

## Comparing open book and Heegaard decompositions of 3-manifolds

*J. Hyam Rubinstein*

### Abstract

We study the maximal value of the Euler characteristic of the pages of all open book decompositions of closed orientable 3-manifolds. In particular, we describe some examples where the minimal genus Heegaard splittings of such 3-manifolds give rise to open book decompositions and other examples where the simplest open book decomposition has larger maximal Euler characteristic of pages than the smallest genus Heegaard splittings. Also, special properties of the Heegaard splitting associated to an open book decomposition are given. Techniques of minimal surface theory and hyperbolic geometry are shown to be useful for such problems.

### 1. Introduction

It is well-known that open book decompositions of 3-manifolds give a natural way of constructing contact structures. This was first observed by Thurston and Winkelnkemper [18]. This correspondence and its converse (going from contact structures to open book decompositions) has been studied recently by Giroux [4]. Some useful basic references on the very active area of contact structures are [2], [3]. On the other hand, open book decompositions were constructed by Alexander [1] but have not attracted much attention in recent years by 3-manifold topologists. (See [10] for a neat way of constructing open book decompositions with connected binding). However, many interesting results have been proven about Heegaard splittings (cf [16]). Our aim in this paper is to sketch some comparisons between open book decompositions and Heegaard splittings, noting that open book decompositions can be viewed as special types of Heegaard splittings. This paper is closely related to [14] (see also [6]) and a more detailed version of applications of minimal surfaces in the study of 3-manifolds is in preparation by the author.

In particular, it would be good to understand how the complexity (maximal Euler characteristic) of open book decompositions of closed orientable 3-manifolds compare with other natural measures of complexity, such as minimal spines, (cf [7], [8]) smallest triangulations ([5]) and minimal Heegaard splittings ([16]).

Our first result is that in the Heegaard genus 2 case, open book decompositions are nearly always more complicated than minimal Heegaard splittings. A natural measure of the complexity of an open book decomposition is to take the Heegaard genus of the

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associated splitting, obtained by thickening a page. So this is  $|\chi| + 1$ , where  $\chi$  is the Euler characteristic of a page. We will call this the *genus* of the open book decomposition. Then the open book genus of  $M$  will be its minimal open book genus. This is essentially the same as the maximal Euler characteristic but is more convenient for our discussions. Note that we will not require the bindings of our open book decompositions to be connected.

Our second result is that in the Dehn surgery space of a knot (or link), by a similar technique to [9], since the Heegaard genus of the surgered manifold remains bounded as the surgery coefficients become unbounded, any low genus open book decompositions come from Heegaard splittings of the complement of the knot or link. So this indicates that either Heegaard genus and open book genus diverge or the smallest genus open book decompositions can be well understood.

Given a triangulation with  $n$  tetrahedra, one might expect to find an open book decomposition of genus ‘similar’ to  $n$ , noting that a triangulation is analogous to a Heegaard diagram. This needs investigation, as does the issue of finding a good algorithmic method of constructing open book decompositions, starting with a triangulation or a Heegaard splitting. Almost normal surface theory gives such a connection between minimal genus Heegaard splittings and triangulations (cf [12],[13], [17]).

We will assume that all 3-manifolds are irreducible, orientable and either closed or the interiors of compact manifolds with tori boundary.

We begin with two basic definitions.

**Definition**

A *Heegaard splitting* of a closed orientable 3-manifold  $M$  is a decomposition of  $M$  into a union of two standard 3-dimensional handlebodies of the same genus (number of handles), by a homeomorphism gluing the two boundaries together.

Similarly, a Heegaard splitting of a compact orientable 3-manifold with tori boundary components, is a decomposition into two compression bodies, which are the result of attaching one handles to a product of a surface and an interval, where all the handles are attached to one boundary surface.

