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Some Asymptotic Results for the Semi-Markovian Random Walk with a Special Barrier

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Abstract

In this study, the semi-Markovian random walk with a special barrier $(X(t))$ is considered and under some weak assumptions the ergodicity of this process is discussed. Moreover, the characteristic function of ergodic distribution of $X(t)$ is given by using a joint distribution of random variables N and Y_N and some exact formulas for the first and second moments of ergodic distribution of the process $X(t)$ are obtained. Based on these results, the asymptotic behaviours of expectation and variance of this process are investigated as $S - s \to \infty$. It is finally proved that the ergodic distribution of the process is close to a uniform distribution over(s, S) as $S - s$ takes sufficiently large values.

Key Words: Semi-Markovian random walk; ergodicity of process; asymptotic behaviour; weakly convergence.

1. Introduction

It is known that numerous interesting problems in the fields of queuing, reliability, stock control, storage, inventory theories and etc. are given by means of the semi-Markovian random walk process with barriers. Many theoretical studies in this topic exist in literature (see References).

Unfortunately, the theoretical results are so complex that it is very difficult to use them for the purpose of applications (see, for example, [1],[2],[3],[4],[8],[20],[21]). However, in recent years, the asymptotic methods for investigation of the processes of queuing and reliability theories are intensively developed (see, [1],[4],[6],[11],[16],[17]).

Particularly, the asymptotic results obtained for the harmonic and ordinary renewal measures indicate that it is possible to get also some asymptotic results for semi-Markovian random walk with a special barrier (see, [2],[7],[9],[17],[22],[24]).

Therefore, this paper will consider a semi-Markovian random walk with a barrier having a special property and investigate the asymptotic behaviour of this process. Due to a special property of the barrier we expect that some important probability characteristics of this process can be expressed by means of a sequence of ladder variables, which is important for applications. In the theory of storage models, such processes are interpreted as a stock's level in a warehouse which works under some special rules. Let's now describe the new physical model proposed in this paper.

Let's observe a warehouse having a sufficiently large volume and suppose that it will be in use for a long time. Let the warehouse work in accordance with the following rules. By demands and supplies arising at random times, the quantity of stock in the warehouse both decreases and increases with random portions. When the quantity of stock in the warehouse falls to a control level $s(0 < s < S)$, as decided previously, it immediately becomes equal to initial quantity S. And the warehouse continues its function in this way.

This stochastic model is called "The extended model of type (s, S)". Some special results connected with such models can be found in literature (see [5],[8],[21],[22],[25],[26]). However, asymptotic properties of such models have not been considered enough in literature. Therefore, in this paper we will investigate some asymptotic properties of a process $X(t)$ which describes the above mentioned storage models and interprets it as the stock's quantity in the warehouse at time t. We now proceed to a mathematical construction of the process $X(t)$.

2. Mathematical Construction of the Process $X(t)$

Let $\{(\xi_n, \eta_n)\}$, $n \geq 1$, be a sequence of independent and identically distributed pairs of random variables defined on any probability space $(\Omega, \Im, \mathrm{P})$where ξ_n's take on positive values, and η_n's take on positive and negative values i.e., $P\{\xi_n > 0\} = 1$, $P\{\eta_n > 0\} > 0$, $P\{\eta_n < 0\} > 0$. Suppose that ξ_1 and η_1 are independent random variables and the distribution function of ξ_1 and η_1 are known, i.e.

$\Phi(t) = P\{\xi_1 \leq t\}, \quad F(x) = P\ \{\eta_1 \leq x\}, \quad t \geq 0, x \in (-\infty, \infty).$

Define a renewal sequence $\{T_n\}$ and random walk $\{Y_n\}$ as follows

$T_n = \sum_{i=1}^{n} \xi_i \quad , \quad Y_n = \sum_{i=1}^{n} \eta_i, \quad n \geq 1, \quad T_0 = Y_0 = 0,$

and a sequence of integer valued random variables N_n as

$N_{n+1} = \inf\{k \geq N_n + 1 : S - Y_k + Y_{N_n} < s\}, n \geq 0, N_0 = 0,$

where s and S are constants $(0 < s < S)$ and $\inf(\emptyset) = +\infty$ is stipulated.

From now on, we denote N_1 as N or $N(\beta)$, where $\beta = S - s > 0$.

Let $\tau_n = T_{N_n}, n \geq 1, \quad \tau_0 = 0$ and $\nu(t) = \max\{ n \geq 0 : T_n \leq t\}, t > 0.$

We can now construct a desired stochastic process $X(t)$ which is as follows

$X(t) = \max\left\{ s, \mathrm{S} - Y_{\nu(t)} + Y_{N_n}\right\}, \text{ if } \tau_n \leq t < \tau_{n+1}, n \geq 0.$

The process X(t) is called "The semi-Markovian random walk with a special barrier on the s-level." The main purpose of this study is to investigate the asymptotic behaviour of some probability characteristics of the process $X(t)$ as $S - s \to \infty$. For this aim we will first discuss the ergodicity of this process.

3. Preliminary Discussions

The ergodic distribution of the process $X(t)$ can be expressed by means of renewal function according to a sequence of random variables $\tau_1, \tau_2, ...$ and very essential results about the asymptotic behaviour of this renewal function exist in renewal theory. Therefore, we think that it is advisable to use the ergodic distribution of process $X(t)$ for the purposes of this study.

Let's state the following proposition on the ergodicity of $X(t)$ as $t \to \infty$.

Proposition 3.1 *Let the initial sequence of random pairs* $\{(\xi_n, \eta_n)\}$, $n \geqslant 1$, *satisfy the following supplementary conditions*

1) $0 < E\xi_1 < \infty$;

2) $E\eta_1 > 0$;

3) η_1 *is non-arithmetic random variable.*

Then the process $X(t)$ is ergodic and for all $x \geqslant s$ the following relation for ergodic distribution function $Q_\beta(x/X)$ holds:

$$Q_\beta(x/X) = \lim_{t\to\infty} P\{X(t) \leqslant x\} = \frac{A(x,\beta)}{EN(\beta)}, \tag{3.1}$$

where

$$A(x,\beta) = \sum_{n=0}^{\infty} P\left\{ Y_i \leqslant \beta, i = \overline{1,n}, Y_n > S - x \right\}, \quad \beta = S - s > 0.$$

Proof. We remember that the process $X(t)$ belongs to a wide class of processes which are well known in literature as the class of stochastic processes with a discrete chance interference (see [8]). Moreover, the ergodic theorem of Smith's "key renewal theorem" type for the latter class exists in this reference (see, [8], p.243). Consequently, the proof of proposition 3.1 can be extracted from this general ergodic theorem. □

Remark. To use the result of the proposition 3.1 in solving various applied problems, the further and more precise calculation of $A(x,\beta)$ is necessary. But it is not easy to investigate $A(x,\beta)$ in detail by applying only direct calculation techniques of probability theory. Usually in applications, a mathematical technique known as the Fourier method of analysis, is used for investigations of problems of this type. The characteristic function of ergodic distribution, which is just a Fourier transform of the latter, will be expressed by means of the joint distribution of pair (N, Y_N). Note that a great number of papers exists in literature in which the numerical characteristics of N and Y_N have been investigated as well as the functional characteristics of them (see [2],[5],[7],[16],[19], etc.). This is one of the main motivations for choosing the pair (N, Y_N) as auxiliary tool of this study.

