On the Centroid of the Prime Gamma Rings II

M. Ali Öztürk and Young Bae Jun

Abstract

The aim of this paper is to study the properties of the extended centroid of the prime Γ-rings. Main results are the following theorems: (1) Let \( M \) be a simple Γ-ring with unity. Suppose that for some \( a \neq 0 \) in \( M \) we have \( a \gamma_1 x \gamma_2 a \beta_1 y \beta_2 a = a \beta_1 y \beta_2 a \gamma_1 x \gamma_2 a \) for all \( x, y \in M \) and \( \gamma_1, \gamma_2, \beta_1, \beta_2 \in \Gamma \). Then \( M \) is isomorphic onto the Γ-ring \( D_{n,m} \), where \( D_{n,m} \) is the additive abelian group of all rectangular matrices of type \( n \times m \) over a division ring \( D \) and \( \Gamma \) is a nonzero subgroup of the additive abelian group of all rectangular matrices of type \( m \times n \) over a division ring \( D \). Furthermore \( M \) is the Γ-ring of all \( n \times n \) matrices over the field \( C_{e} \). (2) Let \( M \) be a prime Γ-ring and \( C_{\Gamma} \) the extended centroid of \( M \). If \( a \) and \( b \) are non-zero elements in \( S = M \Gamma C_{\Gamma} \) such that \( a \gamma_1 x \beta_1 b = b \beta_1 x \gamma_1 a \) for all \( x \in M \) and \( \beta_1, \gamma_1 \in \Gamma \), then \( a \) and \( b \) are \( C_{\Gamma} \)-dependent. (3) Let \( M \) be prime Γ-ring, \( Q \) quotient Γ-ring of \( M \) and \( C_{\Gamma} \) the extended centroid of \( M \). If \( q \) is non-zero element in \( Q \) such that \( q \gamma_1 x \gamma_2 q \beta_1 y \beta_2 q = q \beta_1 y \beta_2 q \gamma_1 x \gamma_2 q \) for all \( x, y \in M \), \( \gamma_1, \gamma_2, \beta_1, \beta_2 \in \Gamma \) then \( S \) is a primitive Γ-ring with minimal right (left) ideal such that \( e \Gamma S \), where \( e \) is idempotent and \( C_{\Gamma} e \) is the commuting ring of \( S \) on \( e \Gamma S \).

Key Words: Γ-division ring, Γ-field, extended centroid, central closure.

1. Introduction


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2. Preliminaries

Let $M$ and $\Gamma$ be two abelian groups. If for all $x, y, z \in M$ and all $\alpha, \beta \in \Gamma$ the conditions

(i) $xay \in M$,
(ii) $(x + y)\alpha z = x\alpha z + y\alpha z, x(\alpha + \beta)z = x\alpha z + x\beta z, x\alpha(y + z) = xay + x\alpha z$,
(iii) $(xay)\beta z = x\alpha(y\beta z)$

are satisfied, then we call $M$ a $\Gamma$-ring. By a right (resp. left) ideal of a $\Gamma$-ring $M$ we mean an additive subgroup $U$ of $M$ such that $UM \subseteq U$ (resp. $MU \subseteq U$). If $U$ is both a right and a left ideal, then we say that $U$ is an ideal of $M$. For each $a$ of a $\Gamma$-ring $M$ the smallest right ideal containing $a$ is called the principal right ideal generated by $a$ and is denoted by $\langle a \rangle_r$. Similarly we define $\langle a \rangle_l$ (resp. $\langle a \rangle$), the principal left (resp. two sided) ideal generated by $a$. An ideal $P$ of a $\Gamma$-ring $M$ is said to be prime if for any ideals $A$ and $B$ of $M$, $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. An ideal $Q$ of a $\Gamma$-ring $M$ is said to be semi-prime if for any ideal $U$ of $M$, $UTU \subseteq Q$ implies $U \subseteq Q$. A $\Gamma$-ring $M$ is said to be prime (resp. semi-prime) if the zero ideal is prime (resp. semi-prime).

