KO-groups of Bounded Flag Manifolds

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Abstract

We exhibit an appropriate suspension of bounded flag manifolds as a wedge sum of Thom complexes of associated complex line bundles. We use the existence of such a splitting to assist our computation of real and complex $K$-groups. Moreover, we compute the $Sq^2$-homology of bounded flag manifolds to make use of relevant Atiyah-Hirzebruch spectral sequence of $KO$-theory.

Key Words: Bounded flag manifolds, $Sq^2$-homology, $KO$-theory, toric variety, stably complex structure

1. Introduction

As explained by Buchstaber and Ray [2], the geometry of bounded flag manifolds plays an important role in complex cobordism, namely that they generate the double cobordism ring $\Omega^*_{DU}$. These objects were originally constructed by Bott and Samelson, and were introduced into complex cobordism by Ray [7].

Bounded flag manifolds also fit into the settings of toric geometry. We showed in [3] that they are smooth projective toric varieties associated to fans arising from crosspolytopes.

By analogy with many stable splitting phenomena discovered in the 80s, we will carry out a programme of exhibiting an appropriate suspension of bounded flag manifolds as a wedge sum of Thom complexes of associated complex line bundles. We then use the existence of such a splitting to assist our computation of real and complex $K$-groups. More generally, Bahri and Bendersky[1] have announced a method for computing $KO$-groups of any toric manifold via the relevant Adam spectral sequence. Our first step overlaps with theirs in that we compute the $Sq^2$-homology of bounded flag manifolds.

We begin with introducing some notations. We follow combinatorial convention by writing $[n]$ for the set of natural numbers $\{1, 2, \ldots, n\}$, and an interval in the poset $[n]$
has the form \([a, b]\) for some \(1 \leq a \leq b \leq n\) which consists of all \(k\) satisfying \(a \leq k \leq b\).

Throughout, \(\omega_1, \ldots, \omega_{n+1}\) will denote the standard basis vectors in \(\mathbb{C}^{n+1}\), and we write \(\mathbb{C}_I\) and \(\mathbb{C}P_I\) for the subspace spanned by the vectors \(\{\omega_i: i \in I\}\) and the projectivization of \(\mathbb{C}_I\) respectively, where \(I \subset [n+1]\).

**Definition 1.1** A flag \(U: 0 < U_1 < \ldots < U_n < \mathbb{C}^{n+1}\) is called bounded if \(\mathbb{C}_{[i-1]} < U_i\) for each \(1 \leq i \leq n\). The set of all bounded flags in \(\mathbb{C}^{n+1}\) is called bounded flag manifold, which is an \(n\)-dimensional smooth complex manifold and will be denoted by \(B(\mathbb{C}^{n+1})\) (or simply by \(B_n\)).

As a consequence of the definition, each factor \(U_i\) of any bounded flag \(U \in B(\mathbb{C}^{n+1})\) is of the form \(\mathbb{C}_{[i-1]} \oplus L_i\), where \(L_i\) is a line in \(\mathbb{C}_i \oplus L_{i+1}\) for \(1 \leq i \leq n\), and \(L_{n+1} = \mathbb{C}_{n+1}\). Therefore, we may display \(U\) as

\[
U: 0 < L_1 < \mathbb{C}_1 \oplus L_2 < \ldots < \mathbb{C}_{[n-1]} \oplus L_n < \mathbb{C}^{n+1}.
\] (1.2)

We define maps \(q_i\) and \(r_i: B(\mathbb{C}^{n+1}) \to \mathbb{C}P_{[i,n+1]}\) by letting \(q_i(U) = L_i\) and \(r_i(U) = L_i^+\), where \(L_i^+\) is the orthogonal complement of \(L_i\) in \(\mathbb{C}_i \oplus L_{i+1}\) for each \(U \in B(\mathbb{C}^{n+1})\), and \(1 \leq i \leq n\).

If \(B(\mathbb{C}_{[n-k+1,n+1]}\) denotes the set bounded flags in \(\mathbb{C}_{[n-k+1,n+1]}\), which we abbreviate to \(B_k\), then there is a sequence of projections

\[
B_n \xrightarrow{\pi_n-1} B_{n-1} \xrightarrow{\pi_{n-2}} \ldots \xrightarrow{\pi_2} B_2 \xrightarrow{\pi_1} B_1 \xrightarrow{\pi_0} \ast,
\]

each of which is the projection of a fiber bundle whose fibers isomorphic to \(\mathbb{C}P^1\) and can be given for any \(0 \leq k \leq n-1\) as follows: \(\pi_k: B_k \to B_{k-1}\) maps each flag

\[
U_k: 0 < L_{n-k+1} < \mathbb{C}_{n-k+1} \oplus L_{n-k+2} < \ldots < \mathbb{C}_{[n-k+1,n-1]} \oplus L_n < \mathbb{C}_{[n-k+1,n+1]}\]

in \(B_k\) to the flag

\[
U_{k-1}: 0 < L_{n-k+2} < \mathbb{C}_{n-k+2} \oplus L_{n-k+3} < \ldots < \mathbb{C}_{[n-k+2,n-1]} \oplus L_n < \mathbb{C}_{[n-k+2,n+1]}\]

in \(B_{k-1}\). There are two inclusions \(i^S_k\) and \(i^N_k: B_{k-1} \to B_k\), which are given respectively by

\[
i^S_k(U_{k-1}): 0 < \mathbb{C}_{n-k+1} < \mathbb{C}_{n-k+1} \oplus L_{n-k+2} < \ldots < \mathbb{C}_{[n-k+1,n-1]} \oplus L_n < \mathbb{C}_{[n-k+1,n+1]},
\]

\[
i^N_k(U_{k-1}): 0 < \mathbb{C}_{n-k+1} < \mathbb{C}_{n-k+1} \oplus L_{n-k+2} < \ldots < \mathbb{C}_{[n-k+1,n-1]} \oplus L_n < \mathbb{C}_{[n-k+1,n+1]},
\]
and

\[ i_k^n(U_{k-1}) : = 0 < L_{n-k+2} < C_{n-k+1} \oplus L_{n-k+2} < \ldots < C_{[n-k+1,n-1]} \oplus L_n < C_{[n-k+1,n+1]} \].