4. The Characteristic Function of the Ergodic Distribution

In this section we will investigate the characteristic function of the ergodic distribution of the process $X(t)$.

Let's now state the main result of this section.

Theorem 4.1 *Under the assumptions of Proposition 3.1, a characteristic function $\Psi_\beta(\theta/X)$ of process $X(t)$ as $t \to \infty$ can be expressed by the probability characteristics of the pair*

(N, Y_N) *as*

$$\Psi_\beta(\theta/X) = \lim_{t\to\infty} E(e^{i\theta X(t)}) = \frac{e^{i\theta S}E\{\exp(-i\theta Y_N)-1\}}{EN(\beta)E\{\exp(-i\theta\eta_1)-1\}},\ \textit{for all } \theta \in (-\infty, +\infty).$$

Proof. It is immediately derived that, for any measurable bounded function $f(x)$ defined on the interval $[0, +\infty)$, the following relation holds, from the Proposition 3.1 with probability 1:

$$\lim_{t\to\infty}\int_0^t f(X(u))du = \lim_{t\to\infty} Ef(X(t)) =$$

$$= \frac{1}{EN(\beta)}\sum_{n=0}^{\infty}\int_s^{\infty} f(x)P\left\{Y_i \le \beta, i = \overline{1,n}; S - Y_n \in dx\right\}. \tag{4.1}$$

Substituting the function $\exp\{-i\theta x\}$ instead of $f(x)$ in Formula (4.1), we get

$$\lim_{t\to\infty} E(e^{-i\theta X(t)}) = \frac{1}{EN(\beta)}\sum_{n=0}^{\infty}\int_s^{\infty} e^{-i\theta x}P\left\{Y_i \le \beta, i = \overline{1,n}; S - Y_n \in dx\right\}$$

Carrying out the necessary calculations we obtain

$$\lim_{t\to\infty} E(e^{-i\theta X(t)}) = \frac{e^{-i\theta S}}{EN(\beta)}\sum_{n=0}^{\infty}\int_s^{\infty} e^{i\theta v}P\left\{Y_i \le \beta, i = \overline{1,n}; Y_n \in dv\right\}. \tag{4.2}$$

In order to continue this investigation we need some further notation first. Put

$c_n(x,\beta) = P\left\{Y_i \le \beta, i = \overline{1,n}; Y_n \le x\right\}, n \ge 1,$

$c_0(x,\beta) = \varepsilon(x), if x \le \beta,$

$d_n(x,\beta) = P\left\{Y_i \le \beta, i = \overline{1,n}; Y_n\rangle\beta, Y_n \le x\right\}, n \ge 1,$

$d_0(x,\beta) = 0, if x > \beta,$

where $\varepsilon(x) = 1$ if $x \geqslant 0$ and $\varepsilon(x) = 0$ if $x < 0$.

Note that the study of $c_n(x,\beta)$ and $d_n(x,\beta)$ is closely connected with the investigation of the random walk $\{Y_n\}$,n≥ 0, prior to the first entry into (β,∞), that is the random walk restricted to $(-\infty,\beta]$. And there is a certain relation between these probability characteristics. To present this relation let's introduce the following transforms of $c_n(x,\beta)$ and $d_n(x,\beta)$

$$\widetilde{c}^*(z,\theta,\beta) = \sum_{n=0}^{\infty} z^n \int_{-\infty}^{\beta} e^{i\theta x}P\left\{Y_i \le \beta, i = \overline{1,n}; Y_n \in dx\right\},$$

$$\widetilde{d}^*(z,\theta,\beta) = \sum_{n=1}^{\infty} z^n \int_{\beta}^{\infty} e^{i\theta x}P\left\{Y_i \le \beta, i = \overline{1,n}; Y_n > \beta, Y_n \in dx\right\},$$

where $\theta \in (-\infty, \infty)$ and $z \in [-1, 1]$.

The following basic identity is well-known in literature (see [7], p.600)

$$1 - \widetilde{d}^*(z, \theta, \beta) = \widetilde{c}^*(z, \theta, \beta)[1 - z\varphi(\theta)], \tag{4.3}$$

where $\varphi(\theta)$ is a characteristic function of variable $\eta_1, i.e., \varphi(\theta) = E(e^{i\theta\eta_1})$.

From (4.3) the following is directly obtained:

$$\widetilde{c}^*(z, \theta, \beta) = \frac{1 - \widetilde{d}^*(z, \theta, \beta)}{1 - z\varphi(\theta)}. \tag{4.4}$$

If the conditions, which are $E\eta_1 > 0$ and $\beta < +\infty$, are satisfied, then it is not difficult to prove

$$\left|\widetilde{d}^*(1, \theta, \beta)\right| \leq 1 \text{ and } |\widetilde{c}^*(1, \theta, \beta)| \leq EN(\beta) < +\infty,$$

where

$$\tilde{d}^*(1, \theta, \beta) = \lim_{z \to 1} \tilde{d}^*(z, \theta, \beta), \tilde{c}^*(1, \theta, \beta) = \lim_{z \to 1} \tilde{c}^*(z, \theta, \beta).$$

For this reason, we can take the limit in (4.4) as $z \to \infty$ and we get

$$\widetilde{c}^*(1, \theta, \beta) = \frac{1 - \widetilde{d}^*(1, \theta, \beta)}{1 - \varphi(\theta)}. \tag{4.5}$$

Furthermore, it is not difficult to show that

$$\widetilde{d}^*(1, \theta, \beta) = \sum_{n=1}^{\infty} \int_{\beta}^{\infty} e^{i\theta x} P\left\{Y_i \leq \beta, i = \overline{1, n-1}; Y_n \rangle \beta, Y_n \in dx\right\} =$$

$$= \sum_{n=1}^{\infty} \int_{\beta}^{\infty} e^{i\theta x} P\left\{N(\beta) = n, Y_n \in dx\right\} = \int_{\beta}^{\infty} e^{i\theta x} P\left\{Y_n \in dx\right\} =$$

$$= E(e^{i\theta Y_N}), \text{ i.e., } \widetilde{d}^*(1, \theta, \beta) = E(e^{i\theta Y_N}). \tag{4.6}$$

