On Derivations of Prime Gamma Rings

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Abstract

We consider some results in a Γ-ring $M$ with derivation which is related to $Q$, and the quotient Γ-ring of $M$.

Key words and phrases: Derivation, gamma ring, prime gamma ring, quotient gamma ring.

1. Introduction

Nobusawa [3] introduced the notion of a Γ-ring, an object more general than a ring. Barnes [1] slightly weakened the conditions in the definition of Γ-ring in the sense of Nobusawa. Öztürk et al. [4, 5] studied extended centroid of prime Γ-rings. In this paper, we consider the main results as follows. (1) Let $M$ be a prime Γ-ring of characteristic 2, $U$ a non-zero ideal of $M$, and $d_1$ and $d_2$ two non-zero derivations of $M$. If $d_1d_2(U) = (0)$, there exists $\lambda \in C_\Gamma$ such that $d_2 = \lambda d_1$ for all $\alpha \in \Gamma$ where $C_\Gamma$ is the extended centroid of $M$. (2) Let $M$ be a prime Γ-ring, $U$ a non-zero right ideal of $M$ and $d$ a non-zero derivation of $M$. If $d(U)\Gamma a = (0)$ where $a$ is a fixed element of $M$, then there exists an element $q$ of $Q$ such that $q\gamma a = 0$ and $q\gamma u = 0$ for all $u \in U$ and $\gamma \in \Gamma$. (3) Let $M$ be a prime Γ-ring with $\text{char} M \neq 2$, $U$ a non-zero right ideal of $M$ and $d_1$ and $d_2$ two non-zero derivations of $M$. If $d_1d_2(U) = (0)$, then there exists two elements $p, q$ of $Q$ such that $qU = (0)$ and $pU = (0)$.

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2. Preliminaries

Let $M$ and $\Gamma$ be (additive) abelian groups. If for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$ the conditions
\begin{enumerate}
\item $ab \in M$,
\item $(a + b)c = ac + cb$,
\item $(ab)c = a(bc)$.
\end{enumerate}
are satisfied, then we call $M$ a $\Gamma$-ring. Let $M$ be a $\Gamma$-ring. The subset
$$Z = \{x \in M \mid x\gamma m = m\gamma x \text{ for all } m \in M \text{ and } \gamma \in \Gamma\}$$
is called the center of $M$. By a right (resp. left) ideal of a $\Gamma$-ring $M$ we mean an additive subgroup $U$ of $M$ such that $U\Gamma M \subseteq U$ (resp. $M\Gamma U \subseteq U$). If $U$ is both a right and a left ideal, then we say that $U$ is an ideal of $M$. For each $a$ of a $\Gamma$-ring $M$ the smallest right ideal containing $a$ is called the principal right ideal generated by $a$ and is denoted by $\langle a \rangle_r$. Similarly we define $\langle a \rangle_l$ (resp. $\langle a \rangle$), the principal left (resp. two sided) ideal generated by $a$. An ideal $P$ of a $\Gamma$-ring $M$ is said to be prime if for any ideals $U$ and $V$ of $M$, $UTV \subseteq P$ implies $U \subseteq P$ or $V \subseteq P$. A $\Gamma$-ring $M$ is said to be prime if the zero ideal is prime.

Theorem 2.1 ([2, Theorem 4]). If $M$ is a $\Gamma$-ring, the following conditions are equivalent:
\begin{enumerate}
\item[(i)] $M$ is a prime $\Gamma$-ring.
\item[(ii)] If $a, b \in M$ and $a\Gamma Mb = (0)$, then $a = 0$ or $b = 0$.
\item[(iii)] If $\langle a \rangle$ and $\langle b \rangle$ are principal ideals of $M$ such that $\langle a \rangle \Gamma \langle b \rangle = (0)$, then $a = 0$ or $b = 0$.
\item[(iv)] If $U$ and $V$ are right ideals of $M$ such that $UTV = (0)$, then $U = (0)$ or $V = (0)$.
\item[(v)] If $U$ and $V$ are left ideals of $M$ such that $UTV = (0)$, then $U = (0)$ or $V = (0)$.
\end{enumerate}
Let $M$ be a prime $\Gamma$-ring such that $M\Gamma M \neq M$. Denote
$$\mathcal{M} := \{(U, f) \mid U(\neq 0) \text{ is an ideal of } M \text{ and } f : U \rightarrow M \text{ is a right } M\text{-module homomorphism}\}.$$
Define a relation $\sim$ on $\mathcal{M}$ by

