On Summand Sum and Summand Intersection Property of Modules

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Abstract

$R$ will be an associative ring with identity and modules $M$ will be unital left $R$–modules. In this work, extending modules and lifting modules with the SSP (or SIP) are studied. A necessary and sufficient condition for a module $M$ to have the SSP is that for every decomposition $M = A \oplus B$ and $f \in \text{Hom}(A, B)$, $\text{Im}(f)$ is a direct summand of $B$. Among others it is shown also that a $(C_3)$ module with the SIP has the SSP, and a $(D_3)$ module with SSP has the SIP.

Key Words: SIP modules, SSP modules, extending modules, lifting modules.

Throughout this work all rings will be associative with identity and modules will be unital left modules. Let $R$ be a ring and $M$ a module. $N \leq M$ will mean $N$ is submodule of $M$. A submodule $N$ of a module $M$ is called small in $M$, denoted by $N<\ll M$, whenever for some submodule $L$ of $M$, $N + L = M$ implies $L = M$. A module $M$ is said to be small if $M$ is small in its injective hull $E(M)$. $0 \neq N \leq M$ is said to be an essential submodule of $M$, denoted by $N \的本质 M$, if for every $0 \neq L \leq M$, $N \cap L \neq 0$. We write $N \leq_d M$ to abbreviate $N$ is a (direct) summand of $M$.

We recall some definitions and properties as follows

(SSP) A module $M$ has the summand sum property (SSP) if the sum of two direct summands is a direct summand of $M$;

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(C₁) Every submodule of $M$ is essential in a summand of $M$;

(C₂) If a submodule $A$ of $M$ is isomorphic to a summand of $M$, then $A$ is summand of $M$; and

(C₃) If $M_1$ and $M_2$ are summands of $M$ such that $M_1 \cap M_2 = 0$ then $M_1 \oplus M_2$ is a summand of $M$.

A submodule $N$ of $M$ is said to be closed in $M$ if there is no proper essential extension of $N$ in $M$ and denoted by $N \leq c M$. Modules with (C₁) are called extending (or CS)-modules. A module $M$ is an extending module if and only if every closed submodule in $M$ is direct summand of $M$. A module $M$ is called quasi-continuous if $M$ has (C₁) and (C₃), and continuous if $M$ has (C₁) and (C₂). We then have

(C₂) $\Rightarrow$ (C₃), SSP $\Rightarrow$ (C₃) and continuous $\Rightarrow$ quasi-continuous.

(SIP) An $R$–module $M$ has the summand intersection property (SIP) if the intersection of two summands is again a summand, and $M$ has the strong summand intersection property (SSIP) if the intersection of any number of summands is again a summand.

Now recall the conditions ($D_i$) dual of the conditions ($C_i$) respectively:

($D_1$) For every submodule $A$ of a module $M$, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $A \cap M_2 << M_2$.

($D_2$) If $A \leq M$ such that $M/A$ is isomorphic to a summand of $M$, then $A$ is a summand of $M$.

($D_3$) If $A$ and $B$ are summands of $M$ with $A + B = M$, then $A \cap B$ is summand of $M$.

Modules with ($D_1$) are called lifting and modules with ($D_1$) and ($D_2$) are called discrete, and modules with ($D_1$) and ($D_3$) are called quasi-discrete modules.

We have the implications ($D_2$) $\Rightarrow$ ($D_3$), SIP $\Rightarrow$ ($D_3$), Discrete $\Rightarrow$ Quasi-discrete.

Modules having the SSP and the SIP were motivated by the works of Kaplansky and Fuchs. Kaplansky proves in his book [6] that if $M$ is a free module over a principal ideal domain $R$, then $M$ has the SIP. And Fuchs suggested the following problem in his book Infinite Abelian Groups.

Problem 9 Characterize the abelian groups in which the intersection of two direct summands is again a summand.
So arose naturally the problem of modules having SSP and their endomorphism rings if they have the SSP or the SIP. Garcia studied this problem in [4] while Wilson studied modules having SIP over Noetherian domains in [11].

In this note we study $D_i$-modules ($i = 1, 2, 3$) with SIP and $C_i$-modules ($i = 1, 2, 3$) with the SSP. We start with Example 1 below.

There exist modules with $D_2$ but have neither the SIP nor the SSP.

**Example 1** Let $F$ be a field and let $R$ denote the following ring:

$$R = \begin{pmatrix} a & 0 & 0 & 0 \\ y & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & x & a \end{pmatrix} : a, b, x, y \in F$$

We consider $R$ as a left $R$-module. Then $R$ satisfies $(D_2)$ since every projective module satisfies $(D_2)$. We show that $R$ does not have neither the SIP nor the SSP. Let

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ b & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & x & 0 \end{pmatrix} : b, x \in F$$

and

$$K = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & x & 0 \end{pmatrix} : b, x \in F$$

be left ideals of $R$. Then $N$ and $K$ are direct summands of $R$, and since $N \cap K = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 \end{pmatrix}$; $x \in F$ is nilpotent the left ideal, $N \cap K$ is not a direct summand of $R$. It is easy to check that the left ideal $N + K = \begin{pmatrix} 0 & 0 & 0 & 0 \\ u & v & 0 & 0 \\ 0 & 0 & v & 0 \\ 0 & 0 & z & 0 \end{pmatrix}$; $u, v, z \in F$ is a proper essential left ideal of $R$ and so not a direct summand of $R$. Then $R$ does not have the SSP.