We consider complex line bundles \( \eta_i \) and \( \eta_i^1 \) over \( B_n \), classified respectively by the maps \( q_{n-i+1} \) and \( r_{n-i+1} \) for every \( 1 \leq i \leq n \), and we set \( \eta_0 \) to be the trivial line bundle with fiber \( \mathbb{C}_{n+1} \). We sometimes refer to them as the associated line bundles on \( B_n \). It follows that

\[ \eta_i \oplus \eta_i^1 \oplus \eta_i^{i-1} \oplus \ldots \oplus \eta_1^1 \cong C_{[n-i,n+1]} \tag{1.3} \]

for every \( i \). As detailed in [7], there is an isomorphism

\[ \tau(B_n) \oplus \mathbb{R}^2 \cong \bigoplus_{i=0}^{n-1} \eta_i \oplus \mathbb{C}, \tag{1.4} \]

giving a stable complex structure on \( B_n \). However, each \( B_n \) can be identified with the total space of the sphere bundle of \( \eta_{n-1} \oplus \mathbb{R} \) over \( B_{n-1} \), and the above \( U \)-structure extends over the associated 3-disk bundle; hence, \( B_n \) represents zero in the complex cobordism ring \( \Omega^U \).

We let \( x_1, \ldots, x_n \in H^2(B_n; \mathbb{Z}) \) denote the respective first Chern classes of \( \eta_1, \ldots, \eta_n \).

**Theorem 1.5** [2] The integral cohomology ring \( H^*(B_n) \) is generated by \( x_1, \ldots, x_n \), and these are subject only to the relations \( x_1^2 = 0 \) and \( x_i^2 = x_{i-1}x_i \) for each \( 2 \leq i \leq n \) and for all \( n > 0 \).

### 2. Stable Splitting and \( Sq^2 \)-Homology of Bounded Flags

Let \( \xi = \{ E, p, B, \mathbb{C}^n \} \) be an \( n \)-dimensional complex vector bundle over a CW-complex \( B \). We let \( D(\xi) \) denote the associated disk bundle consisting of vectors of length at most 1 in each fiber, while \( S(\xi) \) denotes the associated sphere bundle. We then set

\[ T\xi := D(\xi)/S(\xi). \]

The space \( T\xi \) is called the Thom space or the Thom complex of \( \xi \) (see [9]). Alternatively, the Thom complex \( T\xi \) can be constructed as

\[ T\xi \cong CP(\mathbb{C} \oplus \xi)/CP(\xi), \tag{2.6} \]

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where $\mathbb{C}P(\xi)$ is the space obtained by projectivizing each fiber. In particular, if $\xi$ is a line bundle, it then follows that $\mathbb{C}P(\xi)$ is homeomorphic to the base space $B$ so that $T\xi \cong \mathbb{C}P(\mathbb{C} \oplus \xi)/\mathbb{C}$. Furthermore, the Thom class of $\xi$ is defined to be the element (up to sign) $t \in H^n(D(\xi), S(\xi))$ such that $j^*(t)$ is a generator of $H^n(D^n, S^{n-1})$, where $j: (D^n, S^{n-1}) \to (D(\xi), S(\xi))$ is the inclusion of the fiber over some point. In this way, we obtain the Thom isomorphism

$$\Phi^*: H^i(B) \to H^{i+n}(D(\xi), S(\xi)), \quad \Phi^*(z) := p^*(z) \cup t \quad \text{for all } i \in \mathbb{Z}. $$

In the case of bounded flag manifolds, it follows from (2.6) that the Thom complex $T_{\eta_k-1}$ of each $\eta_{k-1}$ is of the form

$$T_{\eta_k-1} \cong \mathbb{C}P(\mathbb{C} \oplus \eta_{k-1})/\mathbb{C}P(\eta_{k-1}) \cong B_k/B_{k-1}. \quad (2.7)$$

Therefore, there is a cofibre sequence

$$B_{k-1} \xrightarrow{i_k} B_k \xrightarrow{q_k} T_{\eta_k-1}, \quad (2.8)$$

where $i_k: B_{k-1} \to B_k$ is either of the inclusions $i^S_k$ or $i^N_k$, and $q_k: B_k \to T_{\eta_k-1}$ is the quotient map. We then have a short exact sequence:

$$0 \to H^{2j}(T_{\eta_k-1}) \xrightarrow{q_k^*} H^{2j}(B_k) \xrightarrow{\pi_k^*} H^{2j}(B_{k-1}) \to 0, \quad (2.9)$$

for each $1 \leq j \leq k-1$ and

$$0 \to H^{2k}(T_{\eta_k-1}) \xrightarrow{q_k^*} H^{2k}(B_k) \to 0. \quad (2.10)$$

Since the composition $\pi_k \circ i_k$ is the identity map on $B_{k-1}$, where $\pi_k: B_k \to B_{k-1}$ is the projection map, then (2.9) splits as abelian groups, that is, the map

$$\Psi: H^{2j}(B_{k-1}) \oplus H^{2j}(T_{\eta_k-1}) \to H^{2j}(B_k)$$

given by $\Psi := \pi_k^* + q_k^*$ is an isomorphism of free abelian groups.

