Characterizations of Artinian and Noetherian Gamma-Rings in Terms of Fuzzy Ideals

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Abstract

Using fuzzy ideals, characterizations of Noetherian Γ-rings are given, and a condition for a Γ-ring to be Artinian is also given.

Key words and phrases: (Artinian, Noetherian) Γ-ring, fuzzy left (right) ideal, Γ-residue class ring.

1. Introduction

The notion of a fuzzy set in a set was introduced by L. A. Zadeh [6], and since then this concept has been applied to various algebraic structures. N. Nobusawa [5] introduced the notion of a Γ-ring, a concept more general than a ring. W. E. Barnes [1] weakened slightly the conditions in the definition of Γ-ring in the sense of Nobusawa. W. E. Barnes [1], S. Kyuno [3] and J. Luh [4] studied the structure of Γ-rings and obtained various generalizations analogous to corresponding parts in ring theory. Y. B. Jun and C. Y. Lee [2] applied the concept of fuzzy sets to the theory of Γ-rings. In this paper, using fuzzy ideals, we discuss characterizations of Noetherian Γ-rings, and we give a condition for a Γ-ring to be Artinian.

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2. Preliminaries

Let \( M \) and \( \Gamma \) be two abelian groups. If for all \( x, y, z \in M \) and all \( \alpha, \beta \in \Gamma \) the conditions

- \( x \alpha y \in M \),
- \( (x + y)\alpha z = x\alpha z + y\alpha z, x(\alpha + \beta)z = x\alpha z + x\beta z, x\alpha (y + z) = x\alpha y + x\alpha z \),
- \( (x\alpha y)\beta z = x\alpha (y\beta z) \)

are satisfied, then we call \( M \) a \( \Gamma \)-ring. By a right (resp. left) ideal of a \( \Gamma \)-ring \( M \) we mean an additive subgroup \( U \) of \( M \) such that \( U \Gamma \subseteq U \) (resp. \( MTU \subseteq U \)). If \( U \) is both a right and a left ideal, then we say that \( U \) is an ideal of \( M \). Let \( U \) be an ideal of a \( \Gamma \)-ring \( M \). If for each \( a + U, b + U \) in the factor group \( M/U \), and each \( \gamma \in \Gamma \), we define \( (a + U)\gamma (b + U) = a\gamma b + U \); then \( M/U \) is a \( \Gamma \)-ring which is called the \( \Gamma \)-residue class ring of \( M \) with respect to \( U \) (see [3]). For any subsets \( A \) and \( B \) of a \( \Gamma \)-ring \( M \), by \( A \subseteq B \) we exclude the possibility that \( A = B \). A \( \Gamma \)-ring \( M \) is said to satisfy the left (right) ascending chain condition of left (right) ideals (or to be left (right) Noetherian) if every strictly increasing sequence \( U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots \) of left (right) ideals of \( M \) is of finite length. A \( \Gamma \)-ring \( M \) is said to satisfy the left (right) descending chain condition of left (right) ideals (or to be left (right) Artinian) if every strictly decreasing sequence \( V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots \) of left (right) ideals of \( M \) is of finite length. A \( \Gamma \)-ring \( M \) is said to be left (resp. right) Noetherian if \( M \) satisfies the left (right) ascending chain condition on left (resp. right) ideals. \( M \) is said to be Noetherian if \( M \) is both left and right Noetherian. A \( \Gamma \)-ring \( M \) is left (resp. right) Artinian if \( M \) satisfies the left (right) descending chain condition on left (resp. right) ideals. \( M \) is said to be Artinian if \( M \) is both left and right Artinian.