We state and prove Lemma 2 for an easy reference.
Lemma 2 [7] Let $M_1$ be a simple module and $M_2$ an uniserial module with composition series $0 \subset U \subset M_2$. Then $M = M_1 \oplus M_2$ is a lifting module.

Proof. Let $L$ be a non-zero submodule of $M$. We show that there exists a submodule $K$ of $M$ such that $M = K \oplus L$. Suppose that $M_1 \not\subset (L + M_2)$. Then $M_1 \not\subset L + M_2$ and $M = L + M_2$. If $L \cap M_2 = M_2$ or $L \cap M_2 = 0$ or $L \cap M_1 = M_1$ we are done. Assume $L \cap M_2 = U$ and $L \cap M_1 = 0$. Then $U \leq L$. Hence $M = L \oplus M_1$. Thus $M$ has $(D_1)$. □

There are modules having the SSP and $(D_1)$ but not the SIP.

Example 3 Let $F$ be a field and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ be the ring of upper triangular matrices over $F$, $N = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$ and $L = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ left ideals of $R$ and $M = R/L$. Let $U = N \oplus M$. Then by [4, Remark on page 81] and Lemma 2, $U$ has the SSP and $(D_1)$ but has not the SIP as left $R$-module.

There are modules having the SIP but not the SSP.

Example 4 Let $F$ be a field and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ be the ring of upper triangular matrices over $F$, $N = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$ and $L = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ left ideals of $R$ and $M = R/L$. Let $U = N \oplus M$. Then by [4, Remark on page 81] the ring $S = \text{End} U$ has the SIP on each side but does not have the SSP on the left.

Example 5 Let $M$ denote the $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}$. Let $N$ be a submodule of $M$. It is easy to check that $N$ is a direct summand of $M$ if and only if $N$ has the form $N = \mathbb{Z}(a,b)$ for some integers $a,b$ with the property that the greatest common divisor of $a$ and $b$ is 1. Consider the submodules $\mathbb{Z}(2,3)$ and $\mathbb{Z}(3,2)$ of $M$. Then they are direct summands of $M$ and $[\mathbb{Z}(2,3)] \cap [\mathbb{Z}(3,2)] = 0$, and it is clear that $\mathbb{Z}(2,3) \oplus \mathbb{Z}(3,2)$ is not a direct summand.
of $M$. Hence $M$ has not the SSP. Also, for any two distinct direct summands $K$ and $N$ of $M$ their intersection $K \cap N$ is always zero. It follows that $M$ has the SIP.

As an easy reference we record the following properties of modules with the SIP and the SSP from [4, 11]

**Proposition 6** (i) $M$ has the SIP (resp. the SSP) if and only if for every pair of summand $S$ and $T$ with $\pi : M \to S$ the projection map, the kernel of the restricted map $\pi|_T$ (resp. the image of the restricted map $\pi|_T$) is summand.

(ii) If $M$ has the SIP (resp. the SSP) and $S \oplus T$ is summand of $M$, then the kernel of any homomorphism from $S$ to $T$ (resp. the image of any homomorphism from $S$ to $T$) is a summand.

**Proposition 7** [3] The $R$-module $M$ has the summand intersection property if and only if for every decomposition $M = A \oplus B$ and every homomorphism $f$ from $A$ to $B$, the kernel of $f$ is a direct summand.

One way of the following Theorem is given as an exercise 39.17 (3) (i) in [12] on page 339 and it is proved in [4]. We prove the other way.

**Theorem 8** The $R$-module $M$ has the summand sum property if and only if for every decomposition $M = A \oplus B$ and every homomorphism $f$ from $A$ to $B$, the image of $f$ is a direct summand of $B$.

**Proof.** The necessity is proved in [4]. For the sufficiency assume that for every decomposition $M = A \oplus B$ and every homomorphism $f$ from $A$ to $B$, the image of $f$ is a direct summand of $B$. Let $N$ and $K$ be direct summands of $M$ and $M = N \oplus N'$ and $M = K \oplus K'$ for some $N' \leq M$ and $K' \leq M$. We prove $N + K$ is direct summand. Let $\pi_K$ and $\pi_{N'}$ denote the projections of $M$ onto $K$ and $N'$, respectively. Let $A$ denote $\pi_{N'}(\pi_K(N))$. Then $A = (N + K') \cap (N + K) \cap N'$ and, by assumption, $A$ is a direct summand and $M = A \oplus L$ for some $L \leq M$. Hence $N' = A \oplus (N' \cap L)$. Then $(N + K) \cap [(N + K') \cap (N' \cap L)] = [(N + K) \cap (N + K') \cap N'] \cap (N' \cap L) = A \cap (N' \cap L) = 0$. To show that $N + K$ is direct summand, it is enough to prove that $M = (N + K) + [(N + K') \cap (N' \cap L)]$. Since $A \leq N + K$ and $A \leq N + K'$, the modular law and $M = N \oplus N' = (N \oplus A) \oplus (N' \cap L)$ imply $N + K = (N \oplus A) \oplus [(N + K) \cap (N' \cap L)]$ and, $N + K' = (N \oplus A) \oplus [(N + K') \cap (N' \cap L)]$. 135
Hence $M = N + K' + K = (N \oplus A) + [(N + K) \cap (N' \cap L)] + [(N + K') \cap (N' \cap L)] \subseteq (N + K) + [(N + K') \cap (N' \cap L)]$. Thus $N + K$ is direct summand and so $M$ has the SSP \( \square \)

We use Theorem 8 to prove the following Theorem 9 and 10 which are Exercises 39.17 (3)(ii) and (iii) in the book [12] on Page 339.

