Diagonal Lift in the Cotangent Bundle and its Applications

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Abstract

The purpose of this paper is to define a diagonal lift $\mathcal{D} g$ of a Riemannian metric $g$ of a manifold $M$ to the cotangent bundle $T^*(M)$ of $M$, to associate with $\mathcal{D} g$ an Levi-Civita connection of $T^*(M)$ in a natural way and to investigate applications of the diagonal lifts.

Key words and phrases: Cotangent bundle, Riemannian metric, diagonal lift, Levi-Civita connection, B-manifold, Killing vector field, geodesic.

1. Introduction

Let $M$ be an $n$-dimensional differentiable manifold of class $C^\infty$ and $T^*(M)$ the cotangent bundle over $M$. If $x^i$ are local coordinates in a neighborhood $U$ of a point $x \in M$, then a covector $p$ at $x$ which is an element of $T^*(M)$ is expressible in the form $(x^i, p_i)$, where $p_i$ are components of $P$ with respect to the natural frame $\partial_i$. We may consider $(x^i, p_i) = (\tilde{x}^j, \tilde{p}_j)$, $i = 1, \ldots, n; \, \tilde{j} = n + 1, \ldots, 2n; \, J = 1, \ldots, 2n$ as local coordinates in a neighborhood $\pi^{-1}(U)$ ($\pi$ is the natural projection $T^*(M)$ onto $M$).

Let now $M$ be a Riemannian manifold with nondegenerate metric $g$ whose components in a coordinate neighborhood $U$ are $g_{ij}$ and denote by $\Gamma^k_{ij}$ the Christoffel symbols formed with $g_{ij}$.

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We denote by $T^r_s(M_n)$ the module over $F(M_n)$ ($F(M_n)$ is the ring of $C^\infty$ functions in $M_n$) all tensor fields of class $C^\infty$ and of type $(r,s)$ in $M_n$. Let $X \in T^1_0(M_n)$ and $w \in T^0_1(M_n)$. Then $C^X$ (complete lift), $H^X$ (horizontal lift) and $V^w$ (vertical lift) have, respectively, components [5]

$$
C^X = \begin{pmatrix} X^h \\ -p_m \partial_h X^m \end{pmatrix}, \quad H^X = \begin{pmatrix} X^h \\ p_m \Gamma^m_{hi} X^i \end{pmatrix}, \quad V^w = \begin{pmatrix} 0 \\ w_h \end{pmatrix}
$$

(1)

with respect to the coordinates $(x^h, x^m)$ in $T^*(M_n)$, where $X^h$ and $w_h$ are respectively local components of $X$ and $w$.

In each coordinate neighborhood $U(x^h)$ of $M_n$, we put

$$
X_{(j)} = \frac{\partial}{\partial x^j}, \quad w^{(j)} = dx^j.
$$

Taking account of (1), we easily see that the components of $H^X_{(j)}$ and $V^w_{(j)}$ are respectively given by

$$
H^X_{(j)} = (A^H_{(j)}) = \begin{pmatrix} \delta^h_j \\ p_m \Gamma^m_{jh} \end{pmatrix}, \quad V^w_{(j)} = (A^V_{(j)}) = \begin{pmatrix} 0 \\ \delta^j_h \end{pmatrix}
$$

with respect to the coordinates $(x^h, x^m)$. We call the set $\{H^X_{(j)}, V^w_{(j)}\}$ the frame adapted to the Riemannian connection $\Gamma$ in $\pi^{-1}(U) \subset T^*(M_n)$. On putting

$$
A_{(j)} = H^X_{(j)}, \quad A_{(j)}^{H} = V^w_{(j)}
$$

we write the adapted frame as $\{A_{(j)}\} = \{A_{(j)}, A_{(j)}^{H}\}$.