$$(U, f) \sim (V, g) \iff \exists W(\neq 0) \subseteq U \cap V \text{ such that } f = g \text{ on } W.$$  

Since $\mathcal{M}$ is a prime $\Gamma$-ring, it is possible to find a non-zero $W$ and so "$\sim$" is an equivalence relation. This gives a chance for us to get a partition of $\mathcal{M}$. We then denote the equivalence class by $\text{Cl}(U, f) = \hat{f}$, where

$$\hat{f} := \{ g : V \to \mathcal{M} \mid (U, f) \sim (V, g) \},$$

and denote by $Q$ the set of all equivalence classes. Then $Q$ is a $\Gamma$-ring, which is called the quotient $\Gamma$-ring of $\mathcal{M}$ (see [4]). The set

$$C_{\Gamma} := \{ g \in Q \mid g\gamma f = f\gamma g \text{ for all } f \in Q \text{ and } \gamma \in \Gamma \}$$

is called the extended centroid of $\mathcal{M}$ (See [4]).

**Lemma 2.2 ([4, p. 476]).** Let $\mathcal{M}$ be a prime $\Gamma$-ring such that $\mathcal{M} \Gamma \mathcal{M} \neq \mathcal{M}$ and $C_{\Gamma}$ the extended centroid of $\mathcal{M}$. If $a_i$ and $b_i$ are non-zero elements of $\mathcal{M}$ such that $\sum a_i \gamma x_i \beta_i b_i = 0$ for all $x_i \in \mathcal{M}$ and $\gamma_i, \beta_i \in \Gamma$, then the $a_i$’s (also $b_i$’s) are linearly dependent over $C_{\Gamma}$. Moreover, if $a \gamma x \beta = b \gamma x \beta a$ for all $x \in \mathcal{M}$ and $\gamma, \beta \in \Gamma$ where $a(\neq 0), b \in \mathcal{M}$ are fixed, then there exists $\lambda \in C_{\Gamma}$ such that $b = \lambda a a$ for all $\alpha \in \Gamma$.

**Theorem 2.3 ([6, Theorem 3.5]).** The $\Gamma$-ring $Q$ satisfies the following properties:

(i) For any element $q \in Q$, there exists an ideal $U_q \subseteq F$ such that $q(U_q) \subseteq M$ (or $q\gamma U_q \subseteq M$ for all $\gamma \in \Gamma$).

(ii) If $q \in Q$ and $q(U) = (0)$ for some $U \subseteq F$ (or $q\gamma U_q = (0)$ for some $U \subseteq F$ and for all $\gamma \in \Gamma$), then $q = 0$.

(iii) If $U \subseteq F$ and $\Psi : U \to M$ is a right $M$-module homomorphism, then there exists an element $q \in Q$ such that $\Psi(u) = q(u)$ for all $u \in U$ (or $\Psi(u) = q\gamma u$ for all $u \in U$ and $\gamma \in \Gamma$).