**Theorem 9** Let $R$ be a ring. The following are equivalent for $R$:

1. $R$ is semisimple
2. Every $R$-module has the SSP
3. Every projective $R$-module has the SSP.

**Proof.** (1) $\implies$ (2) $\implies$ (3) is trivial. Assume that (3) holds. We show that $R$ is semisimple. Let $K$ be a submodule of $R$. Choose a free module $F$ and an epimorphism $\tau$ from $F$ onto $K$. By assumption, the projective module $F \oplus R$ has the SSP. Let $\iota$ denote the injection map from $K$ to $R$ and $f = \iota \tau$ the homomorphism from $F$ to $R$. Then $\text{Im} f = K$ is a direct summand of $R$ by Theorem 8. Hence $R$ is semisimple ring. \( \square \)

**Theorem 10** A ring $R$ is left hereditary if and only if every injective $R$-module has the SSP.

**Proof.** Suppose that $R$ is a left hereditary ring. The every factor module of every injective $R$-module is injective. Let $M$ be an injective module which has a decomposition $M = A \oplus B$. Let $f$ be a homomorphism from $A$ to $B$. Then $A$ is injective. By assumption, $\text{Im} f \cong A/\text{Ker} f$ is injective. Hence $\text{Im} f$ is direct summand of $B$. Thus it follows from Theorem 8 that $M$ has the SSP. To prove the converse assume that every injective $R$-module has the SSP. Let $M$ be an injective module and $N$ a submodule of $M$. By assumption the injective hull $E(M/N)$ of $M/N$ and the injective module $M \oplus E(M/N)$ have the SSP. Let $\phi$ denote the canonical mapping from $M$ onto $M/N$ and $\iota$ the injection of $M/N$ into $E(M/N)$ and $f$ the composition of $\iota \phi$. Then $\text{Im} f = M/N$. By Theorem 8, $M/N$ is direct summand of $E(M/N)$. Hence $M/N$ is injective. Thus $R$ is a left hereditary ring. \( \square \)
Let $N \leq M$. Whenever $N \leq_{\text{ess}} K \leq M$ implies $N = K$, $N$ is called (essentially) closed in $M$ and we denote by $N \leq_{c} M$. A module $M$ is said to be a polyform module if for every $K \leq M$ and $f \in \text{Hom}(K, M)$ $\text{Ker}f \leq_{c} K$ (see [2, 12]).

**Lemma 11** Let $M$ be an extending polyform module. Then $M$ has the SIP.

**Proof.** Let $M$ be an extending polyform module, and let $M = A \oplus B$ be a decomposition of $M$ and $f \in \text{Hom}(A, B)$. Being $M$ polyform, $\text{Ker}(f)$ is closed in $K$. Then $\text{Ker}(f)$ is direct summand as a closed submodule of an extending module $M$. Hence $M$ has the SIP. \(\square\)

A module $M$ is said to be copolyform if for $B \leq A \leq M$ and $A/B \ll M/B$ implies $\text{Hom}(M/B, A/Y) = 0$ for $B \leq Y \leq A$ (see [5]).

**Lemma 12** Let $M$ be a lifting coplyform module. Then $M$ has the SSP.

**Proof.** Let $M$ be lifting coplyform module, and let $A$ and $B$ be direct summands of $M$ and $\pi$ projection from $M$ onto $A$. Let $K$ denote the image $\pi_{B}(B)$ of the restriction of $\pi$ to $B$. Since $A$ is lifting module as a direct summand of $M$, there exists a decomposition $A = K_{1} \oplus K_{2}$ such that $K_{1} \leq K$ and $K \cap K_{2} \ll K_{2}$. Then $K \cap K_{2}$ is also small in $A$ and $M$ and $K = K_{1} \oplus (K \cap K_{2})$. Hence we have a mapping from $M$ onto $K \cap K_{2}$. Since $M$ is coplyform, $K \cap K_{2} = 0$ and so $K = K_{1}$ is direct summand of $A$. \(\square\)

We consider the following conditions for a module $M$.