Therefore, the (4.5) can be rewritten as

$$\widetilde{c}^*(1, \theta, \beta) = \frac{1 - E(\exp(i\theta Y_N))}{1 - E(\exp(i\theta\eta_1))}. \tag{4.7}$$

It is seen from (4.2) that

$$\lim_{t \to \infty} E(e^{-i\theta X(t)}) =$$

$$\frac{e^{-i\theta S}}{EN(\beta)} \sum_{n=0}^{\infty} \int_{-\infty}^{\beta} e^{i\theta x} P\left\{Y_i \leq \beta, i = \overline{1, n}; Y_n \in dx\right\} = \frac{e^{-i\theta S}}{EN(\beta)} \tilde{c}^*(1, \theta, \beta). \tag{4.8}$$

Substituting now Formula (4.7) in (4.8) and replacing the parameter θ by (-θ), we finally obtain

$$\Psi_\beta(\theta/X) = \lim_{t\to\infty} E(e^{i\theta X(t)}) = \frac{e^{i\theta S}}{EN(\beta)} \frac{E\{\exp(-i\theta Y_N) - 1\}}{E\{\exp(-i\theta\eta_1) - 1\}}. \tag{4.9}$$

We have thus expressed the characteristic function $\Psi_\beta(\theta/X)$ of the process X(t) by means of the same one's $Y_{N(\beta)}$ and η_1 as $t \to \infty$.

This completes the proof. □

Remark. Direct calculations of characteristic functions of ergodic distribution of process $X(t)$ are rarely feasible, but much valuable information can be extracted directly from (4.9). Particularly, by using Formula (4.9), it is possible to show that the process $X(t)$ may itself behave as a random variable which is uniformly distributed on the interval (s, S) as $t \to \infty$ and β take on large values. Because of this, in the next section we will investigate the asymptotic behaviour of ergodic distribution function of $X(t)$ as $\beta \to \infty$.

5. The Convergence of the Ergodic Distribution Functions of Process $X(t)$

For the investigation of the asymptotic behaviour of ergodic distribution function of process $X(t)$ as $\beta \to \infty$, we define the auxiliary process $W(t)$ as

$$W(t) = \frac{2X(t)}{\beta} - 1 - \frac{2s}{\beta},$$

and investigate the asymptotic behaviour of its ergodic distribution function.

It is easily seen that the process $W(t)$ is a linear transform of $X(t)$. Therefore, from Proposition 3.1 it immediately follows that the process $W(t)$ is also ergodic under the conditions of Proposition 3.1. Let's denote the characteristic function of ergodic distribution of $W(t)$ by $\Psi_\beta(\theta/W)$ and formulate the following statement.

Lemma 5.1 *Let the initial sequence of random pairs satisfy the assumptions of Proposition 3.1 and the supplementary condition $E(\eta_1^2) < \infty$. Then the following expansion can be written for characteristic function $\Psi_\beta(\theta/W)$ of ergodic distribution of process $W(t)$*

$$\Psi_\beta(\theta/W) = \frac{\sin\theta}{\theta} + O(\frac{1}{\beta}), \quad as \quad \beta \to \infty.$$

Proof. By definition of process $W(t)$ we can write

$$\Psi_\beta(\theta/W) = \lim_{t\to\infty} E[\exp(i\theta W(t)] = \exp[-i\theta(1 + \frac{2s}{\beta})]\Psi_\beta(\frac{2\theta}{\beta}/X). \tag{5.1}$$

For further investigations we denote the characteristic functions of random variables η_1 and $\overline{Y}_N = Y_N - \beta$ by $\varphi_{\eta_1}(t)$ and $\varphi_{\overline{Y}_N}(t)$, respectively. Moreover, we let

$I_1(\theta,\beta) = 1 - \varphi_{\eta_1}(-\frac{2\theta}{\beta})$ and $I_2(\theta,\beta) = 1 - \varphi_{\overline{Y}_N}(-\frac{2\theta}{\beta})$.

By using these notations, the formula (5.1) can be rewritten in the following form

$$\Psi_\beta(\theta/W) = \frac{e^{i\theta} - e^{-i\theta}}{EN(\beta)I_1(\theta,\beta)} + \frac{e^{-i\theta}I_2(\theta,\beta)}{EN(\beta)I_1(\theta,\beta)}. \tag{5.2}$$

Throughout this study, the first and second terms of sum (5.2) will be denoted by $J_1(\theta,\beta)$ and $J_2(\theta,\beta)$, respectively. It is our immediate aim to separately evaluate the $J_1(\theta,\beta)$ and $J_2(\theta,\beta)$.

If $E(\eta_1)^2 < \infty$ then following the inequality is valid for all t (see [7], p.514):

$$|\varphi_{\eta_1}(t) - 1 - itE\eta_1| \leq \frac{t^2}{2}E(\eta_1^2).$$

From this we get the following expansion for $\varphi_{\eta_1}(t)$:

$$\varphi_{\eta_1}(-\frac{2\theta}{\beta}) = 1 - i\frac{2\theta}{\beta}E\eta_1 + O(\frac{1}{\beta^2}), \text{ as } \beta \to \infty.$$

Consequently,

$$I_1(\theta,\beta) = 1 - \varphi_{\eta_1}(-\frac{2\theta}{\beta}) = i\frac{2\theta}{\beta}E\eta_1(1 + O(\frac{1}{\beta})). \tag{5.3}$$

We will now study the asymptotic behaviour of $EN(\beta)$ as $\beta \to \infty$. For this, we need some further notations. Let $(\sum_{i=1}^{n}\nu_i^+;\sum_{i=1}^{n}\chi_i^+), n \geq 0$, be the sequence of strictly ascending ladder epochs (1st component) and ladder heights (2nd component) of the random walk $\{Y_n\}, n \geq 0$, i.e.

$$\nu_0^+ = \chi_0^+ = 0; \quad \nu_1^+ = \inf\ \{\ k \geq 1 : Y_k > 0\}, \quad \chi_1^+ = Y_{\nu_1^+},$$

$$\nu_{n+1}^+ = \inf\left\{\ k \geq 1 : Y_{\sum_{i=1}^{n}\nu_i^+ + k} > Y_{\sum_{i=1}^{n}\nu_i^+}\right\}, \quad \chi_{n+1}^+ = Y_{\sum_{i=1}^{n+1}\nu_i^+} - Y_{\sum_{i=1}^{n}\nu_i^+}, n \geq 1,$$

where $\inf(\emptyset) = \infty$ is stipulated.