(iv) Let $W$ be a submodule (an $(M, M)$-subbimodule) in $Q$ and $\Psi : W \to Q$ a right $M$-module homomorphism. If $W$ contains the ideal $U$ of the $\Gamma$-ring $M$ such that $\Psi(U) \subseteq M$ and $\text{Ann} U = \text{Ann}_r W$, then there is an element $q \in Q$ such that $\Psi(b) = q(b)$ for any $b \in W$ (or $\Psi(b) = q\gamma b$ for any $b \in W$ and $\gamma \in \Gamma$) and $q(a) = 0$ for any $a \in \text{Ann}_r W$ (or $q\gamma a = 0$ for any $a \in \text{Ann}_r W$ and $\gamma \in \Gamma$).
Let $M$ be a $\Gamma$-ring. A map $d : M \to M$ is called a derivation if

$$d(x + y) = d(x) + d(y) \text{ and } d(x\gamma y) = d(x)\gamma y + x\gamma d(y)$$

for all $x, y \in M$ and $\gamma \in \Gamma$.

**Lemma 2.4** ([8, Lemma 3]). Let $M$ be a prime $\Gamma$-ring, $U$ a non-zero ideal of $M$, and $d$ a derivation of $M$. If $a\Gamma d(U) = (0)$ ($d(U)\Gamma a = (0)$) for all $a \in M$, then $a = 0$ or $d = 0$.

**Lemma 2.5** ([8, Lemma 1]). Let $M$ be a prime $\Gamma$-ring and $Z$ the center of $M$.

(i) If $a, b, c \in M$ and $\beta, \gamma \in \Gamma$, then

$$[a\gamma b, c]_{\beta} = a\gamma [b, c]_{\beta} + [a, c]_{\beta}\gamma b + a\gamma (c\beta b) - a\beta (c\gamma b)$$

where $[a, b]_{\gamma}$ is $a\gamma b - b\gamma a$ for all $a, b \in M$ and $\gamma \in \Gamma$.

(ii) If $a \in Z$, then $[a\gamma b, c]_{\beta} = a\gamma [b, c]_{\beta}$ where $[a, b]_{\gamma}$ is $a\gamma b - b\gamma a$ for all $a, b \in M$ and $\gamma \in \Gamma$.

**Lemma 2.6** ([8, Lemma 2]). Let $M$ be a prime $\Gamma$-ring, $U$ a non-zero right (resp. left) ideal of $M$ and $a \in M$. If $U\Gamma a = (0)$ (resp. $a\Gamma U = (0)$), then $a = 0$.

3. Main results

In what follows, let $M$ denote a prime $\Gamma$-ring such that $M\Gamma M \neq M$, $Z$ is the center of $M$, $C_{\Gamma}$ is the extended centroid of $M$ and $[a, b]_{\gamma} = a\gamma b - b\gamma a$ for all $a, b \in M$ and $\gamma \in \Gamma$.

**Lemma 3.1.** Let $M$ be a prime $\Gamma$-ring of characteristic 2. Let $d_1$ and $d_2$ two non-zero derivations of $M$ and right $M$-module homomorphisms. If

$$d_1 d_2(x) = 0 \text{ for all } x \in M,$$

then there exists $\lambda \in C_{\Gamma}$ such that $d_2(x) = \lambda a d_1(x)$ for all $\alpha \in \Gamma$ and $x \in M$.

**Proof.** Let $x, y \in M$ and $\alpha \in \Gamma$. Replacing $x$ by $x\gamma y$ in (3.1), it follows from $\text{char} M = 2$ that for all $x, y \in M$ and $\gamma \in \Gamma$

$$d_1(x) \gamma d_2(y) = d_2(x) \gamma d_1(y).$$
Replacing $x$ by $x\beta z$ in (3.2), we get

$$d_1(x)\beta z \gamma d_2(y) = d_2(x)\beta z \gamma d_1(y)$$ (3.3)

for all $x, y \in M$ and $\gamma \in \Gamma$. Now, if we replace $y$ by $x$ in (3.3), then we obtain

$$d_1(x)\beta z \gamma d_2(x) = d_2(x)\beta z \gamma d_1(x)$$ (3.4)

for all $x \in M$ and $\gamma, \beta \in \Gamma$. If $d_1(x) \neq 0$, then there exists $\lambda(x) \in C_\Gamma$ such that $d_2(x) = \lambda(x)\alpha d_1(x)$ for all $x \in M$ and $\alpha \in \Gamma$ by Lemma 2.2. Thus, if $d_1(x) \neq 0 \neq d_1(y)$, then (3.3) implies that

$$(\lambda(y) - \lambda(x))\alpha d_1(x)\beta z \gamma d_2(x) = 0.$$ (3.5)