**Definition**

An *open book decomposition* of a closed orientable 3-manifold  $M$  consists of a *binding*  $\Sigma$ , which is a finite collection of disjoint simple closed curves, so that the complement  $M \setminus \Sigma$  is a fiber bundle over a circle. The fibers all have boundary  $\Sigma$  with multiplicity one. We will call the *pages* of the decomposition, the fibers together with their boundaries  $\Sigma$ .

Similarly, an open book decomposition of a compact orientable 3-manifold with tori boundary components has a binding so that the complement is a fiber bundle over a circle and all the fibers have boundary equal to the binding plus essential curves on each boundary torus.

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## 2. Genus 2 Heegaard splittings

**Theorem 2.1.** *Most 3-manifolds  $M$  of Heegaard genus 2 have open book genus  $> 2$ .*

*Proof.* Notice that the canonical involution  $g$ , associated with such a genus 2 Heegaard splitting of  $M$ , has branch set a 3-bridge knot or link  $\Gamma$  in  $S^3$ . In fact,  $g$  maps each handlebody of the Heegaard splitting to itself, with fixed set given by three unknotted arcs. The union of these six arcs projects to  $\Gamma$  in  $S^3$ , since  $M/g = S^3$ , where each handlebody projects to a 3-ball. (See [15] for more information about this involution). If  $M$  has an open book decomposition of genus 2, then it is easy to show that the pages are invariant under another canonical involution  $g'$ , with branch set a closed 3-braid  $\Gamma'$  in  $S^3$ . This latter involution maps each page to itself, with 3 fixed points. Note that the pages are once punctured tori. (If the pages were 3 punctured spheres, it is easy to check that either  $M$  is Seifert fibered or  $M$  is a connected sum of lens spaces or copies of  $S^1 \times S^2$ . So we do not need to further consider this case.) Now in [15], it is shown that these two involutions can be made to commute or coincide, after a possible conjugation of one of the involutions by a homeomorphism of  $M$  to itself. So either  $M$  is invariant under the action of  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$  or else the two involutions are equivalent (conjugate under a homeomorphism of  $M$ ). In the latter case, the open book decomposition is also invariant under  $g$ , after applying a homeomorphism of  $M$  to it (which has the same effect as conjugating  $g$  by the homeomorphism).

In the first case, it follows that  $M$  is a 4-fold branched cover over a 2 component link  $\Gamma^*$  or a  $\theta$  graph  $\Theta$ , by [15]. Moreover each component of  $\Gamma^*$  is unknotted, or in the case of a  $\theta$  graph, forming a loop from two of the three arcs of  $\Theta$  gives an unknotted curve, for at least two of the three choices. Constructing a 2-fold branched cover over one of these unknotted curves lifts the other curve (respectively arc) to the 3 braid  $\Gamma^*$  or the 3 bridge knot or link  $\Gamma$ . In particular  $\Gamma$  is either periodic of period 2 or strongly invertible. By this we mean that there is a standard involution  $\sigma$  acting on  $S^3$  with an unknotted circle  $C$  of fixed points, which maps  $\Gamma$  to itself and either  $C \cap \Gamma$  is empty or has two points respectively.

In the second case, it follows that  $\Gamma$  has the structure of a closed 3-braid. and so is again strongly invertible. However 3-bridge knots and links are rarely either periodic or strongly invertible. Moreover very few 3 bridge knots and links are closed 3-braids and so the result is proved.  $\square$

## 3. Open book decompositions and minimal Heegaard splittings

Let  $M$  be an orientable complete hyperbolic 3-manifold with finite volume and  $n$  cusps. So for example,  $M$  could be the complement of a simple knot or link in  $S^3$ . Consider all Dehn surgeries on the cusps of  $M$ . We will denote by  $M_{(p_1, q_1), \dots, (p_n, q_n)}$  the result of doing Dehn filling along curves of slopes  $(p_1, q_1), \dots, (p_n, q_n)$  at the cusps, where as usual the second coordinate is the multiple of the longitude and the first the multiple of the meridian in the case of a knot or link in  $S^3$ .