The pairs $(\nu_n^+, \chi_n^+), n \geq 1$, are mutually independent and identically distributed (see [7], p.392). It is known that under the condition $E\eta_1 > 0$ the equality $P\{\nu_1^+ < \infty\} = 1$ is valid.

It is well known that if $E\eta_1 > 0$ then $E\nu_1^+ < \infty$ (see [7], [17]).

Moreover, we define the renewal process $H(\beta)$ as

$$H(\beta) = \inf \left\{ n \geq 1 : \sum_{i=1}^{n} \chi_i^+ > \beta \right\}.$$

Note that $N(\beta)$ can be rewritten as $N(\beta) = \sum_{i=1}^{H(\beta)} \nu_i^+$. From this it is obviously seen that $N(\beta)$ is a reward renewal process and it can easily be verified by using Wald's identity (see [2], [5], [7], etc.) that

$$EN(\beta) = E\nu_1^+ EH(\beta). \tag{5.4}$$

Note that $EH(\beta)$ is a renewal function associated with $\{\chi_n^+\}, n \geq 0$. Then the asymptotic expansion holds under the condition $E(\chi_1^+)^2 < \infty$ (see [8], p.366)

$$EH(\beta) = \frac{\beta}{E(\chi_1^+)} + \frac{E(\chi_1^+)^2}{2(E\chi_1^+)^2} + o(1), \text{ as } \beta \to \infty.$$

Therefore, we have

$$EN(\beta) = \frac{E(\nu_1^+)}{E(\chi_1^+)}\beta + \frac{E(\nu_1^+)E(\chi_1^+)^2}{2(E\chi_1^+)^2} + o(1), \text{ as } \beta \to \infty.$$

It is well known that if $0 < E\eta_1 < \infty$, then the following equality holds:

$$E(\chi_1^+) = E\eta_1 E\nu_1^+ < \infty.$$

Using (5.4) we get

$$EN(\beta) = \frac{\beta}{E\eta_1} + \frac{E\nu_1^+ E(\chi_1^+)^2}{2(E\chi_1^2)^2} + o(1), \text{ as } \beta \to \infty. \tag{5.5}$$

Note that if the conditions $E\eta_1 > 0$ and $E(\eta_1^2) < \infty$ are satisfied, then the condition $E(\chi_1^+)^2 < \infty$ also is satisfied. We can thus rewrite (5.5) as

$$EN(\beta) = \frac{\beta}{E\eta_1}(1 + O(\frac{1}{\beta})), \text{ as } \beta \to \infty. \tag{5.6}$$

By the asymptotic relations (5.3) and (5.6), it is easily seen that

$$EN(\beta)I_1(\theta, \beta) = 2i\theta(1 + O(\frac{1}{\beta})). \tag{5.7}$$

Coming back to sum (5.2), we conclude that the $J_1(\theta, \beta)$ may be written as

$$J_1(\theta, \beta) = \frac{e^{i\theta} + e^{-i\theta}}{EN(\beta)I_1(\theta, \beta)} = \frac{\sin\theta}{\theta}(1 + O(\frac{1}{\beta})). \tag{5.8}$$

Let us estimate the second term $J_2(\theta, \beta)$ of sum (5.2). It follows from the inequality for Taylor expansion of a characteristic function that if $E|\overline{Y}_N| < \infty$, then following inequality is valid for arbitrary θ (see [7], p.514)

$$\left|\varphi_{\overline{Y}_N}(-\frac{2\theta}{\beta}) - 1\right| \leq \frac{2|\theta|}{\beta} E|\overline{Y}_N|. \tag{5.9}$$

On the other hand, by definition $\overline{Y}_N = Y_N - \beta > 0$, with probability 1. Therefore,

$E\left|\overline{Y}_N\right| = E(Y_N - \beta) = E(\sum_{i=1}^{N(\beta)} \eta_i) - \beta = E\eta_1 EN(\beta) - \beta.$

Taking into account (5.6), we get

$$E\left|\overline{Y}_N\right| = E\eta_1 \frac{\beta}{E\eta_1}(1 + O(\frac{1}{\beta})) - \beta = O(1), \text{ as } \beta \to \infty. \quad (5.10)$$

Therefore, from inequality (5.9) and expansion (5.10) we conclude that

$$1 - \varphi_{\overline{Y}_N}(-\frac{2\theta}{\beta}) = O(\ \frac{1}{\beta}), \text{ as } \beta \to \infty. \quad (5.11)$$

Thus by using (5.7) and (5.11), $J_2(\theta, \beta)$ can be rewritten as

$$J_2(\theta, \beta) = O(\frac{1}{\beta}), \text{ as } \beta \to \infty. \quad (5.12)$$

By using formulas (5.8) and (5.12) we finally have from (5.2)

$$\Psi_\beta(\theta/W) = J_1(\theta, \beta) + J_2(\theta, \beta) = \frac{\sin\theta}{\theta} + O(\frac{1}{\beta}), \text{ as } \beta \to \infty.$$

This completes the proof of Lemma 5.1. □

It is well known that the one-to-one correspondence between distribution functions and characteristic functions is continuous. Since the limit characteristic function $\varphi_1(\theta) = \frac{\sin\theta}{\theta}$ is continuous at $\theta = 0$, then from the continuity theorem (see [19], p.48) the following statement is immediately derived.

Theorem 5.1 *Under the assumptions of Lemma 5.1 the family of ergodic distribution functions $Q_\beta(x/W)$ of process $W(t)$ converges weakly to $Q_1(x)$ which is the uniform distribution function over [-1,1], i.e. for arbitrary x,*

$$Q_\beta(x/W) \to Q_1(x), \text{ as } \beta \to \infty.$$

Proof. The proof of the Theorem 5.1 is directly derived from Lemma 5.1 by using the continuity theorem (see [19], p.48). □

Remark. The assertion of Theorem 5.1 is very important for applications, but Theorem 5.1 does not answer the questions associated with the asymptotic behaviour of the moments of process $X(t)$. Because it is possible that a sequence $\{\pi_n(x)\}$ of distribution functions converges weakly to a limit $\pi(x)$, the moments of the $\pi_n(x)$ do not converge to the moments of $\pi(x)$ (see, [19], p. 53). Therefore, it is advisable to separately investigate the asymptotic behaviour of the moments of ergodic distribution of the process X(t).