**Remark 2.11** Of course, $\Psi$ is not normally multiplicative. For example, recall that the Thom complex of the canonical line bundle $\eta_1 \to B_1 = \mathbb{C}P^1$ can be identified with $\mathbb{C}P^2$. Therefore, we have a cofibre sequence $B_1 \to B_2 \to \mathbb{C}P^2$, while it is obvious that the cohomology ring of $B_2$ is not isomorphic to that of the direct sum of $B_1$ and $\mathbb{C}P^2$.  

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We now suspend (2.8), and consider the pinch map \( \gamma : \Sigma B_k \to \Sigma B_k \lor \Sigma B_k \). We write \( \psi_k \) for the composite

\[
\Sigma B_k \xrightarrow{\gamma} \Sigma B_k \lor \Sigma B_k \xrightarrow{\Sigma \pi_k \lor \Sigma \psi_k} \Sigma B_{k-1} \lor \Sigma T \eta_{k-1}
\]  

(2.12)

for each \( 1 \leq k \leq n \), and note that \( \psi_k^* \) is an isomorphism of cohomology groups

\[
\psi_k^* : H^*(\Sigma B_{k-1}) \oplus H^*(\Sigma T \eta_{k-1}) \cong H^*(\Sigma B_k).
\]  

(2.13)

**Theorem 2.14** The map \( \psi_k : \Sigma B_k \to \Sigma B_{k-1} \lor \Sigma T \eta_{k-1} \) is a homotopy equivalence.

**Proof.** It easily follows from the definitions that the spaces \( \Sigma B_k \) and \( \Sigma B_{k-1} \lor \Sigma T \eta_{k-1} \) are simply connected finite \( CW \)-complexes. Therefore, applying Whitehead’s theorem [11] to (2.13), we obtain the desired result. \( \square \)

Repeated application of Theorem 2.14 yields the following:

**Theorem 2.15** For each \( 1 \leq k \leq n \), there is a homotopy equivalence

\[
\Psi_k : \Sigma B_k \simeq \Sigma T \eta_0 \lor \ldots \lor \Sigma T \eta_{k-1}.
\]  

(2.16)

We note that, if \( t_k \in H^2(T \eta_k; \mathbb{Z}) \) is the Thom class, then it satisfies \( t_k^2 = t_k c_1(\eta_k) \), and the class \( x_{k+1} \) is the pullback of \( t_k \) to \( H^2(B_{k+1}; \mathbb{Z}) \) for any \( 1 \leq k \leq n \).

In order to compute \( KO \)-groups of bounded flag manifolds, we will make use of the relevant Atiyah-Hirzebruch spectral sequences. By a theorem of Thomas [10], some of the differentials in this spectral sequence can be related to Steenrod squares. Moreover, in our case, the \( E_3 \)-term will turn out to be so-called \( Sq^2 \)-homology of the bounded flag manifold. Therefore, to assist such computation, we will determine these homology groups in advance.

Since \( H^*(B_n; \mathbb{Z}_2) \) is concentrated in even dimensions,

\[
\ldots \to H^{2k-2}(B_n; \mathbb{Z}_2) \xrightarrow{Sq^2} H^{2k}(B_n; \mathbb{Z}_2) \to \ldots
\]

is a chain complex because \( Sq^2 Sq^2 = Sq^4 Sq^1 = 0 \). The homology of this chain complex is said to be the \( Sq^2 \) homology of the bounded flag manifold and denoted by \( H_*(B_n; Sq^2) \).

Our main task is now to prove the following theorem.

**Theorem 2.17** The homology group \( H_{2k}(B_n; Sq^2) \) is trivial for all \( n \geq 1 \) and \( k > 1 \).
We divide the proof of this theorem into several steps. Let \( y_1, \ldots, y_n \) be the generators of the group \( H^2(B_n; \mathbb{Z}_2) \) so that they satisfy the relation
\[
y_i^2 = y_i y_{i-1} \text{ for all } i = 2, \ldots, n \text{ and } y_1^2 = 0. \tag{2.18}
\]

If \( y_1, \ldots, y_k \) is a monomial in \( H^{2k}(B_n; \mathbb{Z}_2) \), we denote it simply by \( y_I \), where \( I = \{i_1, \ldots, i_k\} \). In this way, we get a bijection between the non-zero monomials in \( H^{2k}(B_n; \mathbb{Z}_2) \) and the elements of the set \( D_n^k \) consisting of all subsets of \([n]\) with \( k \)-elements. Furthermore, we denote by \( C_n^k \) the \( \mathbb{Z}_2 \)-vector space generated by the set \( D_n^k \); hence, \( C_n^k \) is an isomorphic copy of \( H^{2k}(B_n; \mathbb{Z}_2) \). The idea behind replacing \( H^{2k}(B_n; \mathbb{Z}_2) \) with \( C_n^k \) is just to simplify the notation.

Each non-empty set \( I \in D_n^k \) can be uniquely written as
\[
I = [a_1, b_1] \cup \ldots \cup [a_t, b_t], \tag{2.19}
\]
where \( b_{i-1} + 1 < a_i \) for any \( 2 \leq i \leq t \). For any \( k > 1 \), we define a map \( S^2_q : C_{k-1}^n \rightarrow C_k^n \) by
\[
S^2_q(I) := \sum_{i=1}^{t} (b_i - a_i + 1)[a_1, b_1] \cup \ldots \cup [a_{i-1}, b_i] \cup \ldots \cup [a_t, b_t] \pmod{2} \tag{2.20}
\]
for each \( I \in D_{k-1}^n \) with the conventions that
- if \( a_1 = 1 \), then the first term in the sum is deleted,
- \( S^2_q \) maps the empty set to itself,
and for an arbitrary sum \( I_1 + \ldots + I_l \in C_{k-1}^n \), we insist that
\[
S^2_q(I_1 + \ldots + I_l) = S^2_q(I_1) + \ldots + S^2_q(I_l).
\]

