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On Pseudohyperbolical Curves in Minkowski Space-Time

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Abstract

In this paper, we characterize all spacelike, timelike and null curves lying on the pseudohyperbolic space H_0^3 in the Minkowski space–time E_1^4. Moreover, we prove that there are no timelike and no null curves lying on the pseudohyperbolic space H_0^3 in Minkowski space–time E_1^4.

Key words and phrases: Minkowski space–time, pseudohyperbolic space, curvature.

1. Introduction

A necessary and sufficient conditions for a curve to be a spherical curve in Euclidean space E^3 are given in [6] and [7]. On the other hand, a similar characterizations of a spacelike, a timelike and a null curves lying on the pseudohyperbolic space H_0^2 in the Minkowski space E_1^3 are obtained in [4]. The corresponding Frenet's equations for an arbitrary curve in the Minkowski space–time E_1^4, are given in [5]. By using these equations, in this paper we give some necessary and sufficient conditions for a spacelike curve to lie on the pseudohyperbolic space H_0^3 in Minkowski space–time.

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2. Preliminaries

Minkowski space–time E_1^4 is a Euclidean space E^4 provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system in E_1^4.

Since g is an indefinite metric, recall that a vector $v \in E_1^4$ can have one of three causal characters: it can be *spacelike* if $g(v,v) > 0$ or $v = 0$, *timelike* if $g(v,v) < 0$ and *null (lightlike)* if $g(v,v) = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in E_1^4 can locally be *spacelike, timelike* or *null (lightlike)*, if all of its velocity vectors $\alpha'(s)$ are respectively *spacelike, timelike* or *null*. Also, recall that the norm of a vector v is given by $||\, v\, || = \sqrt{|\, g(v,v)\, |}$. Therefore, v is a unit vector if $g(v,v) = \pm 1$. Next, vectors v, w in E_1^4 are said to be orthogonal if $g(v,w) = 0$. The velocity of the curve $\alpha(s)$ is given by $||\alpha'(s)||$.

The pseudohyperbolic space with center $m = (m_1, m_2, m_3, m_4) \in E_1^4$ and radius $r \in R^+$ in the space–time E_1^4 is the hyperquadric

$$H_0^3(r) = \{a = (a_1, a_2, a_3, a_4) \in E_1^4 \,|\, g(a-m, a-m) = -r^2\}.$$

with dimension 3 and index 0.

Denote by $\{T(s), N(s), B_1(s), B_2(s)\}$ the moving Frenet frame along the curve $\alpha(s)$ in the space E_1^4. Then T,N,B_1,B_2 are, respectively, the tangent, the principal normal, the first binormal and the second binormal vector fields. Spacelike or timelike curve $\alpha(s)$ is said to be parameterized by arclength function s, if $g(\alpha'(s), \alpha'(s)) = \pm 1$. In particular, a null curve $\alpha(s)$ is said to be parameterized by a pseudo–arclength function s if $g(\alpha''(s), \alpha''(s)) = 1$, where pseudo–arclength function s is defined in [1] by $s = \int_0^t (g(\alpha''(t), \alpha''(t))^{\frac{1}{4}}$.

Let $\alpha(s)$ be a curve in the space–time E_1^4, parameterized by arclength function s. Then for the curve α the following Frenet equations are given in [5] :

Case 1. α is spacelike curve:

Then T is spacelike vector, so depending on the causal character of the principal normal vector N, we distinguish subcases 1.1, 1.2 and 1.3 as follows.

Case 1.1. N is spacelike:

Then the first binormal B_1 can have all three causal characters, so we distinguish subcases 1.1.1, 1.1.2 and 1.1.3.