6. Exact Formulas for the First and Second Order Moments of Ergodic Distribution of Process $X(t)$

Taking into account the practical importance of the expectation and variance of the ergodic distribution of X(t), in this section we will extract from the (4.9) the exact expressions for them. For this purpose we introduce the following notations:

$m_k = E(\eta_1^k), M_k = E(Y_N^k), k = 1, 2, 3$

and for shortness of expressions we put

$M_{k1} = \dfrac{M_k}{M_1}; m_{k1} = \dfrac{m_k}{m_1}, k = 2, 3.$

Let us denote the first and second moments of ergodic distribution of process X(t) by

$EX = \lim\limits_{t\to\infty} EX(t),$

$E(X^2) = \lim\limits_{t\to\infty} EX^2(t)$, and

$Var(X) = \lim\limits_{t\to\infty} Var(X(t)).$

We can now state the main result of this section as follows.

Theorem 6.1 *Let the following supplementary conditions are satisfied*

1) $0 < E\eta_1 < \infty$; **2)** $E|\eta_1|^3 < \infty$.

Then the first and second moments of ergodic distribution of the process $X(t)$ exists and can be expressed by moments of Y_N as:

$$EX = S - \frac{M_2}{2M_1} + \frac{m_2}{2m_1}$$

$$E(X^2) = S^2 + \frac{1}{3}\left(\frac{M_3}{M_1} - \frac{m_3}{m_1}\right) - \left(S + \frac{m_2}{2m_1}\right)\left(\frac{M_2}{M_1} - \frac{m_2}{m_1}\right);$$

$$Var(X) = \frac{1}{3}\left[\frac{M_3}{M_1} - \frac{m_3}{m_1}\right] - \frac{1}{4}\left[\frac{M_2^2}{M_1^2} - \frac{m_2^2}{m_1^2}\right].$$

Proof. Note that the conditions (1) and (2) provide the existence and finiteness of first three moments of Y_N for each finite value of parameter S (see [7], p.397). Therefore, Taylor expansions of the characteristic functions of random variables η_1 and Y_N can be written as follows (see, [7], p.514 or [19], p.23).

$$E\{\exp(-i\theta\eta_1)-1\}=-i\theta m_1\left\{1-\frac{i\theta}{2}m_{21}+\frac{(i\theta)^2}{6}m_{31}+o(\theta^2)\right\} \tag{6.1}$$

and

$$E\{\exp(-i\theta Y_N)-1\}=-i\theta M_1\left\{1-\frac{i\theta}{2}M_{21}+\frac{(i\theta)^2}{6}M_{31}+o(\theta^2)\right\},\text{ as } \theta\to 0. \tag{6.2}$$

Dividing the expression (6.2) by (6.1) we can obtain

$$\frac{E\{\exp(-i\theta Y_N)-1\}}{E\{\ \exp(-i\theta\eta_1)-1\}}=$$

$$\frac{M_1}{m_1}\left\{1-\frac{i\theta}{2}[M_{21}-m_{21}]+\frac{(i\theta)^2}{12}[2(M_{31}-m_{31})-3M_{21}(M_{21}-m_{21})]+0(\theta^2)\right\}.$$

By Wald's identity, we have from this

$$\frac{1}{EN(\beta)}\frac{E\{\exp(-i\theta Y_N)-1\}}{E\{\exp(-i\theta\eta_1)-1\}}=1-\frac{i\theta}{2}(M_{21}-m_{21})+$$

$$+\frac{(i\theta)^2}{12}[2(M_{31}-m_{31})-3m_{21}(M_{21}-m_{21})]+0(\theta^2), \tag{6.3}$$

because of that $M_1=E(Y_N)=E\eta_1 EN$ (see, [7], p.397 or [2], p. 20 and etc.).

Let's now write the Taylor expansion of $\exp\{i\theta S\}$ for all finite values S as $\theta\to 0$, i.e.

$$e^{i\theta S}=1+i\theta S+\frac{(i\theta)^2}{2}S^2+o(\theta^2). \tag{6.4}$$

By using (6.3) and (6.4), we can show that, as $\theta\to 0$

$$\lim_{t\to\infty}E(e^{i\theta X(t)})=1-\frac{i\theta}{2}[M_{21}-m_{21}-2S]+$$

$$+\frac{(i\theta)^2}{12}[2(M_{31}-m_{31})-3(M_{21}-m_{21})(m_{21}+2S)+6S^2]+o(\theta^2).$$

From this, it is easily seen that under the conditions of this theorem, the first and second order moments of the ergodic distribution of the process X(t) exist and they can be presented as

$$EX=S-\frac{M_2}{2M_1}+\frac{m_2}{2m_1},$$

$$E(X^2)=S^2+\frac{1}{3}(\frac{M_3}{M_1}-\frac{m_3}{m_1})-(S+\frac{m_2}{2m_1})(\frac{M_2}{M_1}-\frac{m_2}{m_1}).$$

By using the expressions for $E(X)$ and $E(X^2)$, we can hence calculate the $Var(X)$

$$Var(X)=\frac{1}{3}\left[\ \frac{M_3}{M_1}-\frac{m_3}{m_1}\right]-\frac{1}{4}\left[\frac{M_2^2}{M_1^2}-\frac{m_2^2}{m_1^2}\right].$$

This completes the proof of Theorem 6.1. □

These formulas are very important for solving a number of problems of applied sciences. In addition to this, in the following we will study the asymptotic behaviours of expectation and variance of stock's level in this model as $S - s \to \infty$.

7. Asymptotic Expansion for the Expectation of the Ergodic Distribution of the Process $X(t)$

In this previous section, the exact expression for the expectation of ergodic distribution of process $X(t)$ was obtained. But it is well known that the calculation of such expressions is very difficult and unsuitable for purposes of practice. On the contrary, the asymptotic methods of investigation of such problems lead us to some convenient results for applications. Because of this, the asymptotic methods for investigation of moments of ergodic distribution of process $X(t)$ are considered as necessary tools for purposes of praymptotic behaviour of expectation EX of process $X(t)$, when $\beta = S - s \to \infty$. For this aim we will use some additional notations which were introduced in section 5. Consequently, in this section we will investigate the asduced in section 5. Namely, let $(\sum_{i=1}^{n} \nu_i^+; \sum_{i=1}^{n} \chi_i^+), n \geq 0$, be the sequence of strictly ascending ladder epochs (1^{st} component) and ladder heights (2^{nd} component) of the random walk $\{Y_n\}, n \geq 0$.

It is known that under the condition $E\eta_1 > 0$ the equality $P\left\{\nu_1^+ < \infty\right\} = 1$ is valid. If the conditions $E\eta_1 > 0$ and $E|\eta_1|^3 < \infty$ are satisfied, then the first three moments of random variables ν_1^+ and χ_1^+ exist. Let us denote these moments by α_k and μ_k, respectively, i.e.