Since $M$ is a prime $\Gamma$-ring, we conclude by using Lemma 2.4 that $\lambda(y) = \lambda(x)$ for all $x, y \in M$. Hence we proved that there exists $\lambda \in C_\Gamma$ such that $d_2(x) = \lambda(x)\alpha d_1(x)$ for all $x \in M$ and $\alpha \in \Gamma$ with $d_1(x) \neq 0$. On the other hand, if $d_1(x) = 0$, then $d_2(x) = 0$ as well. Therefore, $d_2(x) = \lambda(x)\alpha d_1(x)$ for all $x \in M$ and $\alpha \in \Gamma$. This completes the proof. □

**Proposition 3.2.** Let $M$ be a prime $\Gamma$-ring of characteristic 2 and $d$ a non-zero derivation of $M$. If

$$d(x) \in \mathbb{Z} \text{ for all } x \in M,$$ (3.6)

then there exists $\lambda(m) \in C_\Gamma$ such that $d(m) = \lambda(m)\alpha d(z)$ for all $m, z \in M$ and $\alpha \in \Gamma$ or $M$ is commutative.

**Proof.** From (3.6), we have

$$[d(x), y]_\beta = 0 \text{ for all } x, y \in M \text{ and } \beta \in \Gamma.$$ (3.7)

Replacing $x$ by $x\gamma z$ in (3.7), it follows from Lemma 2.5 that

$$d(x)\gamma[z, y]_\beta + d(z)\gamma[x, y]_\beta = 0$$ (3.8)

for all $x, y, z \in M$ and $\gamma, \beta \in \Gamma$. Replacing $z$ by $d(z)$ in (3.8), we obtain

$$d^2(z)\gamma[x, y]_\beta = 0 \text{ for all } x, y, z \in M \text{ and } \gamma, \beta \in \Gamma.$$ (3.9)
Now, substituting $zm$ for $z$ in (3.9), it follows from $\text{char} M = 2$ that
\begin{equation}
 d^2(z)\alpha m \gamma [x,y]_\beta = 0 \tag{3.10}
\end{equation}
for all $x,y,z,m \in M$ and $\gamma, \beta, \alpha \in \Gamma$. Since $M$ is a prime $\Gamma$-ring, we obtain
\begin{equation}
 d^2(z) = 0 \quad \forall z \in M \quad \text{or} \quad [x,y]_\beta = 0 \quad \forall x,y \in M \quad \text{and} \quad \forall \beta \in \Gamma. \tag{3.11}
\end{equation}
From (3.11), if $d^2(z) = 0$ for all $z \in M$, then replacing $z$ by $z\gamma m$ in this last relation, it follows from $d(x) \in Z$ that
\begin{equation}
 d(z)\gamma d(m) = d(m)\gamma d(z) \quad \text{for all} \quad z,m \in M \quad \text{and} \quad \gamma \in \Gamma. \tag{3.12}
\end{equation}
Replacing $z$ by $za\eta$ in (3.12), it follows from (3.6) that for all $z,m,n \in M$ and $\gamma, \alpha \in \Gamma$
\begin{equation}
 d(z)\alpha n \gamma d(m) = d(m)\alpha n \gamma d(z). \tag{3.13}
\end{equation}
If $d(z) \neq 0$, then there exists $\lambda(m) \in C_\Gamma$ such that $d(m) = \lambda(m)\alpha d(z)$ for all $z,m \in M$ and $\alpha \in \Gamma$ by Lemma 2.2. On the other hand, it follows from (3.11) that if $[x,y]_\beta = 0$ for all $x,y \in M$ and $\beta \in \Gamma$, then $M$ is commutative. This completes the proof.