Case 1.1.1. B_1 is spacelike:

In this case, the Frenet formulae is read

$$\begin{bmatrix} \dot{T} \\ \dot{N} \\ \dot{B}_1 \\ \dot{B}_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where T,N,B_1,B_2 are mutually orthogonal vectors satisfying equations

$$g(T,T) = g(N,N) = g(B_1,B_1) = 1, g(B_2,B_2) = -1.$$

Case 1.1.2. B_1 is timelike:

The Frenet formulae has the form

$$\begin{bmatrix} \dot{T} \\ \dot{N} \\ \dot{B}_1 \\ \dot{B}_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & k_2 & 0 & k_3 \\ 0 & 0 & k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where T, N, B_1, B_2 are mutually orthogonal vectors satisfying equations

$$g(T,T) = g(N,N) = g(B_2,B_2) = 1, g(B_1,B_1) = -1.$$

Case 1.1.3. B_1 is a null:

The Frenet formulae are given by

$$\begin{bmatrix} \dot{T} \\ \dot{N} \\ \dot{B}_1 \\ \dot{B}_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & -k_2 & 0 & -k_3 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where T, N, B_1, B_2 satisfy equations

$$g(T,T) = g(N,N) = 1, g(B_1,B_1) = g(B_2,B_2) = 0,$$
$$g(T,N) = g(T,B_1) = g(T,B_2) = g(N,B_1) = g(N,B_2) = 0, g(B_1,B_2) = 1.$$

Case 1.2. N is timelike:

In this case, the Frenet equations has the form

$$\begin{bmatrix} \dot{T} \\ \dot{N} \\ \dot{B}_1 \\ \dot{B}_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_1 & 0 & k_2 & 0 \\ 0 & k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where T, N, B_1, B_2 are mutually orthogonal vectors satisfying equations

$$g(T,T) = g(B_1,B_1) = g(B_2,B_2) = 1, g(N,N) = -1.$$

Case 1.3. N is a null:

Now the Frenet formulae read

$$\begin{bmatrix} \dot{T} \\ \dot{N} \\ \dot{B}_1 \\ \dot{B}_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & -k_2 \\ -k_1 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where the first curvature k_1 can take only two values: 0 when α is a straight line or 1 in all other cases. In this case, the vectors T,N,B_1,B_2 satisfy equations

$$g(T,T) = g(B_1,B_1) = 1, g(N,N) = g(B_2,B_2) = 0,$$
$$g(T,N) = g(T,B_1) = g(T,B_2) = g(N,B_1) = g(B_1,B_2) = 0, g(N,B_2) = 1.$$

Case 2. α is a timelike curve:

Then T is timelike vector, so the Frenet formulae has the form

$$\begin{bmatrix} \dot{T} \\ \dot{N} \\ \dot{B}_1 \\ \dot{B}_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where T,N,B_1,B_2 are mutually orthogonal vectors satisfying equations

$$g(T,T)=-1, g(N,N)=g(B_1,B_1)=g(B_2,B_2)=1.$$

Case 3. α **is a null curve:**

Then T is null vector, so the Frenet equations are given by

$$\begin{bmatrix} \dot{T} \\ \dot{N} \\ \dot{B}_1 \\ \dot{B}_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_2 & 0 & -k_1 & 0 \\ 0 & -k_2 & 0 & k_3 \\ -k_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where the first curvature k_1 can take only two values: 0 when α is a straight null line or 1 in all other cases. In this case, the vectors T, N, B_1, B_2 satisfy the equations

$$g(T,T)=g(N,N)=g(B_1,B_1)=0, g(B_2,B_2)=1,$$
$$g(T,N)=g(T,B_2)=g(N,B_1)=g(N,B_2)=g(B_1,B_2)=0, g(T,B_1)=1.$$

Recall that the functions $k_1=k_1(s)$, $k_2=k_2(s)$ and $k_3=k_3(s)$ are called respectively the first, the second and the third curvature of curve $\alpha(s)$.

3. The spacelike, timelike and null curves lying on H_0^3

In this section, under the assumption that spacelike, timelike and null curves lie on the pseudohyperbolic space $H_0^3(r)$, we give some theorems characterizing these curves in terms of their three curvatures $k_1(s)$, $k_2(s)$ and $k_3(s)$.