If $M_{1} \leq_{d} M$, $M_{2} \leq_{d} M$ with $M_{1} + M_{2} \leq_{\text{ess}} M$, then $M_{1} + M_{2} = M$ ....... (*)

If $M_{1} \leq_{d} M$, $M_{2} \leq_{d} M$ with $M_{1} \cap M_{2} \ll M$, then $M_{1} \cap M_{2} = 0$ ....... (**) 

**Lemma 13** Let $M$ be a module. If $M$ satisfies (*)&(**) then each direct summand of $M$ satisfies (*)&(**).

**Proof.** Assume that the module $M$ satisfies (*). Let $A$ be a direct summand such that $M = A \oplus B$ for some $B \leq M$ and $A_{1}$ and $A_{2}$ summands of $A$ with $A_{1} + A_{2} \leq_{\text{ess}} A$. Then $A_{2} + B$ and $A_{1}$ are direct summands of $M$ and $A_{1} + (A_{2} + B) \leq_{\text{ess}} M$. Hence $A_{1} + (A_{2} + B) = M$ and so $A_{1} + A_{2} = A$. The remaining is proved dually. \(\square\)
Proposition 14 Let $M$ be an extending module.

1. $M$ has the SSP.

2. $M$ satisfies (*).

3. For any two direct summands $M_1$ and $M_2$ of $M$ and for each homomorphism $f$
   from $M_1$ to $M_2$ with $\text{Im} f \leq_{\text{ess}} M_2$, $\text{Im} f = M_2$.

Then (1) $\iff$ (2) and (3) $\implies$ (1).

Proof. (1) $\implies$ (2) Clear.

(2) $\implies$ (1). Assume that $M$ satisfies (*) and let $M_1$ and $M_2$ be direct summands of $M$. We prove that $M_1 + M_2$ is a direct summand. Being $M$ extending module there exists a direct summand $A$ of $M$ such that $M_1 + M_2$ is essential in $A$ and $M = A \oplus B$ for some submodule $B$ in $M$. By Lemma 13 $A = M_1 + M_2$.

(3) $\implies$ (1). Assume that $M = A \oplus B$ is a decomposition with a homomorphism $f$
from $A$ to $B$. We show that $f(A)$ is a direct summand of $B$. $f(A)$ is either summand of $B$ or contained essentially in a closed submodule $C$.

If $f(A)$ is a direct summand of $B$, there is nothing to prove in this case. Assume that $f(A)$ is contained essentially in a closed submodule $C$ of $B$. By hypothesis $C$ is a direct summand of $M$ and so is that of $B$, and then $B = C \oplus C'$ for some $C' \leq B$. Define the homomorphism $f \oplus 1$ from $A \oplus C'$ to $B$ by $(f \oplus 1)(a + c') = f(a) + c'$ where $a \in A$ and $c' \in C'$. Then $\text{Im}(f \oplus 1) = f(A) \oplus C'$ is essential in $C \oplus C'$. By (3) $f \oplus 1$ is epimorphism and so $f(A) = C$. Therefore, $(A)$ is a direct summand. \qed

Note that in Proposition 14 (1) $\implies$ (3) is not true in general. In fact let $M$ denote the
\Z-module $\Z$ and $M_1 = M_2 = M$. It is known that $M$ is an extending module and has the
SSP. Consider $f$ as the map defined by $f(n) = 2n$ for $n \in M_1$. Then $\text{Im} f = \Z 2 \leq_{\text{ess}} M_2$ and $\text{Im} f \neq M_2$.

Proposition 15 Let $M$ be a lifting module. Then

1. $M$ has the SIP.

2. $M$ satisfies (**).
3. For any two direct summands $M_1$ and $M_2$ of $M$ and for each homomorphism $f$ from $M_1$ to $M_2$ with $\text{Ker}(f) << M_1$, $\text{Ker}(f) = 0$.

Then $(1) \iff (2)$ and $(3) \implies (1)$.

**Proof.** $(1) \implies (2)$. It is trivial.

$(2) \implies (1)$. Assume that $M$ satisfies $(**)$. Let $M_1$ and $M_2$ be direct summands of $M$. We prove $M_1 \cap M_2$ is also a direct summand. We separate two cases:

If $M_1 \cap M_2$ is small in $M$ then by $(**) M_1 \cap M_2 = 0$.

Suppose that $M_1 \cap M_2$ is not small in $M$. Being $M$ lifting module there exists a direct summand $A$ of $M$ such that $A \leq M_1 \cap M_2$, $M = A \oplus B$ and $(M_1 \cap M_2) \cap B << B$ for some $B \leq M$. Then $(M_1 \cap M_2) \cap B << M$, $M_1 \cap B \leq_d B$, $M_2 \cap B \leq_d B$ and $(M_1 \cap B) \cap (M_2 \cap B) << B$. By Lemma 13, $(M_1 \cap B) \cap (M_2 \cap B) = 0$. Hence $M_1 \cap M_2 = A$.

$(3) \implies (1)$. To prove $M$ has the SIP we use Proposition 7 and assume that $M$ has the decomposition $M = A \oplus B$ and a homomorphism $f$ from $A$ to $B$. We show that $\text{Ker}(f)$ is a direct summand. Now we have two cases:

(i) If $\text{Ker}(f) << A$, then by hypothesis we have $\text{Ker}(f) = 0$.