It is easily verified that $2n$ local 1-forms

$$
\tilde{A}^{(i)}_H = \left( \tilde{A}^{(i)}_H \right)_h = \left( \tilde{A}^{(i)}_h \right)_H = \delta^i_h, 0 = dx^i \quad i = 1, \ldots, n,
$$

$$
\tilde{A}^{(i)} = \left( \tilde{A}^{(i)}_H \right)_h = \left( \tilde{A}^{(i)}_h \right)_H = (-p_m \Gamma^m_{hi} \delta_h i) = dp_i - p_m \Gamma^m_{hi} dx^h = \delta^i_p, \quad \tilde{T} = n + 1, \ldots, 2n
$$

(2)

form a coframe $\{\tilde{A}^\alpha\} = \{\tilde{A}^{(i)}, \tilde{A}^{(i)}_H\}$ dual to the adapted frame $\{A_{(j)}\}$, i.e. $\tilde{A}^{(\alpha)} H A_{(j)}^{H} = \delta^{\alpha}_{\beta}$. 492
2. Lift $\mathcal{D} g$ of a Riemannian $g$ to $T^* (M_n)$

On putting locally

$$\mathcal{D} g = g_{ji} \tilde{A}^{(j)} \otimes \tilde{A}^{(i)} + \sum_{j,i=1}^{n} g^{ji} \tilde{A}^{(j)} \otimes \tilde{A}^{(i)}$$

in $T^* (M_n)$, we see that $\mathcal{D} g$ defines a tensor field of type $(0, 2)$ in $T^* (M_n)$, which called the diagonal lift of the tensor field $g$ to $T^* (M_n)$ with respect to $\Gamma$. From (2) and (3) we prove that $\mathcal{D} g$ has components of the form

$$\mathcal{D} g_{j\alpha} = \begin{pmatrix} g_{ji} & 0 \\ 0 & g^{ji} \end{pmatrix}$$

with respect to the coframe $\{ \tilde{A}^{(\alpha)} \}$ (or with respect to the adapted frame $\{ A_{(\alpha)} \}$) in $T^* (M_n)$ and components

$$\mathcal{D} g = \begin{pmatrix} g_{ji} + g^{ks} p_m \Gamma_{jk}^m \Gamma_{is}^f & -g^{is} p_i \Gamma_{js}^f \\ -g^{is} p_i \Gamma_{js}^f & g^{ji} \end{pmatrix}$$

with respect to the local coordinates $(x^j, \tilde{x}^j)$, where $g^{ji}$ denote contravariant components of $g$.

From (4) it easily follows that if $g$ is a Riemannian metric in $M_n$, then $\mathcal{D} g$ is a Riemannian metric in $T^* (M_n)$. The metric $\mathcal{D} g$ is similar to that of the Riemannian extension studied by S. Sasaki in the tangent bundle [4] (for the frame bundle, see [2]).

From (1) and (5) we have

$$\mathcal{D} g (H X, H Y) = g (X, Y).$$

We hence have

**Theorem 1.** Let $X, Y \in T^1_0 (M_n)$. Then the inner product of the horizontal lifts $H X$ and $H Y$ to $T^* (M_n)$ with the metric $\mathcal{D} g$ is equal to the vertical lift of the inner product of $X$ and $Y$ in $M_n$.

From (1) and (5) we have also

$$\mathcal{D} g (V w, V \theta) = V (g (w, \theta)), \quad \forall w, \theta \in T^1_1 (M_n),$$
\[ \mathcal{D}_g(V, C) = -g^{ij} w_j \partial_{w_i} (\partial_{w^i} X^j + \Gamma^i_{\alpha \beta} X^\alpha) \]

\[ = -g^{ij} w_j \langle (\nabla X) \rangle_s \]

\[ = -V (g(w, \langle \nabla X \rangle), \]

\[ \mathcal{D}_g(C, C) = g_{ij} x^i g^j + g^{ij} p_k p_i (\nabla_j X^k)(\nabla_i Y^j) \]

\[ = g_{ij} x^i g^j + g^{ij} \langle \nabla X \rangle_j \langle \nabla Y \rangle_i \]

\[ = V (g(X, Y)) + V (g(\langle \nabla X \rangle, \langle \nabla Y \rangle)), \forall X, Y \in T_0^1(M_n), \forall w \in T_0^0(M_n), \]