Theorem 3.1 *Let $\alpha(s)$ be a unit speed spacelike curve in E_1^4 with spacelike N, B_1 and with curvatures $k_1(s)\neq 0$, $k_2(s)\neq 0$, $k_3(s)\neq 0$ for each $s\in I\subset R$. Then α lies on $H_0^3(r)$ if and only if*

$$(1/k_1)^2+((1/k_2)(1/k_1)')^2-[(1/k_3)(k_2/k_1+((1/k_2)(1/k_1)')')]^2=-r^2. \tag{1}$$

Proof. Let us first suppose that α lies on $H_0^3(r)$ with center m. Then $g(\alpha-m,\alpha-m)=-r^2$, for each $s\in I\subset R$. Differentiating the previous equation four times with respect to

s and by applying Frenet's equations, we get

$$\begin{gathered} g(T,\alpha-m)=0, g(N,\alpha-m)=-1/k_1, g(B_1,\alpha-m)=-(1/k_2)(1/k_1)', \\ g(B_2,\alpha-m)=-(1/k_3)[k_2/k_1+((1/k_2)(1/k_1)')']. \end{gathered} \tag{2}$$

Let us decompose the vector $\alpha - m$ with respect to the pseudo–orthonormal basis $\{T, N, B_1, B_2\}$ by

$$\alpha(s)-m=a(s)T(s)+b(s)N(s)+c(s)B_1(s)+d(s)B_2(s), \tag{3}$$

where $a(s), b(s), c(s), d(s)$ are arbitrary functions. Then we easily find that

$$g(T,\alpha-m)=a, g(N,\alpha-m)=b, g(B_1,\alpha-m)=c, g(B_2,\alpha-m)=-d. \tag{4}$$

Therefore, substituting (2) and (4) into (3) yields (1).

Conversely, let us assume that the relation (1) hold. By taking the derivative of (1) with respect to s, we find

$$\begin{gathered} (1/k_3)[k_2/k_1+((1/k_2)(1/k_1)')'][(k_3/k_2)(1/k_1)'-((1/k_3)(k_2/k_1) \\ +((1/k_2)(1/k_1)')'))']=0. \end{gathered} \tag{5}$$

If there holds $(1/k_3)[k_2/k_1+((1/k_2)(1/k_1)')']=0$, then
substituting this into (1) yields a contradiction. Therefore, it follows that

$$(k_3/k_2)(1/k_1)'-[(1/k_3)(k_2/k_1+((1/k_2)(1/k_1)')')]'=0. \tag{6}$$

Next, we may consider the vector $m \in E_1^4$ given by

$$m=\alpha+(1/k_1)N+(1/k_2)(1/k_1)'B_1-(1/k_3)[k_2/k_1+((1/k_2)(1/k_1)')']B_2. \tag{7}$$

Differentiating (7) with respect to s and by using Frenet formulae, we obtain

$$m'=[(k_3/k_2)(1/k_1)'-((1/k_3)(k_2/k_1+((1/k_2)(1/k_1)')'))']B_2. \tag{8}$$

Then substitution of (6) into (8) implies $m' = 0$ and thus $m =$ constant. Finally, from (7) we find $g(\alpha - m, \alpha - m) = -r^2$, so α lies on $H_0^3(r)$. □

Theorem 3.2 *Let $\alpha(s)$ be a unit speed spacelike curve in E_1^4, with spacelike N, B_1 and with curvatures $k_1(s) \neq 0$, $k_2(s) \neq 0$, $k_3(s) \neq 0$ for each $s \in I \subset R$. Then α lies on $H_0^3(r)$ if and only if*

$$\begin{gathered}(k_3/k_2)(1/k_1)' = [(1/k_3)(k_2/k_1 + ((1/k_2)(1/k_1)')')]' \\ [(1/k_3)(k_2/k_1 + (1/k_2)(1/k_1)')')]^2 > (1/k_1)^2 + ((1/k_2)(1/k_1)')^2.\end{gathered} \tag{9}$$

Proof. Let us first assume that α lies on $H_0^3(r)$. Then by Theorem 3.1. it follows that the relation (1) holds. Differentiating (1) with respect to s, we find that the equation in (9) is satisfied. Moreover, by using (1) we immediately get that the inequality in (9) holds.