$\alpha_k = E(\nu_1^+)^k, \mu_k = E(\chi_1^+)^k, k = 1, 2, 3.$

In addition, to shorten the notation we put

$\mu_{k1} = \dfrac{\mu_k}{(\mu_1)^k}, k = 2, 3,$ and $\sigma_1^2 = Var(\chi_1^+).$

Remember that in Section 5 we defined the renewal process $H(\beta)$ as

$$H(\beta) = \inf\left\{ n \geq 1 : \sum_{i=1}^{n} \chi_i^+ > \beta \right\}.$$

There are some significant points understanding in the literature connected with the moments of the renewal process $H(\beta)$ (see [2], [8], etc.). For example, it is known that (see [7], p.366) if a distribution of χ_1^+ is non-arithmetic and χ_1^+ has an expectation μ_1 and variance σ_1^2, then

$$EH(\beta) = \frac{\beta}{\mu_1} + \frac{1}{2}\mu_{21} + o(1), \text{ as } \beta \to \infty.$$

However, if the third absolute moment of η_1 exists then some sharper result for $EH(\beta)$ can be stated. This statement is precisely given in the following manner.

If $E\eta_1 > 0$ and $E|\eta_1|^3 < \infty$ then

$$EH(\beta) = \frac{\beta}{\mu_1} + \frac{1}{2}\mu_{21} + o(\frac{1}{\beta}) \text{ as } \beta \to \infty, \tag{7.1}$$

(see for example, [17], p.210-211).

Let us now investigate the asymptotic behaviour of $EH^2(\beta)$ as $\beta \to \infty$:

Lemma 7.1 *If $E\eta_1 > 0$ and $E|\eta_1|^3 < \infty$, then the following asymptotic expansion is true as $\beta \to \infty$:*

$$EH^2(\beta) = \frac{\beta^2}{\mu_1^2} + (2\mu_{21} - 1)\ \frac{\beta}{\mu_1} + \frac{1}{6}[9\mu_{21}^2 - 3\mu_{21} - 4\mu_{31}] + o(1).$$

Proof. Put $F_n^+(t) = P\left\{ \sum\limits_{i=1}^{n} \chi_i^+ \leq t \right\}, n \geq 1, t \geq 0$ and $F_0^+(t) = 0$ if $t \leq 0$, 1 if $t > 0$. Then we can write

$$EH^2(\beta) = \sum_{n=1}^{\infty} n^2 (F_{n-1}^+(\beta) - F_n^+(\beta)). \tag{7.2}$$

Finally we can introduce the new notation

$$R(\beta) = EH^2(\beta) - \frac{\beta^2}{\mu_1^2} - (2\mu_{21} - 1)\frac{\beta}{\mu_1} \text{ and put } \varphi^+(\lambda) = E(\exp\ \{ -\lambda\chi_1^+ \}), \lambda > 0.$$

Let $\tilde{R}(\lambda)$ denote the Laplace transform of $R(t)$. With these notations, from (7.2), we have the following relationship

$$\tilde{R}(\lambda) = \frac{1 - \varphi^+(\lambda)}{\lambda} \sum_{n=1}^{\infty} n^2 (\varphi^+(\lambda))^{n-1} - \frac{2}{\lambda^3 \mu_1} - \frac{2\mu_{21} - 1}{\lambda^2 \mu_1}. \tag{7.3}$$

Note that if $|x| < 1$, then the next equality is valid

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{2}{(1-x)^3} - \frac{1}{(1-x)^2}.$$

By using this, we can rewrite (7.3) as

$$\tilde{R}(\lambda) = \frac{2}{\lambda(1-\varphi^{+}(\lambda))^2} - \frac{1}{\lambda(1-\varphi^{+}(\lambda))} - \frac{2}{\lambda^3\mu_1^2} - \frac{(2\mu_{21}-1)}{\lambda^2\mu_1}. \tag{7.4}$$

We will investigate the behaviour of $\tilde{R}(\lambda)$ as $\lambda \to 0$. For this purpose, let us investigate at first the behaviour of $\varphi^{+}(\lambda)$ as $\lambda \to 0$. It is known that under the condition $E(\chi_1^{+})^3 < \infty$ the following asymptotic expansion is true (see [19], p. 23) as $\lambda \to 0$:

$$1-\varphi^{+}(\lambda) = \lambda\mu_1\Big\{1 - \tfrac{\lambda}{2}\mu_{21}\mu_1 + \tfrac{\lambda^2}{6}\mu_{31}\mu_1^2 + o(\lambda^2)\Big\}.$$

Hence, as $\lambda \to 0$

$$(1-\varphi^{+}(\lambda))^{-1} = \frac{1}{\lambda\mu_1}\left\{1 + \frac{\lambda}{2}\mu_{21}\mu_1 + \frac{\lambda^2}{2}[\frac{1}{2}\mu_{21}^2\mu_1^2 - \frac{1}{3}\mu_{31}\mu_1^2] + o(\lambda^2)\right\}.$$

Therefore, from this we easily derive

$$(1-\varphi^{+}(\lambda))^{-2} = \frac{1}{(\lambda\mu_1)^2}\left\{1 + \lambda\mu_{21}\mu_1 + \lambda^2[\frac{3}{4}\mu_{21}^2\mu_1^2 - \frac{1}{3}\mu_{31}\mu_1^2] + o(\lambda^2)\right\}.$$

Substituting these relations in Formula (7.4), we finally get

$$\lambda\tilde{R}(\lambda) = \frac{3}{2}\mu_{21}^2 - \frac{1}{2}\mu_{21} - \frac{2}{3}\mu_{31} + o(1), \text{ as } \lambda \to 0. \tag{7.5}$$

Taking the limit in (7.5) we have

$$\lim_{\lambda\to 0}\lambda\tilde{R}(\lambda) = \frac{1}{6}[9\mu_{21}^2 - 3\mu_{21} - 4\mu_{31}]. \tag{7.6}$$

Applying the Tauber-Abelian theorems to (7.6) we obtain (see [7], p.442)

$$\lim_{\beta\to\infty} R(\beta) = \lim_{\lambda\to 0}\lambda\tilde{R}(\lambda) = \frac{1}{6}[9\mu_{21}^2 - 3\mu_{21} - 4\mu_{31}],$$

which is the same as

$$R(\beta) = \frac{1}{6}[9\mu_{21}^2 - 3\mu_{21} - 4\mu_{31}] + o(1), \text{ as } \beta \to \infty.$$

Therefore, under the conditions of Lemma 1, the following asymptotic expansion holds:

$$EH^2(\beta) = \frac{\beta^2}{\mu_1^2} + (2\mu_{21}-1)\frac{\beta}{\mu_1} + \frac{1}{6}[9\mu_{21}^2 - 3\mu_{21} - 4\mu_{31}] + o(1), \text{ as } \beta \to \infty.$$

This completes the proof of Lemma 1. □

We can now immediately state the next result connected with the variance of $H(\beta)$.