**Theorem 3.3.** Let $M$ be a prime $\Gamma$-ring of characteristic 2, $d_1$ and $d_2$ two non-zero derivations of $M$ and $U$ a non-zero ideal of $M$. If
\begin{equation}
 d_1 d_2(u) = 0 \quad \text{for all} \quad u \in U \tag{3.14}
\end{equation}
then there exists $\lambda \in C_\Gamma$ such that $d_2(x) = \lambda \alpha d_1(x)$ for all $\alpha \in \Gamma$ and $x \in M$.

**Proof.** Let $u,v \in U$ and $\gamma \in \Gamma$. Replacing $u$ by $d_2(u)\gamma v$ in (3.14), we get
\begin{equation}
 d_2^2(u)\gamma d_1(v) = 0 \quad \text{for all} \quad u,v \in U \quad \text{and} \quad \gamma \in \Gamma. \tag{3.15}
\end{equation}
Since $d_1 \neq 0$, it follows from Lemma 2.4 that $d_2^2(u) = 0$ for all $u \in U$, so from $\text{char} M = 2$ that $d_2^2 = 0$. Now, substituting $u\gamma d_2(x)$ for $u$ in (3.14), we get
\begin{equation}
 d_2(u)\gamma d_1(d_2(x)) = 0 \quad \text{for all} \quad u \in U, \quad x \in M \quad \text{and} \quad \gamma \in \Gamma. \tag{3.16}
\end{equation}
Since \( d_2 \neq 0 \), we get \( d_1(d_2(x)) = 0 \) for all \( x \in M \) by Lemma 2.4. Hence there exists \( \lambda \in C_\Gamma \) such that \( d_2 = \lambda x d_1 \) for all \( \alpha \in \Gamma \) by Lemma 3.1.

**Theorem 3.4.** Let \( M \) be a prime \( \Gamma \)-ring, \( U \) a non-zero right ideal of \( M \) and \( d \) a non-zero derivation of \( M \). If

\[
d(u)\gamma a = 0 \quad \text{for all} \quad u \in U \quad \text{and} \quad \gamma \in \Gamma
\]

where \( a \) is a fixed element of \( M \), then there exists an element \( q \) of \( Q \) such that \( q\gamma a = 0 \) and \( q\gamma u = 0 \) for all \( u \in U \) and \( \gamma \in \Gamma \).

**Proof.** Let \( u \in U \), \( x \in M \) and \( \beta \in \Gamma \). Since \( U \) is a right ideal of \( M \), we have \( u\beta x \in U \).

Replacing \( u \) by \( u\beta x \) in (3.17), we get

\[
d(u)\beta x\gamma a + u\beta d(x)\gamma a = 0 \tag{3.18}
\]

for all \( u \in U \), \( x \in M \) and \( \gamma, \beta \in \Gamma \). Hence \( d(u)\beta x\gamma a a_m + u\beta d(x)\gamma a a_m = 0 \) for any \( m \in M \) and \( \alpha \in \Gamma \), and so \( d(u)\beta (\sum x\gamma a a_m) = -(u\beta (\sum d(x)\gamma a a_m)) \). Therefore, for any \( v \in V = M\Gamma a\Gamma M \) which is a non-zero ideal of \( M \), we have

\[
d(u)\beta v = u\beta f(v) \tag{3.19}
\]

for all \( u \in U \). \( f(v) \) is independent of \( u \) but it is dependent on \( v \). Since \( M \) is a prime \( \Gamma \)-ring, \( f(v) \) is well-defined and unique for all \( v \in V \). Note that \( v\alpha y \in V \) for any \( y \in M \), \( v \in V \) and \( \alpha \in \Gamma \). Replacing \( v \) by \( v\alpha y \) in (3.19) we get