(6)

where \( \langle \nabla X \rangle \) is a 1-form with local expression:

\[ \langle \nabla X \rangle = p_t \nabla_s X^t \, dx^s. \]

We recall that any element \( t \in T_0^0(T^*(M_n)) \) is completely determined by its action on lifts of the type \( C X_1, C X_2, \ldots, C X_r \), where \( X_i, i = 1, \ldots, r \) are arbitrary vector fields in \( M_n \) [5, p. 237]. Then \( \mathcal{D} \) is completely determined by (6).

3. Levi-Civita Connection of \( \mathcal{D}_g \)

The components of the adapted frame \( \{ A_{(i)} \} \) are given by

\[ (A_{(i)})^H = (A_j^H, A_j^H) = \begin{pmatrix} \delta^b_j & 0 \\ p_m \Gamma^m_{j b} & \delta^j_b \end{pmatrix}. \]  

(7)

The indices \( \alpha, \beta, \ldots, 1, \ldots, 2n \) indicate the indices with respect to the adapted frame. The inverse of the matrix (7) is given by

\[ (\tilde{A}^b_i)_H = \begin{pmatrix} A^i_H \\ \tilde{A}^i_H \end{pmatrix}, \]

(7')

\( \tilde{A}^i_H \) and \( \tilde{A}^i_H \) being defined by

\[ \tilde{A}^i_H = (\delta^i_h, 0), \tilde{A}^i_H = (-p_m \Gamma^{m}_{h j}, \delta^j_h). \]

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We now consider local vector fields \( A_\beta \) defined in \( \pi^{-1}(U) \) by

\[
A_\beta = A_\beta H \partial_H,
\]

which will give for various types of indices

\[
A_j = \partial_j + \sum_{h=1}^{n} p_m \Gamma_{jh}^m \partial_m, \quad A_T = \partial_T.
\]  

(8)

If \( \Omega_\gamma^\beta \alpha \) denote the non-holonomic object with respect to the vector fields \( A_\beta \), then we have

\[
[A_\gamma, A_\beta] = \Omega_\gamma^\beta \alpha A_\alpha.
\]

According to (7), (7') and (8), the components of the non-holonomic object are given by

\[
\Omega_\gamma^\beta \alpha = (A_\gamma A_\beta C - A_\beta A_\gamma C) A_\alpha^C.
\]

The only non-vanishing components of \( \Omega_\gamma^\beta \alpha \) are

\[
\begin{align*}
\Omega_{jh}^m &= p_m R_{jih}^m, \\
\Omega_{ij}^m &= -p_m R_{jih}^m, \\
\Omega_{j}^i &= -\Gamma_{jh}^i, \\
\Omega_{ij}^h &= \Gamma_{jh}^i,
\end{align*}
\]  

(9)

where \( R_{jih}^m \) are components of the curvature tensor of \( \Gamma \) with metric \( g_{ij} \).

Components of the Riemannian connection determined by the metric \( \mathcal{D}g \) are given by

\[
\mathcal{D}\Gamma^\alpha_{\gamma\beta} = \frac{1}{2} \mathcal{D}g^{\alpha\gamma}(A_\gamma \mathcal{D}g_{\epsilon\beta} + A_\beta \mathcal{D}g_{\gamma\epsilon} - A_\epsilon \mathcal{D}g_{\gamma\beta}) + \frac{1}{2}(\Omega_{\gamma\beta}^\alpha + \Omega_{\alpha\gamma}^\beta + \Omega_{\alpha\beta}^\gamma),
\]