Conversely, let us suppose that (9) holds. It can be easily seen that (9) is the differential of the equation

$$(1/k_1)^2 + ((1/k_2)(1/k_1)')^2 - [(1/k_3)(k_2/k_1 + ((1/k_2)(1/k_1)')')]^2 = \text{constant} = c < 0.$$

Finally, we may take $c = -r^2$, $r \in R^+$, so Theorem 3.1 implies that α lies on $H_0^3(r)$. □

Theorem 3.3 *A unit speed spacelike curve $\alpha(s)$ in E_1^4 with spacelike N, B_1 and with curvatures $k_1(s) \neq 0$, $k_2(s) \neq 0$, $k_3(s) \neq 0$ for each $s \in I \subset R$ lies on $H_0^3(r)$ if and only if there exist a differentiable function $f(s)$ such that*

$$fk_3 = k_2/k_1 + ((1/k_2)(1/k_1)')', \; f' = (k_3/k_2)(1/k_1)', \; f^2 - (f'/k_3)^2 > (1/k_1)^2. \tag{10}$$

Proof. Let us first assume that $\alpha(s)$ lies on $H_0^3(r)$. Then Theorems 3.1. and 3.2. imply that respectively relations (1) and (9) hold. Next, let us define the differentiable function $f(s)$ by

$$f(s) = (1/k_3)[k_2/k_1 + (1/k_2)(1/k_1)')']. \tag{11}$$

Consequently, by using (1), (9) and (11) we easily find that the relations in (10) are satisfied.

Conversely, let us assume that there exist a differentiable function $f(s)$ such that the relations in (10) hold for each $s \in I \subset R$. By using relations in (10), we easily find that both relations in (9) are satisfied. Finally, by Theorem 3.2. it follows that α lies on $H_0^3(r)$. □

Theorem 3.4 *A unit speed spacelike curve $\alpha(s)$ in E_1^4 with spacelike N, B_1 and with curvatures $k_1(s) \neq 0$, $k_2(s) \neq 0$, $k_3(s) \neq 0$ for each $s \in I \subset R$ lies on $H_0^3(r)$ if and only if there exist constants $A, B \in R$ such that the following relations hold:*

$$\begin{array}{c} (1/k_2)(1/k_1)' = [A + \int_0^s (k_2/k_1)\sinh(\int_0^s k_3 ds)ds]\sinh(\int_0^s k_3 ds) \\ -[B + \int_0^s (k_2/k_1)\cosh(\int_0^s k_3 ds)ds]\cosh(\int_0^s k_3 ds), \\ [A + \int_0^s (k_2/k_1)\sinh(\int_0^s k_3 ds)ds]^2 > [B + \int_0^s (k_2/k_1)\cosh(\int_0^s k_3 ds)ds]^2 + (1/k_1)^2. \end{array} \tag{12}$$

Proof. Let us first suppose that α lies on $H_0^3(r)$. By Theorem 3.3. there exist a differentiable function $f(s)$ such that (10) hold. Next, let us define the C^2 function $\theta(s)$ by $\theta(s) = \int_0^s k_3 ds$. Moreover, let us define the C^1 functions $g(s)$ and $h(s)$ by

$$\begin{aligned} g(s) &= -(1/k_2)(1/k_1)'\sinh\theta + f(s)\cosh\theta - \textstyle\int_0^s (k_2/k_1)\sinh\theta ds, \\ h(s) &= -(1/k_2)(1/k_1)'\cosh\theta + f(s)\sinh\theta - \textstyle\int_0^s (k_2/k_1)\cosh\theta ds. \end{aligned} \tag{13}$$

Differentiating functions $\theta(s)$, $g(s)$ and $h(s)$ with respect to s, we find $\theta'(s) = k_3$, $g'(s) = h'(s) = 0$. Hence $g(s) = A$, $h(s) = B$, $A, B \in R$. Substituting this into (13) yields

$$\begin{aligned} -(1/k_2)(1/k_1)'\sinh\theta + f(s)\cosh\theta - \textstyle\int_0^s (k_2/k_1)\sinh\theta ds = A, \\ -(1/k_2)(1/k_1)'\cosh\theta + f(s)\sinh\theta - \textstyle\int_0^s (k_2/k_1)\cosh\theta ds = B. \end{aligned} \tag{14}$$

By multiplying the first of the equations in (14) with $\sinh\theta$ and the second with $-\cosh\theta$ and adding, we find that the equation in (12) holds. Next, by multiplying the first of the equations in (14) with $\cosh\theta$ and the second with $-\sinh\theta$ and adding, we get

$$f(s) = (A + \int_0^s (k_2/k_1) \sinh \theta ds) \cosh \theta - (B + \int_0^s (k_2/k_1) \cosh \theta ds) \sinh \theta. \tag{15}$$