**Example 2.21** Consider the set \( I = \{1, 3, 4, 7, 9\} \in D_5^n \) for some \( n \geq 9 \). Then, we may write \( I \) as \( I = [1, 1] \cup [3, 4] \cup [7, 7] \cup [9, 9] \), and by definition,
\[
S^2_q(I) = 2[1, 1] \cup [2, 4] \cup [7, 7] \cup [9, 9] + [1, 1] \cup [3, 4] \cup [6, 7] \cup [9, 9] + [1, 1] \cup [3, 4] \cup [7, 7] \cup [8, 9] = [1, 1] \cup [3, 4] \cup [6, 7] \cup [9, 9] + [1, 1] \cup [3, 4] \cup [7, 9] = \{1, 3, 4, 6, 7, 9\} + \{1, 3, 4, 7, 8, 9\}.
\]
Definition 2.22 If \( y_{i_1}, \ldots, y_{i_s} \) is a monomial in \( H^{2k}(B_n; \mathbb{Z}_2) \) such that \( Sq^2(I) = I_1 + \ldots + I_s \), then we define \( y_{Sq^2(I)} := y_{I_1} + \ldots + y_{I_s} \).

Lemma 2.23 For any \( I = [a_1, b_1] \cup \ldots \cup [a_t, b_t] \), the Steenrod square \( Sq^2 \) maps the monomial \( y_I \) to \( y_{Sq^2(I)} \), i.e. \( Sq^2(y_I) = y_{Sq^2(I)} \).

Proof. We will proceed by induction on \( t \). When \( t = 1 \), let \( I = [i, i + k - 1] \in D^n_k \) for some \( n \geq 1 \) and \( k > 1 \). Then, by using the relation (2.18) together with the Cartan formula, we have

\[
Sq^2(y_I) = y_{i-i, i+k-1} = y_{Sq^2(I)}.
\]

Assume that the claim holds for \( t - 1 \) so that \( Sq^2(y_J) = y_{Sq^2(J)} \) for all \( J = [c_1, d_1] \cup \ldots \cup [c_{t-1}, d_{t-1}] \). For a given \( I = [a_1, b_1] \cup \ldots \cup [a_t, b_t] \in D^n_k \), we define \( J := I\backslash [a_t, b_t] \). Now, it is easy to verify that if \( Sq^2(J) = \sum_{j=1}^{t-1} J_j \), then

\[
Sq^2(I) = J_1 \cup [a_t, b_t] + \ldots + J_{t-1} \cup [a_t, b_t] + (b_t - a_t + 1)J \cup [a_t - 1, b_t].
\]

Then, it follows from the induction assumption that

\[
Sq^2(y_I) = Sq^2(y_{J_1} \cdot y_{[a_t, b_t]})
\]

\[
= Sq^2(y_{J_1}) \cdot y_{[a_t, b_t]} + y_{J_1} \cdot Sq^2(y_{[a_t, b_t]})
\]

\[
= y_{Sq^2(J)} \cdot y_{[a_t, b_t]} + y_{J_1} \cdot y_{Sq^2([a_t, b_t])}
\]

\[
= (y_{J_1} \cdot y_{[a_t, b_t]} + \ldots + y_{J_{t-1}} \cdot y_{[a_t, b_t]}) + y_{J_1} \cdot y_{Sq^2([a_t, b_t])}
\]

\[
= y_{Sq^2(I)}.
\]

\[\square\]

Proposition 2.24 Let \( J \in D^n_k \) be given, where \( k > 1 \). If \( Sq^2(J) = \emptyset \), then there exists \( I \in D_{k-1}^n \) such that \( Sq^2(I) = J \).

Proof. Assume that \( Sq^2(J) = \emptyset \) for some \( J \in D^n_k \), where \( J = [a_1, b_1] \cup \ldots \cup [a_t, b_t] \).

Then, it follows from the definition of \( Sq^2 \) that either

(i) \( a_1 = 1 \) and \( b_j - a_j \) is odd for all \( j = 2, \ldots, t \), or

(ii) \( a_1 > 1 \) and \( b_j - a_j \) is odd for all \( j = 1, \ldots, t \),

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According to above situations, if we define $I \in D_{k-1}^n$ by either

(i) $I := [a_1, b_1] \cup \ldots \cup [a_j, b_j] \setminus \{b_j - 1\} \cup \ldots [a_i, b_i]$ for some $j = 2, \ldots, t$, or

(ii) $I := [a_1, b_1] \cup \ldots \cup [a_j, b_j] \setminus \{b_j - 1\} \cup \ldots [a_i, b_i]$ for some $j = 1, \ldots, t$,

respectively, it is easy to verify that $Sq^2(I) = J$.

**Example 2.25** Let $J \in D_2^2$ be given by $J = \{1, 3, 4, 5, 6, 8, 9\}$ for some $n \geq 9$ which we can write as $J = [1, 1] \cup [3, 6] \cup [8, 9]$. Then,

$$\begin{align*}
Sq^2(J) &= (3 + 1)[1, 1] \cup [2, 6] \cup [8, 9] + (1 + 1)[1, 1] \cup [3, 6] \cup [7, 9] \\
&= \emptyset.
\end{align*}$$
Following the proof of Proposition 2.24, the image of $I_1 := [1, 1] \cup [3, 4] \cup [6, 6] \cup [8, 9]$ or $I_2 := [1, 1] \cup [3, 6] \cup [9, 9]$ is exactly $J$.

If $I, J$ are two elements in $\mathcal{P}_n^3$, then we define $d(I, J)$, “the difference of $I$ by $J$”, to be the integer $|\{I \setminus I \cap J\}|$.