(ii) Assume that $\text{Ker}(f)$ is not small in $A$. Being $M$ lifting module there exists $C \leq \text{Ker}(f)$ such that $A = C \oplus C'$ and $C' \cap \text{Ker}(f) << A$. Now we define the homomorphism $1 \oplus f : A = C \oplus C' \to C \oplus B$ by $(1 \oplus f)(c + c') = c + f(c')$ where $c \in C$ and $c' \in C'$. Then $\text{Ker}(1 \oplus f) = C' \cap \text{Ker}(f)$. Since $\text{Ker}(1 \oplus f) << A$, we have $\text{Ker}(1 \oplus f) = 0$. On the other hand, $\text{Ker}(f) = C \oplus (C' \cap \text{Ker}(f)) = C$ is a summand of $A$. This gives that $M$ has the SIP. □

Note that the implication $(1) \implies (3)$ in Proposition 15 is not valid in general. Let $M$ denote the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ and $M_1 = M_2 = M$. It is known that $M$ is a lifting module and has the SIP. Let $K$ be a proper submodule of $M$. Then $M/K \cong M$. Consider $\pi$ as the canonical map from $M$ onto $M/K$ defined by $\pi(m) = m + K$ for $m \in M_1$. Let $g$ denote the isomorphism $M/K \cong M$ and set $f = g\pi$. Then $\text{Ker}(f)$ is small in $M$ and non-zero submodule of $M_1$.

Let $M$ be a module. It is well known that for any submodule $N$ of $M$ there exists a closed submodule $K$ such that $N \leq_{ess} K$ and $K$ is called a closure of $N$ in $M$. The module $M$ is called UC-module in case every submodule of $M$ has a unique closure ( see
For $B \leq A \leq M$, $B$ is said to be \textit{coessential} submodule of $A$ or $A$ is \textit{coessential extension} of $B$ if $A/B << M/B$. $A$ is said to be \textit{coclosed} in $M$ if $A$ has no coessential submodule in $M$. Let $B \leq A \leq M$. Then $B$ is called a \textit{coclosure} of $A$ in $M$ if $B$ is coclosed in $M$ and $B$ is coessential in $A$. Suppose that every submodule $A$ of $M$ has a coessential submodule $A^{ec}$ which is contained in every coessential submodule of $A$ in $M$. We call $M$ a \textit{unique coclosure module} or \textit{UCC-module}. Recall that a submodule $A$ of $M$ is said to lie over a direct summand $B$ if $M$ has a decomposition $M = B \oplus C$, such that $B \leq A$ and $A/B << M/B$. It is known that a module $M$ is a UCC-module if and only if every submodule of $M$ lies over a unique direct summand. In this direction Lemma 16 is proved in [4].

**Lemma 16** [4] Let $M$ be a lifting module. Then $M$ has SSP if and only if $M$ is UCC-module.

We state Lemma 17 as a dual to Lemma 16 and a generalization of an exercise mentioned in Anderson-Fuller’s book (page 214, exercise 7). Note that Lemma 17 is also generalizes Proposition 4 of [11].

**Lemma 17** Let $M$ be an extending module. Then the following are equivalent:

1. $M$ is UC-module.
2. $M$ has the SIP.
3. $M$ has the SSIP.

**Proof.** (1) $\implies$ (2) Let $M$ be a UC-module. Let $N$ and $K$ be direct summands of $M$. Then $N \cap K$ is closed in $M$ by Lemma 6 in [10]. By hypothesis $N \cap K \leq_d M$.

(2) $\implies$ (1) Assume that $M$ has the SIP. Let $N \leq M$. Suppose that there are $K \leq M$ and $L \leq M$ such that $N \leq_e K \leq_e M$ and $N \leq_e L \leq_e M$. We prove $K = L$. By hypothesis $K \leq_d M$ and $L \leq_d M$ and by (2) $(K \cap L) \oplus T = M$ for some $T \leq M$. Hence $K = (K \cap L) \oplus (K \cap T)$. Since $N \leq_e K$ and $N \cap (K \cap T) = 0$, $K \cap T = 0$. Hence $K = K \cap L$. Similarly, it is shown that $L = K \cap L$. Therefore $K = K \cap L = L$.

(3) $\implies$ (2). Clear.

(1) $\implies$ (3). Assume that $M$ is UC-module and let $K_i (i \in I)$ be direct summands of $M$. Then every $K_i$ for $i \in I$ is closed, and so by assumption and Lemma 8 (9) in [10]
$\bigcap_{i \in I} K_i$ is closed in $M$. By hypothesis $\bigcap_{i \in I} K_i$ is a direct summand. It completes the proof. □

**Proposition 18** Let $M$ be a quasi-continuous module. The following are equivalent:

1. $M$ has the SSIP.
2. $M$ has the SIP.
3. $E(M)$ has the SIP.
4. $E(M)$ has the SSIP.

**Proof.** (1) $\Leftrightarrow$ (2) and (3) $\Leftrightarrow$ (4) clear from Lemma 17.

(3) $\Rightarrow$ (2) Suppose $E(M)$ has the SIP. Let $A$ and $B$ be direct summands of $M$. Then there exist $A'$ and $B'$ such that $M = A \oplus A'$ and $M = B \oplus B'$. Then we have that $E(M) = E(A) \oplus L$ and $E(M) = E(B) \oplus L'$ for some submodules $L$ and $L'$ of $E(M)$. Since $E(M)$ has the SIP, $E(M) = [E(A) \cap E(B)] \oplus K$ for some $K \leq E(M)$. Therefore, $M = [(E(A) \cap E(B)) \cap M] \oplus (K \cap M)$ by [8, Theorem 2.8]. Now $A \leq_e E(A)$ and $B \leq_e E(B)$ imply $A \leq_e E(A) \cap M$ and $B \leq_e E(B) \cap M$, and since $E(A) \cap M = A \oplus (E(A) \cap M) \cap A'$ and $E(B) \cap M = B \oplus (E(B) \cap M) \cap B'$ it follows that $A = E(A) \cap M$ and $B = E(B) \cap M$. Hence $A \cap B = E(A) \cap E(B) \cap M$ is a direct summand of $M$.