**Theorem 3.1.** *Open book decompositions of the surgered manifolds  $M_{(p_1^k, q_1^k), \dots, (p_n^k, q_n^k)}$  either have unbounded genus or have associated Heegaard splittings constructed by adding trivial handles to one of a finite number of splittings of the cusped manifold  $M$ , as  $\lim_k |p_i^k| + |q_i^k| \rightarrow \infty$ , for all  $i$ , where any sequence of vectors  $V_k$  of Dehn surgery coefficients  $V_k = (p_1^k, q_1^k), \dots, (p_n^k, q_n^k)$  is chosen, which avoids finitely many lower dimensional lattices in the lattice  $\mathbb{Z}^{2n}$ .*

*Proof.* We sketch the proof. The idea is that we can use Gromov-Thurston negatively curved Dehn surgery and convergence of minimal surfaces as in [9], assuming that the open book genus of some sequence of surgered manifolds remains bounded. The basic construction is to remove maximal horotori neighbourhoods of the cusps and replace these by solid tori with appropriate slopes of the meridian disks with negatively curved metrics. Then all the manifolds in the sequence of Dehn surgeries have fixed metrics outside these solid tori. Now we can argue that if there is an open book decomposition of bounded genus in such a manifold, then there is a bounded genus irreducible or strongly irreducible Heegaard splitting. (We just take a regular neighbourhood of a page as one of the handlebodies of a Heegaard splitting and then reduce this by removing trivial handles. An irreducible Heegaard splitting has no trivial handles and if the manifold is non-Haken, such a splitting is strongly irreducible by a classical result of Casson-Gordon. Strongly irreducible splittings have the property that every compressing disk for one handlebody meets every compressing disk for the other handlebody).

Next, by [11] these Heegaard splittings can be realised by minimal surfaces (in the strongly irreducible case) or their pieces obtained by telescoping (in the irreducible case) can be isotoped to minimal surfaces, assuming the first homology group  $H_1(M, \mathbb{Z}_2) = 0$ . (See for example [16] for a description of the process of telescoping of Scharlemann-Thompson). The monotonicity formula for minimal surfaces (cf. [9]) then establishes that these minimal surfaces will miss smaller (but fixed size) negatively curved solid tori, for large enough Dehn surgery on each cusp. It is well-known that minimal surfaces of bounded genus form a compact set in a negatively curved 3-manifold. So we see there are finitely many such minimal surfaces outside the smaller solid tori neighbourhoods, up to isotopy. If  $H_1(M, \mathbb{Z}_2) \neq 0$ , we can have Heegaard surfaces collapsing onto non orientable surfaces with multiplicity 2. In this case, by noting that such non orientable surfaces can be chosen to have minimal genus, the non orientable surfaces can also be isotoped to minimal surfaces and a similar argument to the previous one applies. Therefore, by passing to a subsequence, we may suppose that all the Heegaard surfaces are in the same isotopy class in  $M$ , using compactness of bounded genus minimal surfaces.

To summarise, the relationship between the original bounded genus open book decompositions and the Heegaard splittings is that if a page is thickened to a handlebody, then removing some trivial handles gives the irreducible or strongly irreducible splittings of the above argument. Since these are all isotopic in  $M$ , the same is true for the Heegaard splittings corresponding to the open book decompositions. Next, we claim that the Heegaard splittings in the sequence of Dehn filled manifolds must induce a Heegaard splitting of the cusped manifold  $M$ , by [9].

The explanation is that since we are avoiding finitely many ‘bad’ sublattices of surgeries, the surgery curves (i.e. the cores of the added solid tori) must be cores of the Heegaard handlebodies. Note in [9], one possibility is that the surgery curves are parallel onto the Heegaard surface along unique annuli and only longitudinal Dehn filling relative to this choice of longitude on the surgery curve is allowable. This is how bad sublattices of surgery curves can arise, which must be excluded. The other possibility is that there can be finite numbers of surgeries to be excluded at each cusp, which also yield lower dimensional bad sublattices. Therefore the same is true for the associated Heegaard splittings to the open book decompositions, since these are formed by stabilisation, i.e. adding trivial handles. This works even in the more complicated case where the Heegaard splittings are only irreducible but not strongly irreducible and so telescope, i.e. decompose into incompressible surfaces and strongly irreducible compression body decompositions of the pieces (see [16]). Consequently, by passing to a subsequence of the Dehn surgeries, we can assume that the induced Heegaard splittings corresponding to the open book decompositions all come from the same splitting of the cusped manifold  $M$ .