Lemma 7.2 *If $E(\chi_1^+)^3 < \infty$, then the following asymptotic expansion for variance of $H(\beta)$ is valid*

$$Var(H(\beta)) = \frac{\beta}{\mu_1}\sigma_{11}^2 + \frac{1}{12}[15\mu_{21}^2 - 6\mu_{21} - 8\mu_{31}] + o(1), \text{as } \beta \to \infty,$$

where $\sigma_{11}^2 = \dfrac{\sigma_1^2}{\mu_1^2} = \dfrac{Var(\chi_1^+)}{(E(\chi_1^+))^2}\cdot$

Proof. It is mentioned above that if the condition $E|\eta_1|^3 < \infty$ holds, then the following asymptotic expansion can be written

$$EH(\beta) = \frac{\beta}{\mu_1} + \frac{1}{2}\mu_{21} + o(\frac{1}{\beta}), \beta \to \infty.$$

Therefore, it is easily seen that as $\beta \to \infty$

$$(EH(\beta))^2 = \frac{\beta^2}{\mu_1^2} + \frac{\mu_{21}}{\mu_1}\beta + \frac{1}{4}\mu_{21}^2 + o(1).$$

Taking into account the results of Lemma 7.1, we obtain

$$Var(H(\beta)) = \frac{\beta}{\mu_1}\sigma_{11}^2 + \frac{1}{12}[15\mu_{21}^2 - 6\mu_{21} - 8\mu_{31}] + o(1) \text{ as } \beta \to \infty.$$

This completes the proof of Lemma 7.2. □

Let us give the following theorem which is the main aim of this section.

Theorem 7.1 *If $E\eta_1 > 0$ and $E|\eta_1|^3 < \infty$, then the asymptotic expansion for expectation of ergodic distribution of process $X(t)$ can be given as $\beta \to \infty$:*

$$EX = \frac{S+s}{2} + \frac{m_1}{4}\left[2 - \alpha_1 - 3\frac{\sigma_\eta^2}{m_1^2} - \frac{\sigma_\nu^2}{\alpha_1}\right] - \frac{\mu_1^2}{24\beta}\left[9\mu_{21}^2 - 8\mu_{31}\right] + o(\frac{1}{\beta}),$$

where $\sigma_\nu^2 = Var(\nu_1^+)$ and $\sigma_\eta^2 = Var(\eta_1)$.

Proof. By Theorem 6.1,

$$EX = S - \frac{M_2}{2M_1} + \frac{m_2}{2m_1}, \text{ where } M_k = E(Y_N^k), k = 1, 2. \tag{7.7}$$

Using Wald's identity it is possible to show that

$$E(Y_N) = E(\sum_{i=1}^{H(\beta)} \chi_i^+) = E(\chi_1^+)EH(\beta) = \mu_1 EH(\beta),$$

and

$$Var(Y_N) = \sigma_1^2 EH(\beta) + \mu_1^2 Var(H(\beta)), \text{ (see [5] or [8]).}$$

Therefore,

$E(Y_N^2) = \mu_1^2(EH(\beta))^2 + \sigma_1^2 EH(\beta) + \mu_1^2 Var(H(\beta)).$

Hence,

$$\frac{M_2}{M_1} = \mu_1 \left\{ EH(\beta) + \frac{Var(H(\beta))}{E(H(\beta))} + \sigma_{11}^2 \right\}. \tag{7.8}$$

Using Lemma 7.2 and asymptotic expansion (7.1) for $EH(\beta)$, we can write it as $\beta \to \infty$

$$\frac{Var(H(\beta))}{E(H(\beta))} = \sigma_{11}^2 + \frac{\mu_1}{12\beta}\{15\mu_{21}^2 - 6\mu_{21} - 8\mu_{31} - 6\mu_{21}\sigma_{11}^2\} + o(\frac{1}{\beta}).$$

Therefore, taking into account this equality from (7.8) we finally obtain

$$\frac{M_2}{2M_1} = \frac{\beta}{2} + \frac{1}{4}\mu_1\mu_{21} + \mu_1\sigma_{11}^2 + \frac{\mu_1^2}{24\beta}[15\mu_{21}^2 - 6\mu_{21} - 8\mu_{31} - 6\mu_{21}\sigma_{11}^2] + o(\frac{1}{\beta}). \tag{7.9}$$

By using formula (7.9), we immediately get from (7.7)

$$EX = \frac{S+s}{2} + \frac{m_1}{2} - \frac{\mu_1}{4} + \frac{\sigma_\eta^2}{2m_1} - \frac{5\sigma_1^2}{4\mu_1} - \frac{\mu_1^2}{24\beta}\{15\mu_{21}^2 - 6\mu_{21} - 8\mu_{31} - 6\mu_{21}\sigma_{11}^2\} + o(\frac{1}{\beta}).$$

Carrying out the corresponding calculations, we can rewrite this relation as

$$EX = \frac{S+s}{2} + \frac{m_1}{2} - \frac{\mu_1}{4} + \frac{\sigma_\eta^2}{2m_1} - \frac{5\sigma_1^2}{4\mu_1} - \frac{\mu_1^2}{24\beta}[9\mu_{21}^2 - 8\mu_{31}] + o(\frac{1}{\beta}). \tag{7.10}$$

Note that the following is well known (see [5], [7])

$$\sigma_1^2 = Var(\chi_1^+) = Var(\sum_{i=1}^{\nu_1^+} \eta_i) = E\nu_1^+ Var(\eta_1) + (E\eta_1)^2 Var(\nu_1^+) =$$

$= \alpha_1\sigma_\eta^2 + m_1^2\sigma_\nu^2$ and $\mu_1 = E(\chi_1^+) = m_1\alpha_1$.

By using these equality, from (7.10) we finally have, as $\beta \to \infty$

$$EX = \frac{S+s}{2} + \frac{m_1}{4}\left[2 - \alpha_1 - 3\frac{\sigma_\eta^2}{m_1^2} - \frac{\sigma_\nu^2}{\alpha_1}\right] - \frac{\mu_1^2}{24\beta}\left[9\mu_{21}^2 - 8\mu_{31}\right] + o(\frac{1}{\beta}). \tag{7.11}$$

This completes the proof of Theorem 7.1. □

8. Asymptotic Expansion for the Variance of the Ergodic Distribution of the Process $X(t)$

In this section we will give some asymptotic results connected with the second moment of the process X(t). For this aim let us put

$$\mu_{k1} = \frac{\mu_k}{(\mu_1)^k}, \quad \tilde{m}_{k1} = \frac{m_k}{m_1^k}, k = 2, 3; \text{ and } \quad \sigma_1^2 = Var(\chi_1^+), \sigma_{11}^2 = \frac{\sigma_1^2}{\mu_1^2}$$

and state the following result.