(10)

where \( \Omega_{\gamma\beta}^\alpha = \mathcal{D}g^{\alpha\epsilon} \mathcal{D}g_{\delta\beta} \Omega_{\epsilon\gamma}^\delta, \mathcal{D}g^{\alpha\epsilon} \) are the contravariant components of the metric \( \mathcal{D}g \) with respect to the adapted frame:

\[
(\mathcal{D}g^{\beta\alpha}) = \begin{pmatrix}
g^{ij} & 0 \\
0 & g_{ji}
\end{pmatrix}
\]

(11)
Thus, according to (8), (9), (10) and (11), the components $D_{\Gamma}^{\alpha}_{\gamma}$ with respect to adapted frame are given by

\[
\begin{align*}
D_{\Gamma}^{\alpha}_{\gamma} &= \Gamma^{\alpha}_{\gamma}, \\
D_{\Gamma}^{\alpha}_{\gamma} &= \frac{1}{2} p_{m} R^{h}_{j m}, \\
D_{\Gamma}^{\alpha}_{\gamma} &= \frac{1}{2} p_{m} R^{h}_{j m}, \\
D_{\Gamma}^{\alpha}_{\gamma} &= -\Gamma^{\alpha}_{\gamma}, \quad D_{\Gamma}^{\alpha}_{\gamma} = 0,
\end{align*}
\]

where $R^{h}_{j m} = g^{hl} g^{km} R_{j h k}^{m}$.

The covariant derivative of the diagonal lift $D_{\Gamma}^{\alpha}(\varphi \in T^{1}_{1}(M_{n}))$ has components

\[
D_{\Gamma}^{\alpha}(\varphi \in T^{1}_{1}(M_{n})) = \frac{1}{2} p_{m} R^{h}_{j m}, \\
D_{\Gamma}^{\alpha}(\varphi \in T^{1}_{1}(M_{n})) = -\Gamma^{\alpha}_{\gamma}, \quad D_{\Gamma}^{\alpha}(\varphi \in T^{1}_{1}(M_{n})) = 0
\]

with respect to the adapted frame, where the components of $D_{\Gamma}^{\alpha}$ are given by [5, p. 291]

\[
D_{\Gamma}^{\alpha}(\varphi \in T^{1}_{1}(M_{n})) = \left(\begin{array}{c|c}
\phi^{i}_{j} & 0 \\
0 & -\phi^{j}_{i}
\end{array}\right)
\]

with respect to the adapted frame.

Let us consider a 2n-dimensional Riemannian manifold $M_{2n}$ with the almost complex structure $\varphi$. If tensor of Riemann metric $g_{ij}$ satisfies

\[
g_{m j} \varphi_{i}^{m} = g_{l m} \varphi_{i}^{m},
\]

then we call this Riemann metric a pure metric in an almost complex manifold $M_{2n}$ and call an almost $\mathcal{B}$-manifold an almost complex space with a pure metric.

Now, if a pure metric satisfies

\[
\nabla_{j} F_{i}^{h} = 0 \quad \text{or} \quad \phi_{k} g_{ij} = 0,
\]

where $\phi$ is the Tachibana operator, then we call this manifold a $\mathcal{B}$-manifold (see [1], [3]) ($\nabla_{k}$ denotes the covariant differentiation with respect to the Christoffel symbols formed with $g_{ij}$). Taking account of (8) and (12), we find that $D_{\Gamma}^{\alpha}(\varphi \in T^{1}_{1}(M_{n}))$ has components given
by

\[ D \nabla_k D \varphi^i_j = \nabla_k \varphi^i_j, \]
\[ D \nabla_k D \varphi^i_j = -\nabla_k \varphi^j_i, \]
\[ D \nabla_k D \varphi^i_j = \frac{1}{2} \rho_j \left( R^i_{m \ell} \varphi^m_{\ell j} - R^m_{j \ell} \varphi^i_{\ell m} \right), \]
\[ D \nabla_k D \varphi^i_j = -\frac{1}{2} \rho_j \left( R^i_{k \ell} \varphi^m_{\ell j} + R^m_{j \ell} \varphi^i_{\ell k} \right), \]
\[ D \nabla_k D \varphi^i_j = \frac{1}{2} \rho_j \left( R^i_{km} \varphi^m_{\ell j} + R^m_{j \ell} \varphi^i_{km} \right). \]

all the others being zero, with respect to the adapted frame.