The conclusion is that after excluding certain sublattices of surgeries, any sequence of open book decompositions of bounded genus in the family of Dehn surgered manifolds have associated Heegaard splittings which are obtained by adding trivial handles (this process is often called stabilisation) to a Heegaard splitting of bounded genus of  $M$  and there are finitely many such splittings up to isotopy.  $\square$

#### Remark

Note that the main result of [9] is that with exactly the same hypotheses as for the above theorem, that the genus of minimal Heegaard splittings of the surgered manifolds remains bounded. In fact, all the Heegaard splittings arise from a finite collection of Heegaard splittings of the cusped manifold  $M$ .

We now prove an ‘opposite’ result to Theorem 1, where the Heegaard and open book genus are the same for a large class of manifolds.

**Theorem 3.2.** *Consider the sequence of open book decompositions given by the  $n$ -fold cyclic covering  $M_n$  of  $(1, n)$ -surgery on a simple knot or link which is the binding of an open book decomposition of an irreducible atoroidal manifold  $M$ . For  $n$  sufficiently large, the minimal Heegaard genus of  $M_n$  is the open book genus. In fact, the only small genus irreducible Heegaard splitting of such a manifold is a regular neighbourhood of the page of the open book decomposition.*

*Proof.* The assumption that the knot or link is simple means that the complement of the knot or link is atoroidal, i.e. has no  $\pi_1$ -injective embedded tori or annuli which are not boundary parallel. This allows us to note that the complement of the binding in  $M$  is a hyperbolic 3-manifold, as is also sufficiently large  $(1, n)$ -Dehn surgery on this binding, by well-known results of Thurston.

Notice that  $(1, n)$ -surgery for  $n$  large, gives a manifold with large tubes about short geodesics representing the binding. As  $n \rightarrow \infty$ , the resulting hyperbolic 3-manifolds are

Gromov-Hausdorff converging to the complete hyperbolic cusped manifold  $M$  formed by removing the knot or link binding of the open book structure. The geometry outside the large tubes only varies by a small amount for  $n$  large varying. In particular, the injectivity radius of loops which are not homotopic into the cusps, is bounded from below in the whole sequence of these Dehn surgeries.

Taking then the  $n$ -fold cyclic cover gives a sequence  $M_n$  of manifolds, which is Gromov-Hausdorff converging to the infinite cyclic cover of the complete hyperbolic cusped manifold  $M$ , i.e. is topologically the product of a page and  $\mathbf{R}$ . Hence we can use the bundle argument of [14] to argue that any strongly irreducible or telescoped irreducible Heegaard surface for  $M_n$ , when isotoped to be minimal, will have large genus unless it is the boundary of a regular neighbourhood of a page. Notice first that any bounded genus minimal surface must have bounded area and so cannot intersect large tubes around the binding geodesics of  $M_n$ , by an easy area estimate. Next, any minimal surface which is disjoint from these large tubes but is not the boundary of a regular neighbourhood of a page, can be shown to have area which is too large, contradicting Gauss Bonnet, if the surface has bounded genus. (Any loop which goes around the open book structure, i.e. is transverse to the pages in  $M_n$  and is outside large tubes about the binding, must be very long assuming that  $n$  is large. So if there is such a loop in a minimal surface, it is easy to prove by the coarea formula that the area is large - see [14] for more details). In particular, this establishes that there are no other irreducible Heegaard splittings of low genus in such an open book decomposition.  $\square$

#### 4. Conclusion

##### Remarks

It would be useful to have an algorithm to determine the open book genus, similar to that to find the Heegaard genus ([12], [13], [17]).