**Theorem 2.1** ([4, Theorem 4]). If $M$ is a $\Gamma$-ring, the following conditions are equivalent:

(i) $M$ is a prime $\Gamma$-ring.
(ii) If $a, b \in M$ and $a\Gamma Mb = (0)$, then $a = 0$ or $b = 0$.
(iii) If $\langle a \rangle$ and $\langle b \rangle$ are principal ideals in $M$ such that $\langle a \rangle \Gamma \langle b \rangle = (0)$, then $a = 0$ or $b = 0$.
(iv) If $A$ and $B$ are right ideals in $M$ such that $A\Gamma B = (0)$, then $A = (0)$ or $B = (0)$.
(v) If $A$ and $B$ are left ideals in $M$ such that $A\Gamma B = (0)$, then $A = (0)$ or $B = (0)$.
A $\Gamma$-ring $M$ is said to be \textit{simple} if $M \Gamma M \neq 0$ and $M$ has no ideals other than $0$ and $M$ itself. When a $\Gamma$-ring $M$ has the descending (resp. ascending) chain condition for right ideals, it is abbreviated to $M$ has \textit{min-r condition} (resp. \textit{max-r condition}). The terms \textit{min-l condition} or \textit{max-l condition} on a $\Gamma$-ring $M$ are likewise defined. Let $M$ be a $\Gamma$-ring and let $F$ be the free group generated by $\Gamma$. Then

$$A = \{ \sum_i n_i (\gamma_i, x_i) \in F \mid a \in M \Rightarrow \sum_i n_i a \gamma_i x_i = 0 \}$$

is a subgroup of $F$. Let $R = F/A$ be the factor group, and denote the coset $(\gamma, x) + A$ by $[\gamma, x]$. Clearly, every element of $R$ can be expressed as a finite sum $\sum_i [\gamma_i, x_i]$. Also it can be verified easily that $[\alpha, x] + [\alpha, y] = [\alpha, x + y]$ and $[\alpha, x] + [\beta, x] = [\alpha + \beta, x]$ for all $\alpha, \beta \in \Gamma$ and $x, y \in M$. We define a multiplication on $R$ by

$$\sum_i [\alpha_i, x_i] \sum_j [\beta_j, y_j] = \sum_{i,j} [\alpha_i, x_i \beta_j y_j].$$

Then $R$ forms a ring. If we define a composition on $M \times R$ into $M$ by

$$a \sum_i [\gamma_i, x_i] = \sum_i a \gamma_i x_i, \forall a \in M, \forall \sum_i [\gamma_i, x_i] \in R$$

then $M$ is a right $R$-module, and we call $R$ the \textit{right operator ring} of $M$. Similarly, we can define the \textit{left operator ring} $L$ of $M$. A $\Gamma$-ring $M$ is said to be \textit{right} (resp. \textit{left}) \textit{primitive} if it satisfies:

(i) the right (resp. left) operator ring of $M$ is a right (resp. left) primitive ring

(ii) $M \Gamma x = 0$ (resp. $x \Gamma M = 0$) implies $x = 0$.

A $\Gamma$-ring $M$ is said to be \textit{two-sided primitive} (or simply, \textit{primitive}) if it is both right and left primitive.

\textbf{Theorem 2.2} ([7, Theorem 3.4]). \textit{If $M$ is a $\Gamma$-ring possessing minimal left (resp. right) ideal, then $M$ is primitive if and only if it is prime.}

\textbf{Theorem 2.3} ([7, Theorem 3.6]). \textit{For a $\Gamma$-ring $M$ with min-l condition, the following are equivalent:}

(i) $M$ is prime,
(ii) $M$ is primitive,

(iii) $M$ is simple.

**Theorem 2.4** ([7, Theorem 4.2]). If $M$ is a simple $\Gamma$-ring possessing minimal left (resp. right) ideals, then $M$ is a direct sum of minimal left (resp. left) ideals.