From (11) and (13) we easily find that \( D g \) is pure with respect to the structure \( D \varphi \).

Thus we have

**Theorem 2.** The cotangent bundle of \( \mathcal{B} \)-manifold is \( \mathcal{B} \)-manifold with respect to the metric \( D g \) and the structure \( D \varphi \) if and only if the Riemannian manifold is locally flat.

4. Killing vector fields

A vector field \( X \in T^0_0 \left( M_n \right) \) is said to be an infinitesimal isometry or a Killing vector field of a Riemannian manifold with metric \( g \), if \( \mathcal{L}_X g = 0 \) [5, p78]. In terms of components \( g_{ji} \) of \( g \), \( X \) is a Killing vector field if and only if

\[ \mathcal{L}_X g_{ji} = X^\alpha \nabla_\alpha g_{ji} + g_{\alpha i} \nabla_j X^\alpha + g_{j\alpha} \nabla_i X^\alpha = \nabla_j X_i + \nabla_i X_j = 0, \]

\( X^\alpha \) being components of \( X \), where \( \nabla \) is the Riemannian connection of the metric \( g \).

Let \( \tilde{X} \) be a covector field in \( T^* \left( M_n \right) \) and

\[ \left( \tilde{X}_\alpha \right) = \left( \tilde{X}_h, \tilde{X}_\pi \right) \]

its components with respect to the adapted frame. Then the covariant derivative \( D \nabla \tilde{X} \) has components

\[ D \nabla_\beta \tilde{X}_\alpha = A_{\beta \gamma} \tilde{X}_\gamma + D \Gamma^\gamma_{\beta \alpha} \tilde{X}_\gamma, \]

(14)

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\[ \Gamma_{\beta \alpha} \] being given by (12), with respect to the adapted frame. From (1) we see that components of \( C^X, H^X \) and \( V^w \)

\[ C^X = A^\alpha A C^X, \quad H^X = A^\alpha A H^X, \quad V^w = A^\alpha A V^w \]

with respect to the adapted frame \( C^X, H^X \) and \( V^w \) are given respectively by

\[ \left( C^X \right) = \begin{pmatrix} X^h \cr -p_m \nabla_k X^m \end{pmatrix}, \quad \left( H^X \right) = \begin{pmatrix} X^h \cr 0 \end{pmatrix}, \quad \left( V^w \right) = \begin{pmatrix} 0 \cr w^h \end{pmatrix} \]

by virtue of (7').

The associated covector fields of the complete, horizontal and vertical lifts to \( T^* (M_n) \) with the metric \( Dg \) are given respectively by

\[ \left( C^X_{\beta} \right) = (Dg_{\beta \alpha} C^X) = \left( X_j, -g^{ji} p_m \nabla_i X^m \right), \]
\[ \left( H^X_{\beta} \right) = (Dg_{\beta \alpha} H^X) = \left( X_j, 0 \right), \]
\[ \left( V^w_{\beta} \right) = (Dg_{\beta \alpha} V^w) = \left( 0, w^j \right) \]

with respect to the adapted frame, where \( X_j = g_{ji} X^i, w^j = g^{ji} w_i \).