**Proposition 2.26** Given $\sum_{j \in A} J_j \in \mathcal{P}_n^3$, where $A$ is an arbitrary index set, such that

$$Sq^2(\sum_{j \in A} J_j) = \emptyset,$$  \hspace{1cm} (2.27)

then there exist $\sum_{i \in B} I_i \in \mathcal{P}_{n-1}$ for which

$$Sq^2(\sum_{i \in B} I_i) = \sum_{j \in A} J_j,$$  \hspace{1cm} (2.28)

where $B \subset A$.

**Proof.** Firstly, by consideration of Proposition 2.24, we may assume in (2.27) that $Sq^2(J_j) \neq \emptyset$ for all $j \in A$. The equation (2.27) implies that

$$\sum_{j \in A} Sq^2(J_j) = \emptyset.$$

Suppose that $Sq^2(J_j) = \sum_{s=1}^{t_j} J_{j,s}$ for each $j \in A$, where we write the sum over all $(j, s)$ such that $J_{j,s} \neq \emptyset$. Therefore

$$\sum_{j \in A} Sq^2(J_j) = \sum_{j \in A} \sum_{s=1}^{t_j} J_{j,s} = \emptyset.$$

If we define $U(j, s) := \{(e, f) : J_{j,s} = J_{e,f}\}$ for any $j \in A$ and $1 \leq s \leq t_j$, then it follows that the number of elements in $U(j, s)$ must be even, since we are working over $\mathbb{Z}_2$. It is also clear from the definitions that if $(e, f) \in U(j, s) \setminus \{(j, s)\}$, then $d(J_j, J_e) = 1$. Pick any $(e, f)$ in $U(j, s)$ different than $(j, s)$ and define $d_j$ and $d_e$ to be the elements of $J_j$ and $J_e$ respectively such that $d_j, d_e \notin J_j \cap J_e$, and set

$$U(j) := \{e \in A : (e, f) \in U(j, s) \text{ for some } s \text{ and } f\}.$$

Now, let $I_j := J_j \setminus \{d_j\}$, then $Sq^2(I_j) = \sum_{e \in U(j)} J_e$, which may be obtained easily from the facts that if $I_j = [a_1, b_1] \cup \ldots \cup [a_t, b_t]$, then we have $d_e < b_t$ and $I_j \cup \{d_e\} = J_e$.
We see that $U_f = \{J_i : i \in B\}$, which is equal to

$$\sum_{i \in B} S^2(I_i) = \sum_{i \in B} S^2(I_i) = \sum_{j \in A} J_j,$$

from which we deduce Equation (2.28). This completes the proof. \(\square\)

Example 2.29 Let $J_1 + \ldots + J_4 \in C^4_n$ be given as follows: $J_1 = \{1, 2, 4, 7, 10\}$, $J_2 = \{2, 3, 4, 7, 10\}$, $J_3 = \{2, 4, 6, 7, 10\}$ and $J_4 = \{2, 4, 7, 9, 10\}$. Then,

$$S^2(J_1 + \ldots + J_4) = S^2(J_1) + \ldots + S^2(J_4)$$

$$= [1, 4] \cup [7, 7] \cup [10, 10] + [1, 2] \cup [4, 4] \cup [6, 7] \cup [10, 10] + [1, 2] \cup [4, 4] \cup [7, 7] \cup [9, 10]$$

$$+ [1, 2] \cup [7, 7] \cup [10, 10] + [2, 4] \cup [6, 7] \cup [10, 10] + [2, 4] \cup [7, 7] \cup [9, 10]$$

$$+ [1, 2] \cup [4, 4] \cup [6, 7] \cup [10, 10] + [2, 4] \cup [7, 7] \cup [9, 10] + [2, 4] \cup [7, 7] \cup [9, 10]$$

$$+ [2, 4] \cup [7, 7] \cup [9, 10] + [2, 4] \cup [7, 7] \cup [9, 10]$$

$$= \emptyset.$$

We see that $U(1) = U(2) = U(3) = U(4) = \{1, 2, 3, 4\}$. Therefore, we define $I := J_1 \setminus \{1\}$, which is equal to $[2, 2] \cup [4, 4] \cup [7, 7] \cup [10, 10]$, and

$$S^2(I) = [1, 2] \cup [4, 4] \cup [7, 7] \cup [10, 10]$$

$$+ [2, 4] \cup [7, 7] \cup [10, 10]$$

$$+ [2, 2] \cup [4, 4] \cup [6, 7] \cup [10, 10]$$

$$+ [2, 2] \cup [4, 4] \cup [7, 7] \cup [9, 10]$$

$$= J_1 + J_2 + J_3 + J_4.$$

Proof. [Proof of Theorem 2.17] Combining Lemma 2.23, Propositions 2.24 and 2.26, we see that $ker(S^2) = Im(S^2)$ for any $k > 1$, which completes the proof. \(\square\)

Theorem 2.17 also allows us to compute the $S^2$ homology of the Thom space of the associated line bundle $\eta_i$ over $B_i$ for any $i \geq 1$.