Other significant things to investigate are:

- are there concepts of irreducibility or strong irreducibility for open book decompositions, similarly is there an analogue of the Casson-Gordon result that if an irreducible Heegaard splitting is not strongly irreducible and the manifold is not  $S^3$  with a genus one splitting, then it has an embedded incompressible surface? Is there some type of telescoping of open book decompositions, i.e a splitting along incompressible surfaces?
- for Seifert fibred spaces, can we classify minimal open book decompositions as has been largely done for Heegaard splittings [16]?
- what can be said about the number of components of the binding of the book and how this affects the minimal genus of a page?
- for lens spaces, is it true that minimal open book decompositions are invariant under the canonical involution - is the same true for Seifert fibred spaces? If so, determining the minimal braid number of the branch sets, i.e. the 2-bridge and Montesinos knots and links, will then give the minimal open book decompositions.

Finally we observe some other connections between Heegaard splittings and open book decompositions.

**Theorem 4.1.** *Suppose that  $M$  is a closed orientable 3-manifold and  $S$  is a Heegaard surface for some Heegaard splitting of  $M$ . If  $S$  is associated with an open book decomposition of  $M$ , then  $S$  satisfies the following two conditions;*

- *there is an isotopy of  $M$  taking  $S$  back to itself which interchanges the two handlebodies on either side of  $S$ .*

- *let  $G$  be the subgroup of  $\pi_0(\text{Homeo}(S))$  consisting of isotopy classes of homeomorphisms of  $S$  extending to homeomorphisms of  $M$ . Then the mapping from  $G$  into  $\pi_0(\text{Homeo}(M))$  has a kernel  $K$  of infinite order, unless  $M$  is a Seifert fibered space or a connected sum of lens spaces and copies of  $S^1 \times S^2$ .*

*Proof.* By rotating the pages of an open book decomposition around the fiber bundle structure, one can obtain an isotopy as in the first condition. In fact, since  $S$  consists of two pages of an open book decomposition, with common boundary the binding, such a rotation can be chosen to interchange these two pages. Therefore the two handlebodies on either side of  $S$  are switched also.

For the second condition, notice that if one iterates the isotopy from the first condition, the result is either an infinite family of homeomorphisms which have isotopy classes in the kernel  $K$ , or else some power of the monodromy of the fiber bundle structure is the identity homeomorphism. In the latter case, it is easy to see that the fiber bundle is a Seifert fibered space and filling in the binding curves extends this to a Seifert fibered structure on  $M$ , unless the filling kills the Seifert fibers, in which case  $M$  is a connected sum of lens spaces and copies of  $S^1 \times S^2$ . □

### Question

Is it the case that if a Heegaard splitting surface  $S$  satisfies the conditions in Theorem 4, then there is a binding lying on  $S$  and an open book decomposition so that  $S$  consists of two pages?

Suppose one had a positive answer to this question. Then, given a strongly irreducible Heegaard splitting  $S$  of a closed orientable 3-manifold  $M$  with no open book decomposition associated to  $S$ , if one also knew that  $M$  had finitely many isotopy classes of Heegaard splitting surfaces with the same genus as  $S$ , then finiteness of  $G$  in Theorem 4 would be equivalent to finiteness of  $\pi_0(\text{Homeo}(M))$ . In [13], it is shown that irreducible atoroidal manifolds  $M$  do satisfy such a finiteness condition for Heegaard splitting surfaces.

Irreducibility is the condition that every embedded 2-sphere bounds a 3-ball and the atoroidal condition is that any immersed torus  $T$  in  $M$  satisfies  $\pi_1(T) \rightarrow \pi_1(M)$  has non trivial kernel.

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DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF MELBOURNE, PARKVILLE, AUSTRALIA  
3010

*E-mail address:* rubin@ms.unimelb.edu.au