Then the relations (10) and (15) imply that inequality in (12) holds. □

Conversely, let us suppose that there exist constants $A, B \in R$ such that (12) holds for each $s \in I \subset R$. Differentiating the equation in (12) with respect to s yields

$$\begin{aligned} ((1/k_2)(1/k_1)')' = k_3[(A + \textstyle\int_0^s (k_2/k_1) \sinh(\int_0^s k_3 ds) ds) \cosh(\int_0^s k_3 ds) \\ -(B + \textstyle\int_0^s (k_2/k_1) \cosh(\int_0^s k_3 ds) ds) \sinh(\int_0^s k_3 ds)] - k_2/k_1. \end{aligned} \tag{16}$$

Let us define the differentiable function $f(s)$ by (11). Then by (11) and (16) it follows that (15) holds. Differentiating (15) with respect to s and using (12) we obtain $f' = (k_3/k_2)(1/k_1)'$. Moreover, by using (15), inequality in (12) and by taking the derivative of (15) with respect to s, we find $f^2 - (f'/k_3)^2 > (1/k_1)^2$. Finally, Theorem 3.3. implies that α lies on $H_0^3(r)$.

In the sequal, recall that spacelike curve with spacelike principal normal N and a null binormal B_1 is called *a partially null curve* [5].

Remark 3.1. In the case when α is spacelike curve with timelike principal normal N in the space E_1^4, there holds theorems which are analogous with Theorems 3.1, 3.2, 3.3 and 3.4.

Theorem 3.5 *A partially null unit speed curve $\alpha(s)$ in E_1^4 with curvatures $k_1(s) \neq 0$, $k_2(s) \neq 0$ for each $s \in I \subset R$ lies fully in a lightlike hyperplane of E_1^4 and has $k_3(s) = 0$ for each s.*

Proof. By using the Frenet equations, we easily get $\dot{\alpha} = T$, $\ddot{\alpha} = k_1 N$, $\dddot{\alpha} = -k_1 T + \dot{k_1} N + k_1 k_2 B_1$, $\ddddot{\alpha} = -3k_1 \dot{k_1} T + (\ddot{k_1} - k_1^3) N + (2\dot{k_1} k_1 + k_1 \dot{k_2} + k_1 k_2 k_3) B_1$. It follows that $\dot{\alpha}$, $\ddot{\alpha}$, $\dddot{\alpha}$ are linearly independent vectors and that $\dot{\alpha}$, $\ddot{\alpha}$, $\dddot{\alpha}$, $\ddddot{\alpha}$ are linearly dependent vectors. Moreover, by using the MacLaurin expansion for α given by

$$\alpha(s) = \alpha(0) + \dot{\alpha}(0) \frac{s}{1!} + \ddot{\alpha}(0) \frac{s^2}{2!} + \dddot{\alpha}(0) \frac{s^3}{3!} + \ldots,$$

we find that α lies fully in a lightlike hyperplane π of the space E_1^4 spanned by $\{\dot\alpha(0), \ddot\alpha(0), \dddot\alpha(0)\}$. Therefore, α satisfies the equation of π, namely $g(\alpha(s)-p, q) = 0$, where $p, q \in E_1^4$ and q is a constant null vector. Differentiating the previous equation three times with respect to s and by using Frenet equations, we find $g(B_1, q) = 0$. Hence $q = \lambda B_1$, $\lambda \in R$. Then $\dot q = \lambda k_3 B_1 = 0$ and therefore $k_3(s) = 0$ for each s. □

Theorem 3.6 *A partially null unit speed curve α in E_1^4 with curvatures $k_1(s) \neq 0$, $k_2(s) \neq 0$ for each $s \in I \subset R$ lies on $H_0^3(r)$ if and only if $(1/k_2)(1/k_1)' =$ constant $\neq 0$.*

Proof. Let us first suppose that α lies on $H_0^3(r)$ with center m. By Theorem 3.5. it follows that $k_3 = 0$ for each s and by definition there holds $g(\alpha - m, \alpha - m) = -r^2$. Differentiating the previous equation three times with respect to s and by using Frenet equations, we get $g(B_1, \alpha - m) = (-1/k_2)(1/k_1)'$. By taking the derivative of the last equation with respect to s, we find $((1/k_2)(1/k_1)')' = 0$ and thus $(1/k_2)(1/k_1)' =$ constant $= c_0$. If $c_0 = 0$, then $g(B_1, \alpha - m) = 0$. This means that null vector B_1 and timelike vector $\alpha - m$ are orthogonal in E_1^4, which is a contradiction. Thus $c_0 \neq 0$.