Corollary 2.30 The group $H_{2k}(T\eta_i; S^2)$ is trivial for all $i$ and $k \geq 1$. 456
Proof. This follows from the existence of the homotopy equivalence

\[ \psi_{i+1} : \Sigma B_{i+1} \to \Sigma B_i \vee \Sigma T\eta_i \]  

(2.31)

for all \( i \geq 1 \) and \( k > 1 \). When \( k = 1 \), it is easy to see the map \( Sq^2 : H^2(T\eta_i; \mathbb{Z}_2) \to H^4(T\eta_i; \mathbb{Z}_2) \) is an injection. \( \square \)

3. **KO-Groups of Bounded Flags**

Throughout, we will consider the spectra \( KO \) and \( K \) representing real and complex \( K \)-theory respectively. There are natural transformations: complexification \( c : KO^*(X) \to K^*(X) \), realification \( r : K^*(X) \to KO^*(X) \), and conjugation \( - : K^*(X) \to K^*(X) \). The formulas

\[ r \cdot c = 2 : KO^*(X) \to KO^*(X), \]
\[ c \cdot r = 1 + - : K^*(X) \to K^*(X), \]

are well known, where \( c \) is a ring homomorphism, but \( r \) is not. The coefficient rings \( K_\ast(S^0) \) and \( KO_\ast(S^0) \) are given as follows.

\[ KO_\ast \cong \mathbb{Z}[e, x, y, y^{-1}]/\{2e, e^3, e \cdot x, x^2 - 4y\} \quad \text{and} \quad K_\ast \cong \mathbb{Z}[z, z^{-1}], \]

(3.32)

where \( z \) is represented by the complex Hopf bundle over \( S^2 \), and \( e, x \) and \( y \) are represented by the real Hopf bundle over \( S^1 \), the symplectic Hopf bundle over \( S^1 \), and the canonical bundle over \( S^8 \), respectively.

We recall that \( \eta_1, \ldots, \eta_n \) are the associated line bundles over \( B_n \) with the first Chern classes \( x_1, \ldots, x_n \) respectively, described as in Section 1. In order to compute the real and complex \( K \)-theory of \( B_n \), we first recall Theorem 2.15, the stable splitting of \( B_n \), which is possible after one-suspension:

\[ B_n \simeq \bigvee_{i=0}^{n-1} T\eta_i. \]

(3.33)

It is important to note that in the above decomposition, we consider each \( \eta_i \) over \( B_i \) rather than \( B_n \) for every \( 1 \leq i \leq n-1 \), while \( \eta_0 \) is the trivial line bundle over a point, whose Thom complex may be identified with \( \mathbb{C}P^1 \). If we recall the Thom isomorphism \( \Phi : H^{2k}(B_i; \mathbb{Z}) \to H^{2k+2}(T\eta_i; \mathbb{Z}) \) for \( 1 \leq k \leq n \) and the fact that \( B_i \) is a toric variety[3]
arising from an \( i \)-crosspolytope, we see that the group \( H^{2k+2}(T\eta_i; \mathbb{Z}) \) is of rank \( h_k \) when \( k \geq 1 \), where \( h = (h_0, \ldots, h_i) \) is the \( h \)-vector of the \( i \)-crosspolytope given by \( h_k = \binom{i}{k} \) for any \( 0 \leq k \leq i \). Generators are given by \( t_I x_I \), where \( I \subseteq [i] \) is any subset of cardinality \( k \).

Firstly, we compute the complex K-theory. Let \( \gamma_i \) be the element in \( \tilde{K}^2(B_n) \) such that \( z \gamma_i = \eta_i - 1 \in \tilde{K}^0(B_n) \) for each \( 1 \leq i \leq n \), and let \( \theta_n \) be the line bundle over \( T\eta_n \) such that \( c_1(\theta_n) = t_n \) in \( H^2(T\eta_n; \mathbb{Z}) \); then \( t^K_n = z^{-1}(\theta_n - 1) \) is the Thom class in \( K^2(T\eta_n) \). Then the corresponding Thom isomorphism expresses \( K^*(B_n) \) as a free module over \( K^*(B_n) \) on generators 1 and \( t^K_n \).

**Proposition 3.34** The multiplicative structure of \( K^*(T\eta_n) \) is determined by \( (t^K_n)^2 = t^K_n \gamma_n \).

**Proof.** The construction of \( t^K_n \) ensures that
\[
\text{ch}(t^K_n) = (e^{ut_n} - 1)/u = t_n(e^{ux_n} - 1)/x_nu,
\]
since \( t_n^2 = t_n x_n \), where \( H_*(K) \cong \mathbb{Q}[u, u^{-1}] \). Therefore,
\[
\text{ch}((t^K_n)^2) = (\text{ch}(t^K_n))^2 = (e^{ut_n} - 1) \cdot t_n(e^{ux_n} - 1)/x_nu^2, \\
= (e^{ut_n} - 1) \cdot (e^{ux_n} - 1)/u^2, \\
= \text{ch}(t^K_n) \cdot \text{ch}(\gamma_n).
\]
On the other hand, the Atiyah-Hirzebruch spectral sequence for \( T\eta_n \) collapses, since \( T\eta_n \) has a cell-decomposition concentrated in even dimensions; the Chern character \( ch \) is therefore monic and the result follows. \( \square \)

**Theorem 3.35** For each \( n \geq 1 \), the complex K-theory \( K^*(B_n) \) of bounded flag manifold is given by
\[
K^*(B_n) \cong K_\ast[\gamma_1, \ldots, \gamma_n]/(\gamma_i^2 - \gamma_i \gamma_{i-1}, \gamma_1^2).
\]

**Proof.** This follows from (3.33) and Proposition 3.34. \( \square \)

In order to compute the \( KO \)-groups of \( B_n \), we recall that the \( E_2 \) and \( E_\infty \) terms of the Atiyah-Hirzebruch spectral sequence of \( KO \)-theory are given by
\[
E_2^{p,q} \cong H^p(X; KO_q(S^0)), \quad (3.37)
\]
\[
E_\infty^{p,q} \cong G_p KO^{p+q}(X) = F_p^{p+q}(X)/F_{p+1}^{p+q}(X), \quad (3.38)
\]
where $F^m_p(X) = \text{Ker}[\widetilde{KO}^m(X) \rightarrow \widetilde{KO}^m(X^{p-1})]$ and $X^{p-1}$ is the $(p-1)$-skeleton of $X$. As for the differentials $d^{p,q}_r: E^r_{p,q} \rightarrow E^r_{p+r,q-r+1}$, it follows from Theorem 4.2 of [10] that

$$d^{p,-8t}_2 = Sq^2 \circ \rho: H^p(X; \mathbb{Z}) \rightarrow H^{p+2}(X; \mathbb{Z}_2),$$

$$d^{p,-8t-1}_2 = Sq^2: H^p(X; \mathbb{Z}_2) \rightarrow H^{p+2}(X; \mathbb{Z}_2),$$

where $\rho$ is the mod 2 reduction map.