(2) $\Rightarrow$ (3) Assume $M$ has the SIP and let $A$ and $B$ be direct summands of $E(M)$ and $E(M) = A \oplus A'$ and $E(M) = B \oplus B'$ for some $A' \leq E(M)$ and $B' \leq E(M)$ and $A = E(A)$ and $B = E(B)$. By [8, Theorem 2.8] $A \cap M$ and $B \cap M$ are direct summands of $M$. By assumption $A \cap B \cap M$ is direct summand of $M$, and so $(A \cap B \cap M) \oplus L = M$ for some $L \leq M$. Since $A \cap M \leq_e A$ and $B \cap M \leq_e B$ $A \cap B \cap M \leq_e A \cap B$. Hence $E(M) = E(A \cap B \cap M) \oplus E(L) = E(A \cap B) \oplus E(L)$. Therefore, $A = E(A \cap B) \oplus (E(L) \cap A)$ and $B = E(A \cap B) \oplus (E(L) \cap B)$. Then $E(A \cap B) \leq A \cap B \leq E(A \cap B)$ implies $A \cap B = E(A \cap B)$ is a direct summand of $E(M)$. □

It is proved in [4] that a quasi-injective (resp. quasi-projective) module with the SIP (resp. the SSP) has the SSP (resp. the SIP). In this direction, we prove the following Lemma.
Lemma 19 Let $M$ be a module.

1. Let $M$ be a $(C_3)$ module. If $M$ has the SIP then $M$ has the SSP.

2. Let $M$ be a $(D_3)$ module. If $M$ has the SSP then $M$ has the SIP.

Proof. (1). Let $M$ be a $(C_3)$ module. Assume $M$ has the SIP. Let $N$ and $T$ be a direct summands of $M$. We show that $N + T$ is direct summand of $M$. Since $M$ has the SIP then there exists $L \leq M$ such that $(N \cap T) \oplus L = M$. By modularity law, we get that $N = (N \cap T) \oplus (L \cap N)$ and $T = (N \cap T) \oplus (L \cap T)$. Then we have $N + T = (N \cap T) + [(L \cap N) \oplus (L \cap T)]$. Next we prove that $(N \cap T) [((L \cap N) \oplus (L \cap T))] = 0$. For if, $x \in (N \cap T) \cap [(L \cap N) \oplus (L \cap T)]$, then $x = n_1 + n_2$ where $n_1 \in L \cap N$ and $n_2 \in L \cap T$. We have $n_2 = x - n_1 \in [(N \cap T) + (L \cap N)] \cap (L \cap T) \leq N \cap (L \cap T) = 0$. Hence $n_2 = 0$ and $x = n_1$. Now $x = n_1 \in (N \cap T) \cap (L \cap N) = N \cap T \cap L = 0$. Thus $N + T = (N \cap T) \oplus (L \cap N) \oplus (L \cap T) = T \oplus (L \cap N)$. Since $M$ has the SIP and $L, N$ are direct summands then $L \cap N$ is a direct summand and so by $(C_3)$ it follows that $N + T = T \oplus (L \cap N)$ is a direct summand of $M$. Thus $M$ has the SSP.

(2). Let $M$ be a $(D_3)$ module. Assume $M$ has the SSP. Let $X$ and $Y$ be direct summands of $M$. We prove that $X \cap Y$ is a direct summand of $M$. Since $M$ has the SSP then $X + Y$ is a direct summand, and so there exists $Z \leq M$ such that $M = (X + Y) \oplus Z$. Since $X$, $Y$ and $Z$ are direct summands and $M$ has the SSP then $X + Z$ and $Y + Z$ are direct summands, and since $M$ is $(D_3)$ and $M = (X + Z) + (Y + Z)$ then $(X + Z) \cap (Y + Z)$ is direct summand, and so there exists $U \leq M$ such that $M = [(X + Z) \cap (Y + Z)] \oplus U$. Now $(X + Z) \cap (Y + Z) = [X \cap (Y + Z)] + Z$ and $X \cap (Y + Z) \leq X \cap Y$ and $M = [(X + Z) \cap (Y + Z)] \oplus U$ imply $M = (X \cap Y) \oplus Z \oplus U$. □

Corollary 20 Let $M$ be a module having the SIP. Then $M$ is $(C_3)$ module if and only if $M$ has the SSP.

Proof. Let $M$ be a module having the SIP. Assume that $M$ is $(C_3)$ module. Then by Lemma 19 $M$ has the SSP. The converse is clear since every module having the SSP is a $(C_3)$ module. □
Note that the converse statements (1) and (2) in Lemma 19 need not be true in general. There are $(C_3)$ modules with the SSP but not the SIP. Namely the module in Example 3 is a module having the SSP and therefore $(C_3)$ but does not have the SIP.