\[
d(u)\beta (v\alpha y) = u\beta f(v\alpha y) \quad \text{for all} \quad y \in M, \tag{3.20}
\]

and so by using (3.19) and (3.20), we have

\[
(d(u)\beta v)\alpha y = u\beta f(v\alpha y) \quad \Rightarrow \quad (u\beta f(v))\alpha y = u\beta f(v\alpha y)
\]

\[
\Rightarrow \quad u\beta f(v)\alpha y = u\beta f(v\alpha y)
\]

\[
\Rightarrow \quad u\beta (f(v)\alpha y - f(v\alpha y)) = 0,
\]

which implies from Lemma 2.6 that

\[
f(v\alpha y) = f(v)\alpha y \tag{3.21}
\]
for all \( y \in M, v \in V \) and \( \alpha \in \Gamma \). It follows from (3.21) that \( f : V \to M \) is a right \( M \)-module homomorphism. In this case, \( q = Cl(V, f) \in Q \). Moreover, \( f(v) = q \beta v \) for all \( v \in V \) and \( \alpha \in \Gamma \) by Theorem 2.3. Let \( x \in M, v \in V, u \in U \) and \( \gamma, \beta \in \Gamma \). Replacing \( v \) by \( x\gamma v \) in (3.19), we get

\[
d(u)\beta(x\gamma v) = u\beta f(x\gamma v) = u\beta (q \beta x \gamma v). \tag{3.22}
\]

Also, replacing \( u \) by \( u \gamma x \) in (3.19), we get

\[
d(u)\gamma x \beta v = u\gamma x \beta q \beta v - u \gamma d(x) \beta v. \tag{3.23}
\]

Now, replacing \( \beta \) by \( \gamma \) and replacing \( \gamma \) by \( \beta \) in (3.23), we get

\[
d(u)\beta x \gamma v = u\beta x \gamma q \gamma v - u \beta d(x) \gamma v. \tag{3.24}
\]

Thus, from (3.22) and (3.24) we obtain

\[
u \beta(q \beta x - x \gamma q + d(x)) \gamma v = 0 \tag{3.25}
\]

for all \( x \in M, v \in V, u \in U \) and \( \gamma, \beta \in \Gamma \). Hence \( d(x) = x \gamma q - q \beta x \) for all \( x \in M \) and \( \gamma, \beta \in \Gamma \) by Lemma 2.6. Now, we shall prove that \( q \) can be chosen in \( Q \) such that \( q \gamma a = 0 \) and \( q \gamma u = 0 \) for all \( u \in U \) and \( \gamma \in \Gamma \). Let \( u \in U \) and \( x \in M, d(u) = q\alpha u - u \beta q \) and \( d(x) = q \beta x - x \alpha q \). Then we have \( 0 = d(u \beta x) \gamma a = (q\alpha (u \beta x) - (u \beta x) \alpha \gamma) \gamma a \). Thus, \( q\alpha u \beta x \gamma a = u \beta x \alpha q \gamma a \). If \( q \gamma a = 0 \), then \( q\alpha u \beta x \gamma a = 0 \), and so since \( M \) is prime \( \Gamma \)-ring, we get \( q\Gamma U = (0) \). On the other hand, if \( q \gamma a \neq 0 \), then \( q \gamma u \neq 0 \). In fact, if \( q \gamma u = 0 \), then \( q \gamma a = 0 \) since \( q\alpha u \beta x \gamma a = u \beta x \alpha q \gamma a \). Thus, we may suppose that \( q \gamma a \neq 0 \) and \( q \gamma u \neq 0 \) for all \( u \in U \) and \( \gamma \in \Gamma \). In this case, we get