**Theorem 3.40** The groups $\widetilde{KO}^{2j+1}(T_{\eta_i})$ are trivial for any $i \geq 0$ and $j \in \mathbb{Z}$, except for $\widetilde{KO}^{8j-7}(T_{\eta_0})$, which is isomorphic to $\mathbb{Z}_2$.

**Proof.** For $\widetilde{KO}^{-1}(T_{\eta_i}) \cong \widetilde{KO}^7(T_{\eta_i})$, the $E_2$-term of the spectral sequence is given by $E_2^{p+7,-p} \cong H^{p+7}(T_{\eta_i}; \widetilde{KO}_p(S^0))$, which is trivial except for $p \equiv 1 \pmod{8}$, where $-7 < p \leq 2i - 5$. Assume that $p = 8t + 1$ for some $t \geq 0$. Then, by (3.39), we can replace the sequence

$$\ldots \rightarrow E_2^{8t+6, -8t} \xrightarrow{d_2^{8t+6, -8t}} E_2^{8t+8, -8t-1} \xrightarrow{d_2^{8t+8, -8t-1}} E_2^{8t+10, -8t-2} \rightarrow \ldots$$

with

$$\ldots \rightarrow H^{8t+6}(T_{\eta_i}; \mathbb{Z}) \xrightarrow{Sq^2} H^{8t+8}(T_{\eta_i}; \mathbb{Z}_2) \xrightarrow{Sq^2} H^{8t+10}(T_{\eta_i}; \mathbb{Z}_2) \rightarrow \ldots$$

By Corollary 2.30, it follows that $E_3^{8t+8, -8t-1} \cong 0$; hence the spectral sequence collapses to the $E_3$-term, and the result follows. The proofs of the other cases are similar to that of $\widetilde{KO}^{-1}(T_{\eta_i})$. On the other hand, since we may identify $T_{\eta_0}$ with $\mathbb{C}P^1$, it follows from (3.32) that $\widetilde{KO}^{-7}(T_{\eta_0}) \cong \mathbb{Z}_2$. 

**Corollary 3.41** The groups $\widetilde{KO}^{2j+1}(B_n)$ are trivial for any $j \in \mathbb{Z}$, except for $\widetilde{KO}^{8j-7}(B_n)$, which is isomorphic to $\mathbb{Z}_2$.

**Proof.** The splitting given by (3.33) induces an isomorphism of groups

$$\widetilde{KO}^{2j+1}(B_n) \cong \bigoplus_{i=0}^{n-1} \widetilde{KO}^{2j+1}(T_{\eta_i}),$$

for $j \in \mathbb{Z}$. Thus, the claim follows from Theorem 3.40. 

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Theorem 3.43  The groups $\tilde{KO}^2(T\eta_i)$ are free of rank $2^{i-1}$ for any $i \geq 1$ and $j \in \mathbb{Z}$.

Proof. To show that these groups are free, we consider the fibration $U \to U/O$, and from the fact that $BO \times \mathbb{Z} = \Omega(U/O)$, we obtain the exact Bott sequence for any CW-complex $X$ in the form

$$\cdots \to \tilde{KO}^k(X) \to \tilde{K}^k(X) \to \tilde{KO}^{k+2}(X) \to \tilde{KO}^{k+1}(X) \to \cdots. \quad (3.44)$$

Applying the sequence (3.44) for $T\eta_i$ and $k = -1$, we get the exact sequence

$$\cdots \to \tilde{K}^{-1}(T\eta_i) \to \tilde{KO}^1(T\eta_i) \to \tilde{KO}^0(T\eta_i) \to \tilde{K}^0(T\eta_i) \to \tilde{KO}^2(T\eta_i) \to \cdots. \quad (3.45)$$

By Theorem 3.40, the group $\tilde{KO}^1(T\eta_i) \cong \tilde{KO}^{-1}(T\eta_i)$ is trivial for any $i \geq 1$; hence, $\tilde{KO}^0(T\eta_i)$ is a free group. A similar argument will apply to the other cases.

To find the rank of these groups, we consider the related Atiyah-Hirzebruch spectral sequences. For example, for $\tilde{KO}^0(T\eta_i)$, we have $E_2^{p,-p} \cong H^p(T\eta_i; \tilde{KO}_p(S^0))$, which is trivial except for $p \equiv 0, 2, 4 \pmod{8}$, where $0 < p \leq 2i + 2$.