We now compute the Lie derivatives of the metric \( Dg \) with respect to \( C^X, H^X \) and \( V^w \) by means of (14) and (15). The Lie derivatives of \( Dg \) with respect to \( C^X, H^X \) and \( V^w \) have respectively components

\[ \left( \mathcal{L}_{C^X} Dg_{\beta \alpha} \right) = \left( D\nabla_\beta C^X \alpha + D\nabla_\alpha C^X \beta \right) \]
\[ = \left( \begin{array}{cc}
\nabla_j X_i + \nabla_i X_j & \nabla_j \nabla_k X_i + R_{ijkt} X^t \\
-p_m g^{kj} g^{lm} (\nabla_i \nabla_k X^l + R_{ijkt} X^t) & -(g^{is} \nabla_s X^j + g^{is} \nabla_s X^t)
\end{array} \right) \]

(16)

\[ \left( \mathcal{L}_{H^X} Dg_{\beta \alpha} \right) = \left( D\nabla_\beta H^X \alpha + D\nabla_\alpha H^X \beta \right) = \left( \begin{array}{cc}
\nabla_j X_i + \nabla_i X_j & -p_m g^{kj} R_{ijkt} m X^t \\
-p_m g^{kj} R_{ijkt} m X^t & 0
\end{array} \right) \]

\[ \left( \mathcal{L}_{V^w} Dg_{\beta \alpha} \right) = \left( D\nabla_\beta V^w \alpha + D\nabla_\alpha V^w \beta \right) = \left( \begin{array}{cc}
0 & g^{js} \nabla_s X \cr g^{js} \nabla_s X \cr 0
\end{array} \right) \]
with respect to the adapted frame in $T^*(M_n)$.

Since we have

$$\nabla_i \nabla_k X^\ell + R_{\ell i k} X^\ell = 0, \quad \mathcal{L}_X g^{ij} = -(g^{ij} \nabla_s X^i + g^{is} \nabla_s X^j) = 0$$

as a consequence of $\mathcal{L}_X g_{ji} = \nabla_j X_i + \nabla_i X_j = 0$ (see [6, p.17]), we conclude by means of (16) that the complete lift $^C X$ is a Killing vector field in $T^*(M_n)$ if and only if $X$ is a Killing vector field in $M_n$.

We next have

$$R_{\ell i k}^m X^\ell = 0$$

as a consequence of the vanishing of the second covariant derivative of $X$. Conversely, the conditions $\mathcal{L}_X g_{ji} = \nabla_j X_i + \nabla_i X_j = 0$ and $R_{\ell i k}^m X^\ell = 0$ imply that the second covariant derivative of $X$ vanishes. Summing up these results, we have

**Theorem 3.** Necessary and sufficient conditions in order that the

a) complete $^C X \in T^1_0(T^*(M_n))$,
b) horizontal $^H X \in T^1_0(T^*(M_n))$ and
c) vertical $^V w \in T^1_0(T^*(M_n))$

lifts to $T^*(M_n)$ with the metric $^D g$, of a vector field $X$ and covector field $w$ in $M_n$ be a Killing vector field in $T^*(M_n)$ are that,

a) $X$ is a Killing vector field in $M_n$,
b) $X$ is a Killing vector field with vanishing second covariant derivative in $M_n$ and
c) $w$ is parallel in $M_n$.

### 5. Geodesics in $T^*(M_n)$ with metric $^D g$

Let $C$ be a curve in $M_n$ expressed locally by $x^h = x^h(t)$ and $w_h(t)$ be a covector field along $C$. Then, in the cotangent bundle $T^*(M_n)$, we define a curve $\tilde{C}$ by

$$x^h = x^h(t), \quad x^h \overset{\text{def}}{=} p_h = w_h(t)$$

(17)

If the curve $C$ satisfies at all the points the relation

$$\frac{\delta w_a}{dt} = \frac{dw_a}{dt} - \Gamma^i_{jh} \frac{dx^j}{dt} w_i = 0,$$

(18)
then the curve $\tilde{C}$ is said to be a horizontal lift of the curve $C$ in $M_n$. Thus, if the initial condition $w_h = w_0$ for $t = t_0$ is given, there exists a unique horizontal lift expressed by (17).

