On Non-Existence of Korovkin’s Theorem in the Space of $L_p$–locally Integrable Functions

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Abstract

It is shown that a Korovkin-type theorem does not hold in the weighted space of $L_p$–locally integrable functions on the whole real axis.

Key words and phrases: Linear positive operators, Korovkin-type theorem, Weighted $L_p(\text{loc})$ space

1. The problem of convergence of sequences of linear positive operators in the space of functions, which are continuous on a finite interval $[a, b]$ and bounded on the whole real axis, was systematically investigated in Korovkin’s monograph [1]. Many generalizations and extensions of Korovkin’s classical theorem are known (we refer to monograph [2] for a bibliography). In particular, it was shown in papers [3] and [4]† that Korovkin’s theorem does not hold in the weighted spaces of functions $f$, which are continuous on the whole axis and satisfy the inequality $|f(x)| \leq M_f \rho(x)$, where $M_f$ is a positive constant depending on the function $f$ and $\rho(x) \geq 1$ is a continuous and increasing function on $(-\infty, \infty)$. This space is a linear normed space endowed with the norm

$$\|f\|_\rho = \sup_{-\infty < x < \infty} \frac{|f(x)|}{\rho(x)}.$$  

The aim of this paper is to investigate the existence of Korovkin-type theorems in the space of $L_p$–locally integrable functions. Note that the problem of convergence of

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† A. D. Gadjiev = A. D. Gadžiev (also, in other translated papers: A. D. Gadzhiev, A. D. Gadziev).
sequences of linear positive operators, acting from $L_p(a, b)$ to $L_p(a, b)$, has been studied by many authors. We refer the reader to the papers [5] – [10]. Note that all results mentioned are restricted to the case of the finite interval $[a, b]$.

We will consider the problem of convergence of sequences of linear positive operators in the space of locally integrable functions on the whole real axis.

Let $w(x) = 1 + x^2$, $-\infty < x < \infty$, and denote by $L_{p,w}(\text{loc})$ the space of measurable functions $f$ satisfying the inequality

$$
\left( \int_{-\infty}^{x} |f(t)|^p \, dt \right)^{1/p} \leq M_f w(x), \quad -\infty < x < \infty,
$$

where $p \geq 1$ and $M_f$ is a constant depending on the function $f$. Setting

$$
\|f\|_{p,w} = \sup_{-\infty < x < \infty} \left( \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} |f(t)|^p \, dt \right)^{1/p} w(x),
$$

we see that $L_{p,w}(\text{loc})$ is a linear normed space with this norm.

We will deal with the following problem.

Let $L_n$, $n = 1, 2, \ldots$, be a sequence of linear positive operators, acting from $L_{p,w}(\text{loc})$ to $L_{p,w}(\text{loc})$ and satisfying the following two conditions:

i) The norms of these operators are uniformly bounded;

ii) For $m = 0, 1, 2$

$$
\lim_{n \to \infty} \|L_n(t^m; x) - x^m\|_{p,w} = 0.
$$

(1.1)

Is it possible to assert then that for each function $f \in L_{p,w}(\text{loc})$

$$
\lim_{n \to \infty} \|L_n f - f\|_{p,w} = 0?
$$

An affirmative solution to this problem would lead to a Korovkin-type theorem in $L_{p,w}(\text{loc})$.

However, we are going to show that the answer is negative.
2. Main result

Our main result is the following.

**Theorem 1.** There exists a sequence of linear positive operators \( L_n \), acting from \( L_{p,w}(\text{loc}) \) to \( L_{p,w}(\text{loc}) \) and satisfying conditions i), ii), and there exists a function \( f^* \in L_{p,w}(\text{loc}) \) for which

\[
\lim_{n \to \infty} \|L_n f^* - f^*\|_{p,w} \geq 2^{1-\frac{1}{p}}.
\]

**Proof.** We define a sequence of operators \( L_n \) by the formulas

\[
L_n(f,x) = \begin{cases} 
\frac{x^2}{(x+\frac{1}{2})^2}f(x+\frac{1}{2}), & \text{if } (n-\frac{1}{2}) \leq x \leq n \\
f(x), & \text{otherwise.}
\end{cases}
\]

Obviously that \( L_n \) are linear positive operators, acting from \( L_{p,w}(\text{loc}) \) to \( L_{p,w}(\text{loc}) \) and

\[
\|L_n f\|_{p,w} \leq 4 \|f\|_{p,w}.
\]

Since

\[
\|L_n (y^m, t) - t^m\|_{p,w} \leq \sup_{(n-\frac{1}{2}) \leq x \leq n} \frac{(x + \frac{1}{2})^m}{1 + x^2} \leq \frac{(n + \frac{1}{2})^m}{1 + (n - \frac{1}{2})^2}
\]

for \( m = 0, 1 \) and \( L_n(t^2, x) = x^2 \), conditions (i) holds.

Consider the function

\[
f^*(x) = \begin{cases} 
x^2, & \text{if } x \in \bigcup_{k=1}^{\infty} [k - \frac{1}{2}, k) \\
-x^2, & \text{if } x \in \bigcup_{k=0}^{\infty} (k, k + \frac{1}{2}] \\
0, & \text{if } x < 0
\end{cases}
\]

which obviously belongs to \( L_{p,w}(\text{loc}) \). For \( n - \frac{1}{2} \leq y \leq n \) obviously \( f^*(y) = y^2 \), \( f^*(y + \frac{1}{2}) = -(y + \frac{1}{2})^2 \) and therefore
3. In this section we will give an affirmative statement on approximation in $L_{p,w}(\text{loc})$.

