ is nilpotent by Lemma 1 of [7]. $\Box$

**Theorem 2** Let $G$ be a locally graded torsion-free group and let every proper subgroup $H$ of $G$ be a $B_n$-group for some natural number $n$ depending on $H$. If every proper subgroup of $G$ is hypercentral of length at most $\omega$, then $G$ is nilpotent.

**Proof.** Since every proper subgroup of $G$ is hypercentral of length at most $\omega$, $H = \bigcup_{i=0}^{\infty} Z_i(H)$ for all $H < G$; since $Z_i(H)$ is nilpotent, for all $i \geq 0$, $\langle x \rangle$ is subnormal in $H$, for all $x \in H$. Thus $H$ is a Baer group. By Lemma 6.1 of [6], $H$ is nilpotent. Let $F$ be a finitely generated non-trivial subgroup of $G$. If $F \neq G$ then $F$ is nilpotent by the above. If $F = G$, then $G$ is a finitely generated locally graded group and so $G$ is nilpotent by Theorem 2 of [16]. Therefore $G$ is locally nilpotent group. Finally, we conclude that $G$ is nilpotent by Theorem 2.1 of [15]. $\Box$
Theorem 3 Let $G$ be a locally nilpotent group and let every proper subgroup $H$ of $G$ be a $B_n$-group for some natural number $n$ depending on $H$. If every proper subgroup of $G$ is hypercentral of length at most $\leq \omega$, then $G$ is soluble.

Proof. Suppose that $G$ is not soluble. Let $T$ be the periodic part of $G$. $T$ is a subgroup of $G$ by 12.1.1 of [13]. If $T = 1$, then $G$ is nilpotent by Theorem 2. Therefore $G$ is soluble. If $G = T$, then every proper subgroup of $G$ is nilpotent by Theorem 1. By Theorem 3.3.(i),(ii) of [15], $G$ is a Fitting $p$-group. $G \neq G'$ by Theorem 1.1 of [1]. Therefore $G$ is soluble. If $1 \neq T \neq G$, then $T$ is hypercentral of length at most $\leq \omega$. Therefore $T$ is nilpotent by Theorem 1. Since $G/T$ is torsion-free, $G/T$ is soluble by Theorem 1. Since $T$ and $G/T$ are soluble, $G$ is soluble. This is a contradiction. 

Theorem 4 Let $G$ be a locally nilpotent group and let every proper $H$ be a $B_n$-group for some natural number $n$ depending on $H$. If $G$ is hypercentral of length at most $\leq \omega$, then $G$ is nilpotent.

Proof. Suppose that $G$ is not nilpotent. $G$ is soluble by Theorem 3. Every proper subgroup of $G$ is nilpotent by the proof of Theorem 3. By hypothesis and Theorem 3.1.(i),(ii) of [15], every subgroup of $G$ is subnormal. By Theorem 2.7 of [8] $G$ is nilpotent. If $G$ has a maximal subgroup, then $G$ is a metabelian Chernikov $p$-group and $G$ is hypercentral of length at most $\leq \omega + 1$ in [9]. This is a contradiction. 

Corollary 5 Let $G$ be a locally nilpotent group and let every proper subgroup $H$ of $G$ be a $B_n$-group for some natural number $n$ depending on $H$. If every proper subgroup of $G$ is hypercentral of length at most $\leq \omega$, then either $G$ is hypercentral or $G$ is an $N_0$-group.

Proof. Suppose that $G$ is not hypercentral. Then $G$ is not nilpotent. $G$ is soluble by Theorem 3 and every proper subgroup of $G$ is nilpotent by the proof of Theorem 3. If $G$ has a maximal subgroup, then $G$ is a metabelian Chernikov $p$-group and $G$ is hypercentral of length at most $\leq \omega + 1$ in [9]. This is a contradiction.
Theorem 6  Let $G$ be a locally soluble torsion-free group and let every proper subgroup $H$ of $G$ be a $B_n$-group for some natural number $n$ depending on $H$. Then either $G$ is locally nilpotent or $G$ is finitely generated.

Proof. Suppose that $G$ is not finitely generated. Let $F$ be a finitely generated subgroup of $G$. Since $G \neq F$, $F$ is nilpotent by Corollary 2 of Theorem 10.57 of [12]. Thus $G$ is finitely generated. $\square$

References


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