\[
q\alpha u \beta x \gamma a = u \beta x \alpha q \gamma a \tag{3.26}
\]

for all \( x \in M, u \in U \) and \( \gamma, \beta, \alpha \in \Gamma \). It follows from Lemma 2.2 that there exists \( \lambda \in C_{\Gamma} \) such that \( q \gamma a = \lambda \delta a \) and \( q \gamma u = \lambda \delta u \) for all \( u \in U \) and \( \gamma, \delta, \alpha \in \Gamma \). Hence, if \( q' = q - \lambda \), then \( q' \Gamma a = 0 \) and \( q' \Gamma U = (0) \). This completes the proof.

**Theorem 3.5.** Let \( M \) be a prime \( \Gamma \)-ring with \( \text{char} M \neq 2 \), \( U \) a non-zero right ideal of \( M \) and \( d \) a non-zero derivation of \( M \). Then the subring of \( M \) generated by \( d(U) \) contains no non-zero right ideals of \( M \) if and only if \( d(U) \Gamma U = (0) \).
Proof. Let \( A \) be the subring generated by \( d(U) \). Let \( S = A \cap U, u \in U, s \in S \) and \( \gamma \in \Gamma \). Then \( d(s\gamma u) = d(s)\gamma u + s\gamma d(u) \in A \), and so we have \( d(s)\gamma u \in S \). Thus \( d(S)\Gamma U \) is a right ideal of \( M \). In this case, \( d(S)\Gamma U = (0) \) by hypothesis. \( d(u\gamma a) = d(u)\gamma a + u\gamma d(a) \in S \) and \( d(u)\gamma a \in S \) where \( u \in U, a \in A \). Thus, we have \( u\gamma d(a) \in S \). Therefore, \( 0 = d(u\gamma d(a))\beta a = (u\gamma d^2(a) + d(u)\gamma d(a))\beta a \). Since \( M \) is a prime \( \Gamma \)-ring, it follows from Lemma 2.6 that

\[
u\gamma d^2(a) + d(u)\gamma d(a) = 0 \tag{3.27}
\]

for all \( u \in U; \gamma \in \Gamma \) and \( a \in A \). Replacing \( u \) by \( u\beta v \) where \( v \in U, \beta \in \Gamma \) in (3.27), we get, for all \( u, v \in U, \beta, \gamma \in \Gamma \) and \( a \in A \)

\[
d(u)\beta v\gamma d(a) = 0. \tag{3.28}
\]

Since \( M \) is a prime \( \Gamma \)-ring, we get \( d(U)\Gamma U = (0) \) or \( d(A)\Gamma U = (0) \). If \( d(A)\Gamma U = (0) \), then \( d^2(U)\Gamma U = (0) \). Let \( u, v \in U \) and \( \beta \in \Gamma \). Then \( 0 = d(d(u\beta v)) = u\beta d^2(v) + d(u)\beta d(v) + d(v)\beta d(u) + d^2(u)\beta v \), and so we have \( d(u)\beta d(v) = 0 \) for all \( u, v \in U \) and \( \beta \in \Gamma \) by \( \text{char} M \neq 2 \). Replacing \( u \) by \( u\gamma w \) where \( w \in U, \gamma \in \Gamma \) in last relation, we have \( d(u)\gamma w\beta d(v) = 0 \) which yields \( d(u)\gamma v = 0 \) for all \( u, v \in U \) and \( \gamma \in \Gamma \).

Conversely assume that \( d(U)\Gamma U = (0) \). Then \( AU = (0) \). Since \( M \) is a prime \( \Gamma \)-ring, \( A \) contains no non-zero right ideals. \( \square \)

Theorem 3.6. Let \( M \) be a prime \( \Gamma \)-ring with \( \text{char} M \neq 2 \), \( U \) a non-zero right ideal of \( M \) and \( d_1 \) and \( d_2 \) two non-zero derivations of \( M \). If \( d_1d_2(U) = (0) \), then there exists two elements \( p, q \) of \( Q \) such that \( q\Gamma U = (0) \) and \( p\Gamma U = (0) \).