There are $(D_3)$ modules having the SIP but not the SSP.

**Example 21** Let $K$ be a field and $M$ denote the left $R$-module $R = \begin{pmatrix} K & 0 & K \\ 0 & K & 0 \\ 0 & 0 & K \end{pmatrix}$.

Let $e_{ij}$ denote the matrix units in $R$. Then it is easy to check that $A = R(e_{11} + e_{13})$, $B = Re_{22}$, $A \oplus B$, $C = R(e_{11} + e_{22})$, $D = R(e_{13} + e_{22} + e_{33})$, $E = R(e_{13} + e_{33})$, $F = Re_{11}$ and $G = R(e_{11} + e_{33})$ are only direct summands of $M$ and their intersections are also direct summands and $A \oplus B \oplus F$ is an essential submodule of $M$. Then $M$ has the SIP. Also $M$ has $(D_3)$ as a projective module over $R$. Now $A \cap C = 0$ and $A \oplus C = A \oplus B \oplus F$ is not a direct summand. Hence $M$ does not have the SSP.

It is proved in [4] that for any ring $R$ and any module $M$, $M$ has the SSP and the SIP if and only if $S = \text{End} M$ has the SSP. Now we prove Theorem 22 that also generalizes Corollary 2.4 in [4].

**Theorem 22** Let $M$ be a module. Then

1. If $M$ has $(D_3)$ then $M$ has the SSP if and only if $S = \text{End} M$ has the SSP.
2. If $M$ has $(C_3)$ then $M$ has the SIP if and only if $S = \text{End} M$ has the SSP.

**Proof.** (i) Assume $S$ has the SSP. Then $M$ has the SSP and SIP.

Assume $M$ has the SSP. Since $M$ has $(D_3)$ then by lemma 19, $M$ has the SIP. Then $S$ has the SSP.

(ii) Assume $M$ has the SIP. Then by lemma 19, $M$ has the SSP and so $M$ has the SIP and SSP implies $S$ has the SSP.

Assume $S$ has the SSP. Then $M$ has the SSP and SIP by [4, Theorem 2.3].

Let $M$ be a module. The submodule $Z(M) = \{ m \in M : l(m) \leq_{css} M \}$ is called singular submodule of $M$. In case $Z(M) = 0$, $M$ is called nonsingular module.
Corollary 23 Assume $M$ is nonsingular quasi-continuous module with $S = \text{End}(M)$. Then $S$ has the SSP as a right $S$-module.

Proof. Let $M$ be a nonsingular quasi-continuous module with a decomposition $M = A \oplus B$ and $f \in \text{Hom}(A, B)$. Since $Z(M) = 0$ it is easy to prove that Ker($f$) is closed in $M$. Hence Ker($f$) is direct summand of $M$ since $M$ is an extending module. By Proposition 7 $M$ has the SIP, and by Lemma 19, $M$ has the SSP. Then from [4, Theorem 2.3], $S$ has the SSP as a right $S$-module. □

Let $M$ be a module. Let $N << M$. Then $N$ is a small module, that is $N$ is small submodule of $E(N)$ and also $E(M)$. In the subsequent $Z^*(M)$ will denote the submodule \( \{ m \in M : Rm << E(M) \} \) of $M$ (see [9]).

Corollary 24 Let $M$ be a quasi-discrete module with $Z^*(M) = 0$ and $S = \text{End}(M)$. Then $S$ has the SIP as a right $S$-module.

Proof. Let $M$ be a quasi-discrete module and assume $Z^*(M) = 0$ and $A$ a submodule of $M$. Then there exists a direct summand $B$ such that $M = B \oplus B'$ with $B \leq A$ and $A \cap B'$ is small in $M$, and hence $A \cap B' \leq Z^*(M) = 0$. It follows that $A = B$ and $A$ is direct summand. Thus $M$ is semisimple module and so $M$ has the SIP and the SSP. By [12, 37.7] $S$ is regular ring in the sense of von Neumann. Let $I = eS$ and $I' = fS$ be right ideals of $S$ that are direct summands of $S$ for some idempotents $e$ and $f$ of $S$. Then $eM \cap fM$ is direct summand of $M$ as $M$ has the SIP. If $\alpha$ is the orthogonal projection of $M$ on $eM \cap fM$ then it is easy to check that $\alpha S = eS \cap fS$. Thus $eS \cap fS$ is a direct summand of $S$. □

Lemma 25 Let $R$ be a commutative Noetherian ring and $M = M_1 \oplus M_2$ with indecomposable submodules $M_1$ and $M_2$. Assume that $M$ has the $(C_3)$ and the SIP, then

1. $\text{Hom}(M_1, M_2) = 0$ or

2. $M_1$ is isomorphic to $M_2$ and there is some prime ideal $A \leq R$ with $\text{ann}(x) = A$ for every nonzero $x \in M_1$.  

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Proof. Take $0 \neq f \in \text{Hom}(M_1, M_2)$. Since $\text{Ker}(f)$ is a direct summand of $M_1$ we have $\text{Ker}(f) = 0$. Similarly $\text{Im}f$ is direct summand of $M_2$ since $M_1 \oplus M_2$ has the SSP. Hence $f$ is onto and so $M_1$ is isomorphic to $M_2$.