Conversely, if $(1/k_2)(1/k_1)' =$ constant $\neq 0$ let us consider the vector

$$m = \alpha + (1/k_1)N - [((1/k_1)^2 + r^2)/((2/k_2)(1/k_1)')]B_1 + (1/k_2)(1/k_1)'B_2, \tag{17}$$

where $r \in R^+$. Differentiating the previous equation with respect to s yields $m' = 0$, so $m =$ constant. Finally, (17) implies that $g(\alpha - m, \alpha - m) = -r^2$ and hence α lies on $H_0^3(r)$. □

Theorem 3.7 *Let $\alpha(s)$ be a partially null unit speed curve in E_1^4 with curvatures $k_1(s) \neq 0$, $k_2(s) \neq 0$ for each $s \in I \subset R$. Then α lies on $H_0^3(r)$ if and only if*

$$\sinh(\int_0^s k_2 ds) \int_0^s (1/k_1)' \sinh(\int_0^s k_2 ds)ds - \cosh(\int_0^s k_2 ds) \int_0^s (1/k_1)' \cosh(\int_0^s k_2 ds)ds = 0. \tag{18}$$

Proof. Let us first assume that α lies on $H_0^3(r)$. By Theorem 3.6. it follows that $(1/k_2)(1/k_1)' = \text{constant} \neq 0$. Let us put $c_0 = (1/k_2)(1/k_1)'$ and $\theta(s) = \int_0^s k_2(s)ds$. Then we have

$$\sinh\theta \int_0^s (1/k_1)' \sinh\theta ds - \cosh\theta \int_0^s (1/k_1)' \cosh\theta ds = 0.$$

Conversely, let us assume that (18) holds. Let us put $\theta(s) = \int_0^s k_2(s)ds$. By taking the derivative of (18) with respect to s, we find

$$\cosh\theta \int_0^s (1/k_1)' \sinh\theta ds - \sinh\theta \int_0^s (1/k_1)' \cosh\theta ds = (1/k_2)(1/k_1)'. \tag{19}$$

Differentiating (19) with respect to s yields $((1/k_2)(1/k_1)')' = 0$. Hence $(1/k_2)(1/k_1)' = \text{constant} = c_0$, $c_0 \in R$. If $c_0 = 0$, then subtracting (19) of (18) implies a contradiction. Therefore, $c_0 \neq 0$ so by Theorem 3.6. curve α lies on $H_0^3(r)$.

Recall that a spacelike curve with a null principal normal N in the space E_1^4 is called *a pseudo null curve* [5]. For such curves, we have the following two theorems. □

Theorem 3.8 *Let $\alpha(s)$ be a pseudo null unit speed curve with curvatures $k_1(s) = 1$, $k_2(s) \neq 0$, $k_3(s) \neq 0$ for each $s \in I \subset R$. Then α lies on $H_0^3(r)$ if and only if $k_3/k_2 = constant < 0$.*

Proof. Let us first suppose that α lies on $H_0^3(r)$ with center m. By definition we have $g(\alpha - m, \alpha - m) = -r^2$. Differentiating the previous equation four times with respect to s and by using Frenet equations, we get

$$g(\alpha - m, T) = 0, g(N, \alpha - m) = -1, g(B_1, \alpha - m) = 0, g(B_2, \alpha - m) = -k_3/k_2. \tag{20}$$