We now review some fuzzy logic concepts. A fuzzy set \( \mu \) in a \( \Gamma \)-ring \( M \) is called a fuzzy left (resp. right) ideal of \( M \) ([2]) if it satisfies

- (F11) \( \mu(x - y) \geq \min\{\mu(x), \mu(y)\} \)
- (F12) \( \mu(x\gamma y) \geq \mu(y) \) (resp. \( \mu(x\gamma y) \geq \mu(x) \))

for all \( x, y \in M \) and \( \gamma \in \Gamma \). A fuzzy set \( \mu \) in a \( \Gamma \)-ring \( M \) is called a fuzzy ideal of \( M \) if \( \mu \) is both a fuzzy left and a fuzzy right ideal of \( M \). We note from [2] that if \( \mu \) is a fuzzy
left (right) ideal of a Γ-ring $M$ then $\mu(0) \geq \mu(x)$ for all $x \in M$, and $\mu$ is a fuzzy ideal of a Γ-ring $M$ if and only if it satisfies (FI1) and

(FI3) $\mu(x\gamma y) \geq \max\{\mu(x), \mu(y)\}$ for all $x, y \in M$ and $\gamma \in \Gamma$.

3. Main results

**Theorem 3.1.** Let $U$ be an ideal of a Γ-ring $M$. If $\mu$ is a fuzzy left (right) ideal of $M$, then the fuzzy set $\tilde{\mu}$ of $M/U$ defined by

$$\tilde{\mu}(a + U) = \sup_{x \in U} \mu(a + x)$$

is a fuzzy left (right) ideal of the Γ-residue class ring $M/U$ of $M$ with respect to $U$.

**Proof.** Let $a, b \in M$ be such that $a + U = b + U$. Then $b = a + y$ for some $y \in U$, and so

$$\tilde{\mu}(b + U) = \sup_{x \in U} \mu(b + x) = \sup_{x \in U} \mu(a + y + x) = \sup_{x + y = z \in U} \mu(a + z) = \tilde{\mu}(a + U).$$

Hence $\tilde{\mu}$ is well-defined. For any $x + U, y + U \in M/U$ and $\gamma \in \Gamma$, we have

$$\tilde{\mu}((x + U) - (y + U)) = \tilde{\mu}((x - y) + U) = \sup_{z \in U} \mu((x - y) + z)$$

$$= \sup_{z = u - v \in U} \mu((x - y) + (u - v))$$

$$= \sup_{u, v \in U} \mu((x + u) - (y + v))$$

$$\geq \sup_{u, v \in U} \min\{\mu(x + u), \mu(y + v)\}$$

$$= \min\left\{\sup_{u \in U} \mu(x + u), \sup_{v \in U} \mu(y + v)\right\}$$

$$= \min\left\{\tilde{\mu}(x + U), \tilde{\mu}(y + U)\right\}$$

and

$$\tilde{\mu}((x + U)\gamma(y + U)) = \tilde{\mu}(x\gamma y + U) = \sup_{z \in U} \mu(x\gamma y + z)$$

$$\geq \sup_{z \in U} \mu(x\gamma y + x\gamma z)$$

because $x\gamma z \in U$

$$= \sup_{z \in U} \mu(x\gamma (y + z)) \geq \sup_{z \in U} \mu(y + z)$$

$$= \tilde{\mu}(y + U).$$
Similarly, $\tilde{\mu}(x + U)\gamma(y + U) \geq \tilde{\mu}(x + U)$. Hence $\tilde{\mu}$ is a fuzzy left (right) ideal of $M/U$.

\begin{proof}
Let $\mu$ be a fuzzy left ideal of $M$. Using Theorem 3.1, we find that $\tilde{\mu}$ defined by $\tilde{\mu}(a + U) = \sup_{x \in U} \mu(a + x)$ is a fuzzy left ideal of $M/U$. Since $\mu(0) = \mu(u)$ for all $u \in U$, we get $\mu(a + u) \geq \min\{\mu(a), \mu(u)\} = \mu(a)$.