It remains to show the conditions on annihilators. Let $x, y \in M_1$ be nonzero and assume that there is $a \in \text{ann}(x)$ but $a$ is not in $\text{ann}(y)$. Define $g : M_1 \rightarrow M_2$ by $g(m) = f(am)$ for $m \in M_1$. Then $x \in \text{Ker}(g)$ and $y$ is not in $\text{Ker}(g)$. Hence $\text{Ker}(g) \neq 0$ and $g \neq 0$. This is a contradiction. Hence $a \in \text{ann}(x)$ implies $a \in \text{ann}(y)$ or $\text{ann}(x) = \text{ann}(y)$. Then $\text{ann}(x)$ is prime follows from [6, Theorem 6].

Theorem 26 Let $M$ have a decomposition $M = M_1 \oplus M_2$ with $M_1$ local module and $M_2$ simple module.

1. Assume $\text{Hom}(M_1, M_2) \neq 0$. Then $M$ has not the SIP.

2. Assume $\text{Hom}(M_2, M_1) \neq 0$. Then $M$ has not the SSP.

Proof. (1). Assume that $M = M_1 \oplus M_2$ has the SIP. Let $f \in \text{Hom}(M_1, M_2)$ be a nonzero homomorphism. Then $\text{Ker}(f) \neq 0$. Since $M$ has the SIP, by Proposition 7 $\text{Ker}(f)$ is a direct summand of $M_1$. This gives a contradiction. Therefore, $M$ have not the SIP.

(2). Suppose that $M = M_1 \oplus M_2$ has the SSP. Let $f \in \text{Hom}(M_2, M_1)$ be a nonzero homomorphism. Then $\text{Im}f \neq M_1$. Since $M$ has the SSP, by Theorem 8 $\text{Im}f$ is a direct summand of $M_1$. This is not possible. It follows that $M$ has not the SSP.

Corollary 27 Let $M$ have a decomposition $M = M_1 \oplus M_2$ with $M_1$ uniserial module and $M_2$ simple module.

1. Assume $\text{Hom}(M_1, M_2) \neq 0$. Then $M$ has not the SIP.

2. Assume $\text{Hom}(M_2, M_1) \neq 0$. Then $M$ has not the SSP.

Proof. Clear.

The following example is known. We study here as an illustration of Theorem 26.
Example 28 Let $p$ be a prime integer. Let $M_1 = \mathbb{Z}/Zp^2$ and $M_2 = \mathbb{Z}/Zp$ be $\mathbb{Z}-$modules and $M = M_1 \oplus M_2$. Then $M$ has neither the SIP nor the SSP.

Proof. Let $f : M_1 \rightarrow M_2$ be defined by $f(x + Zp^2) = y + Zp$ where $x + Zp^2 \in M_1$ and $y + Zp \in M_2$ and $y$ is the remainder when $x$ is divided by $p$. Then $\text{Ker}(f) = M_1 p$ which is not a direct summand of $M_1$. Hence $M$ has not the SIP. Let $f : M_2 \rightarrow M_1$ be defined by $f(x + Zp) = px + Zp^2$ where $x + Zp \in M_2$. Then $\text{Im}(f) = M_1 p$ which is not a direct summand. Hence $M$ has not the SSP. \hfill \square

Theorem 29 Let $M$ be a module with $S = \text{End}(M)$.

1. If $M$ is $(C_2)$-module then $M \oplus M$ has the SIP if and only if $S$ is regular ring.

2. If $M$ is $(D_2)$-module then $M \oplus M$ has the SSP if and only if $S$ is regular ring.

Proof. (1). Let $M$ be $(C_2)$-module. Necessity: Assume that the module $M \oplus M$ has the SIP. Let $f \in S$. Then $f$ is a homomorphism from a direct summand of $M \oplus M$ to a direct summand of $M \oplus M$. By assumption and Proposition 7, $\text{Ker}(f)$ is direct summand of $M$. Then $\text{Im}(f)$ is isomorphic to a direct summand of $M$. By $(C_2)$, $\text{Im}(f)$ is direct summand of $M$. Thus $S$ is a regular ring from [12, 37.7]. Sufficiency: Suppose that $S = \text{End}(M)$ is a regular ring. By [12, 37.9 (c)], $\text{End}(M \oplus M)$ is also regular ring as a $2 \times 2$ matrix ring over the regular ring $S$, and so $\text{Ker}(f)$ of every $f \in \text{End}(M \oplus M)$ is a direct summand of $M \oplus M$. Hence $M \oplus M$ has the SIP by Proposition 7. Thus $M$ has the SIP as a direct summand of $M \oplus M$.

(2). Let the module $M$ has $(D_2)$. Necessity: Assume now that $M \oplus M$ has the SSP. Let $f \in S$. By assumption and by Propostion 8, $\text{Im}(f)$ is a direct summand of $M$. Since $\text{Im}(f) \cong M/\text{Ker}(f)$ and $M$ has the $(D_2)$, $\text{Ker}(f)$ is a direct summand of $M$. By [12, 37.7] $S$ is a regular ring. The proof of sufficiency of (2) is proved in the same way as the sufficiency of (1). This completes the proof. \hfill \square

References


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