Proof. If \( d_1d_2(U) = (0) \), then \( d_1(A) = (0) \) where \( A \) is a subring generated by \( d_2(U) \). Since \( d \neq 0 \), \( A \) contains no non-zero right ideals of \( M \). Thus, from Theorem 3.5, we have \( d_2(u)\gamma v = 0 \) for all \( u, v \in U \) and \( \gamma \in \Gamma \). Also, there exists \( q \in Q \) such that \( q\Gamma U = (0) \) by Theorem 3.4. Therefore \( d_2(u\gamma v) = u\gamma d_2(v) \) for all \( u, v \in U \) and \( \gamma \in \Gamma \). In this case, \( 0 = d_1d_2(u\gamma v) = d_1(u\gamma d_2(v)) = d_1(u)\gamma d_2(v) \), and since \( M \) is a prime \( \Gamma \)-ring, we get \( d_2(u)\gamma v = 0 \) for all \( u, v \in U \) and \( \gamma \in \Gamma \). Again, by Theorem 3.4, there exists \( p \in Q \) such that \( p\Gamma U = (0) \). This completes the proof. \( \square \)
Remark 3.7. (a) Consider the following example. Let $R$ be a ring. A derivation $d : R \to R$ is called an inner derivation if there exists $a \in R$ such that $d(x) = [a, x] = ax - xa$ for all $x \in R$. Let $S$ be the $2 \times 2$ matrix ring over Galois field $\{0, 1, w, w^2\}$, with inner derivations $d_1$ and $d_2$ defined by

$$d_1(x) := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad d_2(x) := \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix}, \quad x$$

for all $x \in S$. Then the characteristic of $S$ is 2 and we have $d_1 \neq 0$, $d_2 \neq 0$, $d_1d_2 = 0$ and $d_2^2 = 0$. Also, if we take

$$M := M_{1\times 2}(S) = \{(a, b) | a, b \in S\} \text{ and } \Gamma := \left\{ \begin{pmatrix} n \\ 0 \end{pmatrix} | n \text{ is an integer} \right\},$$

then $M$ is a prime $\Gamma$-ring of characteristic 2. Define an additive map $D_1 : M \to M$ by $D_1(x, y) = (d_1(x), d_1(y))$. Since $(x, y) \begin{pmatrix} n \\ 0 \end{pmatrix} (a, b) = (nxa, nxb)$, therefore $D_1$ is a derivation on $M$. Similarly $D_2 : M \to M$ given by $D_2(x, y) = (d_2(x), d_2(y))$ is a derivation. In this case, we have $D_1 \neq 0$, $D_2 \neq 0$, $D_1D_2 = 0$ and $D_2^2 = 0$ (see [7]). Thus we know that there exist two derivations $D_1$, $D_2$ of $M$ such that $D_1D_2(M) = (0)$ but $D_1(M)\Gamma M \neq (0)$ and $D_2(M)\Gamma M \neq (0)$. Therefore the condition of $\text{char}M \neq 2$ in Theorems 3.5 and 3.6 is necessary.

(b) In Theorems 3.4 and 3.6, if $a\gamma(c\beta b) = a\beta(c\gamma b)$ for all $a, b, c \in M$ and $\gamma, \beta \in \Gamma$, then $d(x) = [q, x]_\gamma = q\gamma x - x\gamma q$ for all $x \in M$, $\gamma \in \Gamma$ and for some $q \in Q$ is inner derivation and also $d_1(x) = [q, x]_\gamma$ and $d_2(x) = [q, x]_\beta$ for all $x \in M$, $\gamma, \beta \in \Gamma$ and for some elements $q, p \in Q$ are inner derivations by Lemma 2.5(i).

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