Again, $\mu(a) = \mu(a + u - u) \geq \min\{\mu(a + u), \mu(u)\} = \mu(a + u)$. Hence $\mu(a + u) = \mu(a)$ for all $u \in U$, that is, $\tilde{\mu}(a + U) = \mu(a)$. Therefore the correspondence $\mu \mapsto \tilde{\mu}$ is injective. Now let $\tilde{\mu}$ be any fuzzy left ideal of $M/U$ and define a fuzzy set $\mu$ in $M$ by $\mu(a) = \tilde{\mu}(a + U)$ for all $a \in M$. For every $x, y \in M$ and $\gamma \in \Gamma$, we have

$$
\begin{align*}
\mu(x - y) &= \tilde{\mu}((x - y) + U) = \tilde{\mu}(x + U) - (y + U) \\
&\geq \min\{\tilde{\mu}(x + U), \tilde{\mu}(y + U)\} = \min\{\mu(x), \mu(y)\},
\end{align*}
$$

and $\mu(x\gamma y) = \tilde{\mu}(x\gamma y + U) = \tilde{\mu}(x + U)\gamma(y + U) \geq \tilde{\mu}(y + U) = \mu(y)$. Thus $\mu$ is a fuzzy left ideal of $M$. Note that $\mu(z) = \tilde{\mu}(z + U) = \tilde{\mu}(U)$ for all $z \in U$, which shows that $\mu(z) = \mu(0)$ for all $z \in U$. This completes the proof.
\end{proof}

\textbf{Theorem 3.3.} If every fuzzy left ideal of a $\Gamma$-ring $M$ has finite number of values, then $M$ is left Artinian.

\begin{proof}
Suppose that every fuzzy left ideal of a $\Gamma$-ring $M$ has finite number of values and $M$ is not left Artinian. Then there exists strictly descending chain $U_0 \supset U_1 \supset U_2 \supset \cdots$ of left ideals of $M$. Define a fuzzy set $\mu$ in $M$ by

$$
\mu(x) = \begin{cases} 
\frac{n}{n+1} & \text{if } x \in U_n \setminus U_{n+1}, \ n = 0, 1, 2, \cdots, \\
1 & \text{if } x \in \bigcap_{n=0}^{\infty} U_n,
\end{cases}
$$

where $U_0$ stands for $M$. Let us prove that $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in M$. Let $x, y \in M$. Then $x - y \in U_n \setminus U_{n+1}$ for some $n$ ($n = 0, 1, 2, \cdots$), and so either $x \notin U_{n+1}$
or $y \notin U_{n+1}$. So for definiteness, let $y \in U_k \setminus U_{k+1}$ for $k \leq n$. It follows that

$$\mu(x - y) = \frac{n}{n+1} \geq \frac{k}{k+1} = \min\{\mu(x), \mu(y)\}.$$ 

Next, let us show that $\mu(x\gamma y) \geq \mu(y)$ for all $x, y \in M$ and $\gamma \in \Gamma$. There exists a non-negative integer $n$ such that $x\gamma y \in U_n \setminus U_{n+1}$. Then $y \notin U_{n+1}$, and hence $y \in U_k \setminus U_{k+1}$ for $k \leq n$. Hence

$$\mu(x\gamma y) = \frac{n}{n+1} \geq \frac{k}{k+1} = \mu(y).$$

Therefore $\mu$ is a fuzzy left ideal of $M$ and $\mu$ has infinite number of different values. This contradiction proves that $M$ is a left Artinian $\Gamma$-ring.

**Theorem 3.4.** A $\Gamma$-ring $M$ is left Noetherian if and only if the set of values of any fuzzy left ideal of $M$ is a well ordered subset of $[0, 1]$.

**Proof.** Suppose that $\mu$ is a fuzzy left ideal of $M$ whose set of values is not a well ordered subset of $[0, 1]$. Then there exists a strictly decreasing sequence $\{\lambda_n\}$ such that $\mu(x_n) = \lambda_n$. Denote by $U_n$ the set $\{x \in M \mid \mu(x) \geq \lambda_n\}$. Then $U_1 \subset U_2 \subset U_3 \subset \cdots$ is a strictly ascending chain of left ideals of $M$, which contradicts that $M$ is left Noetherian.