(i) Let $p \equiv 8t$ for some $t \geq 1$. Then, from the sequence

$$\cdots \to E_2^{st-2,-st+1} \overset{d_2^{st-2,-st+1}}{\to} E_2^{st,-st} \overset{d_2^{st,-st}}{\to} E_2^{st+2,-st-1} \to \cdots, \quad (3.46)$$

we have that $E_2^{st-2,-st+1} \cong 0$, since $\tilde{KO}_{st-1}(S^0) \cong \tilde{KO}_{r}(S^0) \cong 0$. By (3.39), we can replace (3.46) with the sequence

$$0 \to H^{st}(T\eta_i; \mathbb{Z}) \overset{Sq^2\varphi}{\to} H^{st+2}(T\eta_i; \mathbb{Z}_2) \to \cdots,$$

from which we obtain that

$$E_3^{st,-st} \cong \text{Ker}[Sq^2\varphi: H^{st}(T\eta_i; \mathbb{Z}) \to H^{st+2}(T\eta_i; \mathbb{Z}_2)].$$

Since the differential $d_k: E_k^{p,-p} \to E_{k+1}^{p+k,-p-k+1}$ (total degree 1) is a zero map (compare to Theorem 3.40) for any $k \geq 3$, the group $E_3^{st,-st}$ will survive to $E_\infty^{st,-st}$. Moreover, the group $E_\infty^{st,-st}$ is isomorphic to $h_{st}$ copies of $\mathbb{Z}$ for each $t \geq 1$, where $h = (h_0, \ldots, h_i)$ is the $h$-vector of the $i$-crosspolytope.

(ii) Let $p \equiv 2 \pmod{8}$, then $p = 8t + 2$ for some $t \geq 0$. Then, from the sequence

$$\cdots \to E_2^{st,-st-1} \to E_2^{st+2,-st-2} \to E_2^{st+4,-st-3} \to \cdots,$$
we see that $E_{2}^{st+4, -st-3} \cong 0$, and

$$E_{3}^{st+2, -st-2} \cong H^{st+2}(T\eta; \mathbb{Z}/2) / \text{Im}[Sq^2 : H^{st}(T\eta; \mathbb{Z}_2) \to H^{st+2}(T\eta; \mathbb{Z}_2)],$$

which is a finite group for any $t \geq 0$.

(iii) Let $p \equiv 4 \pmod{8}$, then $p = 8t + 4$ for some $t \geq 1$. Then, from the sequence

$$\ldots \to E_{2}^{st+2, -st-3} \to E_{2}^{st+4, -st-4} \to E_{2}^{8r+6, -st-5} \to \ldots,$$

we deduce that $E_{3}^{st+4, -8r-4} \cong H^{st+4}(T\eta; \mathbb{Z})$. Similar to the case (i), the group $E_{3}^{st+4, -8r-4}$ will survive to $E_{\infty}$ so that the group $E_{\infty}^{st+4, -8r-4}$ is of rank $h_{4t+1}$ for each $t \geq 1$.

As a conclusion, since all groups in our filtration are free, all the extension problems are trivial, and from the well-known formula

$$h_{1} + h_{3} + h_{5} + \ldots = \left(\frac{i}{1}\right) + \left(\frac{i}{3}\right) + \ldots = 2^{i-1},$$

the group $\tilde{KO}^0(T\eta)$ is isomorphic to $\mathbb{Z}^{2^{i-1}}$ for any $i \geq 1$. For the other cases, we can obtain the results in the same way as the proof of $j = 0$.

\begin{corollary} \label{cor:KO_Bn}
(a) $\tilde{KO}^0(B_n) \cong \mathbb{Z}_2 \oplus \mathbb{Z}^{2^{n-1}-1}$,

(b) $\tilde{KO}^{-2}(B_n) \cong \tilde{KO}^{-6}(B_n) \cong \mathbb{Z}^{2^{n-1}}$,

(c) $\tilde{KO}^{-4}(B_n) \cong \mathbb{Z}^{2^{n-1}-1}$.
\end{corollary}

\begin{proof}
Once again, we apply to (3.33), from which we obtain an isomorphism

$$\tilde{KO}^{2j}(B_n) \cong \bigoplus_{i=0}^{n-1} \tilde{KO}^{2j}(T\eta),$$

for any $j \in \mathbb{Z}$. On the other hand, it follows from (3.32) that $\tilde{KO}^0(\mathbb{C}P^1) \cong \mathbb{Z}_2$, $\tilde{KO}^{-2}(\mathbb{C}P^1) \cong \tilde{KO}^{-6}(\mathbb{C}P^1) \cong \mathbb{Z}$ and $\tilde{KO}^{-4}(\mathbb{C}P^1) \cong 0$. Now, the claims follow from Theorem 3.43 and the formula

$$2^0 + 2^1 + 2^2 + \ldots + 2^{n-2} = 2^{n-1} - 1.$$
Let us explain what we have gained so far. We first recall that from the fibration

$$O/U \xrightarrow{L} BU \xrightarrow{r} BO,$$

we obtain the associated exact Bott sequence for $B_n$:

$$\ldots \to \tilde{K}O^{-1}(B_n) \to \tilde{K}O^{-2}(B_n) \xrightarrow{\chi} \tilde{K}O^0(B_n) \xrightarrow{r} \tilde{K}O^0(B_n) \to \tilde{K}O^0(B_n) \to \ldots,$$

which links the real and complex $K$-theory through the realification homomorphism $r$. Here, $\chi$ is induced by $f$ and may be identified with $z^{-1} \cdot c$ by composing the complexification homomorphism with multiplication by $z^{-1}$.

When combined with Corollary 3.41, it reduces to a short exact sequence

$$0 \to \tilde{K}O^{-2}(B_n) \xrightarrow{\chi} \tilde{K}O^0(B_n) \xrightarrow{r} \tilde{K}O^0(B_n) \to 0.$$  \hfill (3.50)

Therefore, if $\mathcal{K}_n$ denotes the kernel of $r: \tilde{K}O^0(B_n) \to \tilde{K}O^0(B_n)$, then it follows that $\mathcal{K}_n \cong \tilde{K}O^{-2}(B_n) \cong \mathbb{Z}^{2^n-1}$ by Corollary 3.47. We note that since $\chi$ is a monomorphism in this case, the group $\mathcal{K}_n$ can be identified with the set of stably complex structures on $B_n$.

References


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