Moreover, differentiating the last equation in (20) with respect to s, we find

$$-g(T, \alpha - m) - k_3 g(B_1, \alpha - m) = -(k_3/k_2)'. \tag{21}$$

Therefore, substituting (20) into (21) we obtain $(k_3/k_2)' = 0$. Thus $k_3/k_2 =$ constant $= c_0$, $c_0 \in R$. We shall prove that $c_0 < 0$. Let us decompose the vector $\alpha - m$ by

$$\alpha(s) - m = a(s)T(s) + b(s)N(s) + c(s)B_1(s) + d(s)B_2(s), \tag{22}$$

where $a(s), b(s), c(s), d(s)$ are arbitrary functions. Then by (20) and (22) it follows that $g(T, \alpha - m) = a = 0$, $g(N, \alpha - m) = d = -1$, $g(B_1, \alpha - m) = c = 0$, $g(B_2, \alpha - m) = b = -c_0$. Substituting this into (22) we find $\alpha - m = -c_0 N - B_2$ and thus $g(\alpha - m, \alpha - m) = 2c_0 = -r^2$. Hence $c_0 < 0$.

Conversely, let us suppose that $k_3/k_2 =$ constant < 0. Then we may put $k_3/k_2 = -r^2/2, r \in R^+$. Let us consider vector $m = \alpha - (r^2/2)N + B_2$. Differentiating the previous equation with respect to s, we find $m' = 0$ and thus $m =$ constant. Moreover, $g(\alpha - m, \alpha - m) = -r^2$, so α lies on $H_0^3(r)$. □

Theorem 3.9 *Let $\alpha(s)$ be a pseudo null unit speed curve with curvatures $k_1(s) = 1$, $k_2(s) \neq 0$, $k_3(s) \neq 0$ for each $s \in I \subset R$. Then α lies on $H_0^3(r)$ if and only if the following two relations hold:*

$$\sinh(\int_0^s k_3 ds) \int_0^s k_2 \sinh(\int_0^s k_3 ds) ds - \cosh(\int_0^s k_3 ds) \int_0^s k_2 \cosh(\int_0^s k_3 ds) ds = 0, \tag{23}$$

$$\cosh(\int_0^s k_3 ds) \int_0^s k_2 \sinh(\int_0^s k_3 ds) ds < \sinh(\int_0^s k_3 ds) \int_0^s k_2 \cosh(\int_0^s k_3 ds) ds.$$

Proof. Let us first assume that α lies on H_0^3. By Theorem 3.8 we have $k_2/k_3 =$ constant < 0. Let us put $k_2/k_3 = -c_0^2$, $c_0 \in R$ and $\theta(s) = \int_0^s k_3(s)ds$. Then we easily find

$$\sinh\theta \int_0^s k_2 \sinh\theta ds - \cosh\theta \int_0^s k_2 ds = 0.$$

Differentiating the previous equation with respect to s, we get

$$\cosh\theta\int_0^s k_2\sinh\theta ds-\sinh\theta\int_0^s k_2\cosh\theta ds=-c_0^2=k_2/k_3, \tag{24}$$

and hence

$$\cosh\theta\int_0^s k_2\sinh\theta ds<\sinh\theta\int_0^s k_2\cosh\theta ds.$$

Conversely, if (23) holds, let us put $\theta(s)=\int_0^s k_3(s)ds$. By taking the derivative of (23) with respect to s, we find

$$\cosh\theta\int_0^s k_2\sinh\theta ds-\sinh\theta\int_0^s k_2\cosh\theta ds=k_2/k_3. \tag{25}$$

Differentiating (25) with respect to s yields $(k_2/k_3)'=0$. Hence $k_2/k_3=\text{constant}=c_0$, $c_0\in R$. Moreover, (23) and (25) imply $c_0<0$. Finally, by Theorem 3.8 it follows that α lies on $H_0^3(r)$. □

Theorem 3.10 *There are no timelike and no null curves $\alpha(s)$ lying on the pseudohyperbolic space $H_0^3(r)$ in E_1^4.*

Proof. If $\alpha(s)$ is timelike unit speed curve lying on $H_0^3(r)$ with center m, then $g(\alpha-m,\alpha-m)=-r^2$. Differentiating the previous equation with respect to s, we get $g(T,\alpha-m)=0$. It follows that T and $\alpha-m$ are two timelike mutually orthogonal vectors in E_1^4, which is a contradiction. If $\alpha(s)$ is a null curve lying on $H_0^3(r)$, then in a similar way it follows that null vector T and timelike vector $\alpha-m$ are orthogonal vectors in E_1^4, which is a contradiction. □

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