First of all, let $w_{\alpha}(x) = 1 + |x|^{2+\alpha}$, $\alpha > 0$, and let $L_{p,w_\alpha}(\text{loc})$ be the space of measurable functions $f$ with the finite norm

$$
\|f\|_{p,w_\alpha} = \sup_{-\infty < \xi < \infty} \frac{1}{w_\alpha(\xi)} \left( \int_{-\infty}^{\infty} |f(t)|^p dt \right)^{1/p}.
$$

Obviously, for any numbers $a, b$ ($a < b$)

$$
L_p(-\infty, \infty) \subset L_{p,w_\alpha}(\text{loc}) \subset L_{p,w}(\text{loc}) \subset L_p(a, b).
$$

Let also $CB(-\infty, \infty)$ be the space of all continuous and bounded functions $f$ on the whole real axis with the norm

$$
\|f\|_{CB} = \sup_{-\infty < x < \infty} |f(x)|.
$$
Lemma 1. Let \( f \in L_{p,w}(\text{loc}) \). Then given \( \varepsilon > 0 \) there exists a function \( g \in CB \left( -\infty, \infty \right) \) such that
\[
\| f - g \|_{p,w, \alpha} < \varepsilon
\]
for any \( \alpha > 0 \).

Proof. Using the inequality
\[
\sup_{|x| \leq x_0} \frac{1}{w_\alpha(x)} \left( \int_{x - \frac{1}{2}}^{x + \frac{1}{2}} |f(t)|^p \, dt \right)^{\frac{1}{p}} \leq \left( \int_{(x_0 - \frac{1}{2})}^{(x_0 + \frac{1}{2})} |f(t)|^p \, dt \right)^{\frac{1}{p}},
\]
and the well known Lusin Theorem, we can find a continuous function \( g_1 \) such that
\[
\sup_{|x| \leq x_0} \frac{1}{w_\alpha(x)} \left( \int_{x - \frac{1}{2}}^{x + \frac{1}{2}} |f(t) - g_1(t)|^p \, dt \right)^{\frac{1}{p}} < \varepsilon \tag{3.2}
\]
holds for any \( \varepsilon > 0 \).

Since by the definition of \( L_{p,w}(\text{loc}) \)
\[
\sup_{|x| > x_0} \frac{1}{w_\alpha(x)} \left( \int_{x - \frac{1}{2}}^{x + \frac{1}{2}} |f(t)|^p \, dt \right)^{\frac{1}{p}} \leq M_f \sup_{|x| > x_0} \frac{w(x)}{w_\alpha(x)}, \tag{3.3}
\]
we can choose \( x_0 > 0 \) so large that the inequality
\[
\sup_{|x| > x_0} \frac{w(x)}{w_\alpha(x)} < \varepsilon \tag{3.4}
\]
holds for any \( \varepsilon > 0 \).

Therefore, denoting by \( g \) a continuous and bounded function on the whole real axis, which coincides with \( g_1 \) on \( (-x_0 - \frac{1}{2}, x_0 + \frac{1}{2}) \), we complete the proof by using (3.2), (3.3) and (3.4).

Lemma 2. Let \( L_n \) be a sequence of linear positive operators acting from \( L_{p,w}(\text{loc}) \) to \( L_{p,w}(\text{loc}) \) and satisfying conditions (i) and (ii). Then for any \( f \in CB \left( -\infty, \infty \right) \)
\[
\lim_{n \to \infty} \| L_n f - f \|_{p,w, \alpha} = 0.
\]

211
Proof. We have
\[
\lim_{n \to \infty} \|L_n f - f\|_{p,w} \leq \|L_n (|f(y) - f(t)|, t)\|_{p,w} + \|f\|_{CB} \|L_n 1 - 1\|_{p,w}
\]
and the last term tends to zero by (1.1).

Consider the first term on the right hand side. Since \( f \) is continuous and bounded we can write the inequality [1] as
\[
|f(y) - f(t)| < \varepsilon + \frac{2 \|f\|_{CB}}{\delta^2} (y - t)^2
\]
and for \( x_0 \) satisfying (3.4) the following inequality holds:
\[
\|L_n (|f(y) - f(t)|, t)\|_{p,w} \leq (2 \|f\|_{CB} + 1) \|L_n 1\|_{p,w} \varepsilon
\]
\[
+ \frac{2 \|f\|_{CB}}{\delta^2} \sup_{|x| \leq x_0} \frac{1}{w(x)} \left( \int_{x - \frac{1}{\delta}}^{x + \frac{1}{\delta}} L_n^p \left( (y - t)^2, t \right) dt \right)^{\frac{1}{p}}
\]
It remains to note that by condition (i) the last term tends to zero as \( n \to \infty \) and the \( \|L_n 1\|_{p,w} \) are uniformly bounded.

**Theorem 2.** Let \( L_n \) be a sequence of linear positive operators acting from \( L_{p,w}(\text{loc}) \) to \( L_{p,w}(\text{loc}) \) as well as from \( L_{p,w_a}(\text{loc}) \) to \( L_{p,w_a}(\text{loc}) \) and satisfying conditions (i) and (ii). Then for any function \( f \in L_{p,w}(\text{loc}) \)
\[
\lim_{n \to \infty} \|L_n f - f\|_{p,w_a} = 0
\]
and the result fails to be true for \( \alpha = 0 \).

**Proof.** Using Lemma 1 and the uniform boundedness of \( \|L_n\| \) we have \( \|L_n\| \leq M \) and
\[
\|L_n f - f\|_{p,w_a} \leq \|L_n (f - g, t)\|_{p,w_a} + \|L_n g - g\|_{p,w_a} + \|f - g\|_{p,w_a}
\]
\[
\leq (M + 1) \|f - g\|_{p,w_a} + \|L_n g - g\|_{p,w_a}.
\]

The proof now follows from the Lemma 1 and Lemma 2. The last assertion of the theorem follows from Theorem 1.

212
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References


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