Theorem 8.1 *Let the conditions of proposition 3.1 and supplementary condition* $E|\eta_1|^3 < \infty$ *be satisfied.*

Then the variance of ergodic distribution of the process X(t) has the following asymptotic expansion as $\beta \to \infty$*:*

$$Var(X) = \frac{\beta^2}{12} + \frac{13\mu_2}{12\mu_1}\beta + \frac{1}{48}[c_1\mu_1^2 + 4c_2m_1^2] + o(1),$$

where $c_1 = 91\sigma_1^4 + 134\sigma_{11}^2 - 16\mu_{31} - 7$ *and* $c_2 = 3\tilde{m}_{21}^2 - 4\tilde{m}_{31}$.

Proof. In section 6, we have shown that

$$Var(X) = \frac{1}{3}[\frac{M_3}{M_1} - \frac{m_3}{m_1}] - \frac{1}{4}[\frac{M_2^2}{M_1^2} - \frac{m_2^2}{m_1^2}]. \tag{8.1}$$

By using Wald's identity it is not difficult to see that (see [5])

$$M_1 = E(Y_N) = E(\sum_{i=1}^{H(\beta)} \chi_i^+) = E\chi_1^+ EH(\beta) = \mu_1 EH(\beta), \tag{8.2}$$

$$M_2 = E(Y_N^2) = \mu_1^2 EH^2(\beta) + (\mu_2 - \mu_1^2)EH(\beta), \tag{8.3}$$

$$M_3 = E(Y_N^3) = \mu_1^3 EH^3(\beta) + 3\mu_1(\mu_2 - \mu_1^2)EH^2(\beta)+$$
$$(\mu_3 - 3\mu_1\mu_2 + 2\mu_1^3)EH(\beta). \tag{8.4}$$

Note that under the conditions of this Theorem 8.1, the first three moments of random variable χ_1^+ and renewal process $H(\beta)$ exist. Therefore, the formulas (8.2), (8.3) and (8.4) can be written.

Using the formulas (8.2), (8.3) and (8.4) we obtain

$$\frac{M_2}{2M_1} = \frac{\mu_1}{2}\frac{EH^2(\beta)}{EH(\beta)} + \frac{\mu_2 - \mu_1^2}{2\mu_1}, \tag{8.5}$$

and

$$\frac{M_3}{3M_1}=\frac{\mu_1^2}{3}\frac{EH^3(\beta)}{EH(\beta)}+(\mu_2-\mu_1^2)\frac{EH^2(\beta)}{EH(\beta)}+[\frac{\mu_3}{3\mu_1}-\mu_2+\frac{2}{3}\mu_1^2]. \tag{8.6}$$

On the other hand, we can rewrite the formula (8.1) as follows

$$Var(X)=\frac{M_3}{3M_1}-(\frac{M_2}{2M_1})^2+(\frac{m_2}{2m_1})^2-\frac{m_3}{3m_1}. \tag{8.7}$$

Taking into account the expressions (8.5) and (8.6) we have from (8.7)

$$Var(X)=\frac{\mu_1^2}{3}\frac{EH^3(\beta)}{EH(\beta)}-\frac{\mu_1^2}{4}(EH(\beta))^2+\frac{\mu_1-\mu_1^2}{2}EH(\beta)-\frac{\mu_1^2}{2}Var(H(\beta))+$$

$$+\frac{\mu_2-\mu_1^2}{2}\frac{Var(H(\beta))}{EH(\beta)}-\frac{\mu_1^2}{4}[\frac{Var(H(\beta))}{EH(\beta)}]^2+\frac{\mu_3}{3\mu 1}-\frac{m_3}{3m_1}-(\frac{\mu_2}{2\mu_1})^2$$

$$+(\frac{m_2}{2m_1})^2-\frac{\mu_2}{2}+\frac{5}{12}\mu_1^2. \tag{8.8}$$

Using the Tauberian and Abelian theorems (see [8], p.442), it is possible to get as $\beta\to\infty$,

$$EH(\beta)=\frac{\beta}{\mu_1}+\frac{\mu_{21}}{2}+o(\frac{1}{\beta}), \tag{8.9}$$

$$Var(H(\beta))=\frac{\beta}{\mu_1}(\mu_{21}-1)+A+o(1), \tag{8.10}$$

$$EH^3(\beta)=(\frac{\beta}{\mu_1})^3+(\frac{\beta}{\mu_1})^2+\frac{\beta}{\mu_1}B+o(\beta), \tag{8.11}$$

where $A=\frac{1}{12}[15\mu_{21}^2-6\mu_{21}-8\mu_{31}]$, $B=9\mu_{21}^2-3\mu_{31}-5$.

Carrying out the corresponding calculations, from (8.9), (8.10) and (8.11) we can show as $\beta\to\infty$

$$\frac{Var(H(\beta))}{EH(\beta)}=(\mu_{21}-1)+O(\frac{1}{\beta})=(\mu_{21}-1)+o(1), \tag{8.12}$$

and

$$\frac{EH^3(\beta)}{EH(\beta)}=(\frac{\beta}{\mu_1})^2+4\frac{\beta}{\mu_1}\mu_{21}+(7\mu_{21}^2-3\mu_{31}-5)+o(1). \tag{8.13}$$

Taking into account these expansions ((8.9)-(8.13)) in Formula (8.8) we obtain the following asymptotic expansion for $Var(X)$ as $\beta\to\infty$:

$$Var(X)=\frac{\beta^2}{12}+\frac{13}{12}\frac{\mu_2}{\mu_1}\beta+$$

$$+\frac{1}{12}\left\{91(\frac{\mu_2}{2\mu_1})^2-4\ \frac{\mu_3}{\mu_1}-12\mu_2-12\mu_1^2+3(\frac{m_2}{m_1})^4-4\frac{m_3}{m_1}\right\}+o(1). \tag{8.14}$$

With the notations

$c_1 = 91\sigma_{11}^4 + 134\sigma_{11}^2 - 16\mu_{31} - 7, \quad c_2 = 3\tilde{m}_{21}^2 - 4\tilde{m}_{31},$

we can rewrite the formula (8.14) as

$$Var(X) = \frac{\beta^2}{12} + \frac{13\mu_2}{12\mu_1}\beta + \frac{1}{48}[c_1\mu_1^2 + 4c_2 m_1^2] + o(1).$$

This completes the proof of Theorem 8.1. □

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