We now consider differential equations of the geodesics of the cotangent bundle $T^\ast(M_n)$ with the metric $Dg$. If $t$ is the arc length of a curve $x^A = x^A(t)$ in $T^\ast(M_n)$, equations of geodesics in $T^\ast(M_n)$ have the usual form

$$\frac{\delta^2 x^A}{dt^2} = \frac{d^2 x^A}{dt^2} + \frac{D}{C^B} \Gamma_{C}^{A} \frac{dx^C}{dt} \frac{dx^B}{dt} = 0$$

(19)

with respect to the induced coordinates $(x^i, x^\ast_j) = (x^i, \rho_i)$ in $T^\ast(M_n)$.

We find it more convenient to refer equations (19) to the adapted frame $\{A_\alpha, A_\gamma\}$. Using (2), we now write

$$\theta^h = A^{(h)}_A dx^A = dx^h,$$

$$\theta^\ast = A^{(\ast)}_A dx^A = \delta p_h,$$

and put

$$\frac{\theta^h}{dt} = A^{(h)}_A \frac{dx^A}{dt} = \frac{dx^h}{dt},$$

$$\frac{\theta^\ast}{dt} = A^{(\ast)}_A \frac{dx^A}{dt} = \delta p_h \frac{dt}{dt} = \delta p_h - \Gamma_{j}^{i} \frac{dx^j}{dt} p_i,$$

along a curve $x^A = x^A(t)$, i.e., $x^h = x^h(t)$, $p_h = p_h(t)$ in $T^\ast(M_n)$.

If we therefore write down the form equivalent to (19), namely,

$$\frac{d}{dt} \left( \frac{\theta^\ast}{dt} \right) + \frac{D}{C^\alpha} \Gamma_{\beta}^{\alpha} \frac{\theta^\beta}{dt} = 0$$

with respect to the adapted frame and take account of (12), then we have

$$\begin{align*}
\frac{d^2 x^h}{dt^2} + p_m R^h_{\ j} \frac{dx^j}{dt} \frac{dp_m}{dt} &= 0, \\
\frac{d^2 p_h}{dt^2} + \frac{1}{2} p_m R^h_{\ j} \frac{dx^j}{dt} \frac{dp_m}{dt} &= 0
\end{align*}$$

(20)
Since we have

\( R_{jik}^m \frac{dx^j}{dt} \frac{dx^i}{dt} = 0 \)

as a consequence of \( R_{(jik)}^m = 0 \), we conclude by means of (20) that a curve \( x^i = x^i(t) \), \( p_h = p_h(t) \) in \( T^*(M_n) \) with the metric \( Dg \) is a geodesic in \( T^*(M_n) \), if and only if

\[
\begin{align*}
\frac{d^2 x^h}{dt^2} + p_m R_{ji}^m \frac{dx^i}{dt} \frac{\delta p_h}{\delta x^i} &= 0, \quad (a) \\
\frac{d^2 p_h}{dt^2} &= 0 \quad (b)
\end{align*}
\]

(21)

If a curve satisfying (21) lies on a fibre given by \( x^h = \text{const} \), then (20, (b)) reduces to

\[
\frac{d^2 p_h}{dt^2} = 0
\]

so that \( p_h = a_h t + b_h \), \( a_h \) and \( b_h \) being constant. Thus we have

**Theorem 4.** If geodesic \( x^h = x^h(t) \), \( p_h = p_h(t) \) lies in a fibre of \( T^*(M_n) \) with the metric \( Dg \), the a geodesic is expressed by linear equations \( x^h = c^h \), \( p_h = a_h t + b_h \), where \( a_h, b_h \) and \( c^h \) are constant.

From (18) and (21), we have

**Theorem 5.** The horizontal lift of a geodesic in \( M_n \) is always geodesic in \( T^*(M_n) \) with the metric \( Dg \).

References


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