Conversely, assume that the set of values of any fuzzy left ideal of $M$ is a well ordered subset of $[0, 1]$ and $M$ is not a left Noetherian $\Gamma$-ring. Then there exists a strictly ascending chain

$$U_1 \subset U_2 \subset U_3 \subset \cdots \quad (3.1)$$

of left ideals of $M$. Note that $U := \bigcup_{i \in \mathbb{N}} U_i$ is a left ideal of $M$, where $\mathbb{N}$ is the set of all natural numbers. Define a fuzzy set $\mu$ in $M$ by

$$\mu(x) = \begin{cases} 
0 & \text{if } x \notin U_i, \\
\frac{1}{k} & \text{where } k = \min\{i \in \mathbb{N} \mid x \in U_i\}.
\end{cases}$$

It can be easily seen that $\mu$ is a fuzzy left ideal of $M$. Since the chain (3.1) is not terminating, $\mu$ has a strictly descending sequence of values, contradicting that the value set of any fuzzy left ideal is well ordered. Consequently, $M$ is left Noetherian. \hfill \Box
Lemma 3.5. ([2, Theorem 3]) A fuzzy set $\mu$ in a $\Gamma$-ring $M$ is a fuzzy left (right) ideal of $M$ if and only if for every $\lambda \in [0, 1]$, the set $U(\mu; \lambda) := \{ x \in M \mid \mu(x) \geq \lambda \}$ is a left (right) ideal of $M$ when it is nonempty.

Lemma 3.6. Let $S = \{ \lambda_n \in (0, 1) \mid n \in \mathbb{N} \} \cup \{ 0 \}$, where $\lambda_i > \lambda_j$ whenever $i < j$. Let $\{ U_n \mid n \in \mathbb{N} \}$ be a family of left ideals of a $\Gamma$-ring $M$ such that $U_1 \subset U_2 \subset U_3 \subset \cdots$. Then a fuzzy set $\mu$ in $M$ defined by

$$
\mu(x) = \begin{cases} 
\lambda_1 & \text{if } x \in U_1, \\
\lambda_n & \text{if } x \in U_n \setminus U_{n-1}, \ n = 2, 3, \cdots, \\
0 & \text{if } x \in M \setminus \bigcup_{n=1}^{\infty} U_n,
\end{cases}
$$

is a fuzzy left ideal of $M$.

Proof. Using Lemma 3.5, the proof is straightforward.

Theorem 3.7. Let $S = \{ \lambda_1, \lambda_2, \cdots, \lambda_n, \cdots \} \cup \{ 0 \}$ where $\{ \lambda_n \}$ is a fixed sequence, strictly decreasing to 0 and $0 < \lambda_n < 1$. Then a $\Gamma$-ring $M$ is left Noetherian if and only if for each fuzzy left ideal $\mu$ of $M$, $\text{Im}(\mu) \subset S$ implies that there exists $n_0 \in \mathbb{N}$ such that $\text{Im}(\mu) \subset \{ \lambda_1, \lambda_2, \cdots, \lambda_{n_0} \} \cup \{ 0 \}$.

Proof. If $M$ is left Noetherian, then $\text{Im}(\mu)$ is a well ordered subset of $[0, 1]$ by Theorem 3.4 and so the condition is necessary by noticing that a set is well ordered if and only if it does not contain any infinite descending sequence. Conversely, if possible let $M$ be not left Noetherian. Then there exists a strictly ascending chain of left ideals of $M$ $U_1 \subset U_2 \subset U_3 \subset \cdots$. Define a fuzzy set $\mu$ in $M$ by

$$
\mu(x) = \begin{cases} 
\lambda_1 & \text{if } x \in U_1, \\
\lambda_n & \text{if } x \in U_n \setminus U_{n-1}, \ n = 2, 3, \cdots, \\
0 & \text{if } x \in M \setminus \bigcup_{n=1}^{\infty} U_n.
\end{cases}
$$

Then, by Lemma 3.6, $\mu$ is a fuzzy left ideal of $M$. This contradicts our assumption. Hence $M$ is left Noetherian. \hfill \Box

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