**Theorem 2.5** ([5, Theorem 3.23]). Let $M$ be a semi-prime $\Gamma$-ring with min-$r$ condition and let $M = I_1 \oplus I_2 \oplus \cdots \oplus I_m = J_1 \oplus J_2 \oplus \cdots \oplus J_n$, where $I_1, I_2, \ldots, I_m, J_1, J_2, \ldots, J_n$ are minimal right ideals. Then $m = n$.

The integer $m = n$ in Theorem 2.5 is called the right dimension of the semi-prime $\Gamma$-ring with min-$r$ condition and denoted by $\dim(M_R)$. One can define the left dimension of a $\Gamma$-ring in a similar way. If $M$ is simple, then $M$ is semi-prime (see [5]).

For an additive group $G$, denote by $G_{m;n}$ the additive group of all matrices over $G$. Let $M$ be a $\Gamma$-ring and let $M_{m;n}$ and $\Gamma_{n;m}$ denote, respectively, the sets of $m \times n$ matrices with entries from $M$ and of $n \times m$ matrices with entries from $\Gamma$. For $(a_{ij}), (b_{ij}) \in M_{m;n}$ and $(\gamma_{ij}) \in \Gamma_{n;m}$, define $(a_{ij})(\gamma_{ij})(b_{ij}) = (c_{ij})$, where $c_{ij} = \sum_p \sum_q a_{ip}\gamma_{pq}b_{qj}$. Then $M_{m;n}$ forms a $\Gamma_{n;m}$-ring.

**Theorem 2.6** ([6, Theorem 4.2]). Let $M$ be a simple $\Gamma$-ring with min-$r$ and min-$l$ conditions and $\Gamma_0 = \Gamma/\kappa$, where $\kappa := \{ \gamma \in \Gamma \mid M\gamma M = 0 \}$. Then the $\Gamma_0$-ring $M$ is isomorphic to the $\Gamma'$-ring $D_{n;m}$, where $D_{n;m}$ is the additive abelian group of all rectangular matrices of type $n \times m$ over a division ring $D$ and $\Gamma'$ is a nonzero subgroup of the additive abelian group of all rectangular matrices of type $m \times n$ over a division ring $D$ and $m = \dim(M_L)$ and $n = \dim(M_R)$.

**Lemma 2.7** ([12, Lemma 3]). Let $M$ be a prime $\Gamma$-ring such that $M\Gamma M \neq M$ and quotient $\Gamma$-ring $Q$ of $M$. Then, for each non-zero $q \in Q$ there is a non-zero ideal $U$ of $M$ such that $q(U) \subset M$.

**Lemma 2.8** ([12, p. 476]). Let $M$ be a prime $\Gamma$-ring such that $M\Gamma M \neq M$ and $C_{\Gamma}$ the extended centroid of $M$. If $a_i$ and $b_i$ are non-zero elements of $M$ such that $\sum a_i\gamma_i x\beta_i b_i = 0$ for all $x \in M$ and $\gamma_i, \beta_i \in \Gamma$, then the $a_i$’s (also $a_i$’s) are linearly dependent over $C_{\Gamma}$. Moreover, if $a\gamma x\beta b = b\gamma x\beta a$ for all $x \in M$ and $\gamma, \beta \in \Gamma$ where $a(\neq 0), b \in M$ are fixed, then there exists $\lambda \in C_{\Gamma}$ such that $b = \lambda a a$ for all $a \in \Gamma$. 

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3. Centroids

Let $M$ be a prime $\Gamma$-ring such that $M\Gamma M \neq M$. Denote

$$\mathcal{M} := \{(U, f) \mid U(\neq 0) \text{ is an ideal of } M \text{ and } f : U \to M \text{ is a right } M\text{-module homomorphism}\}.$$  

Define a relation $\sim$ on $\mathcal{M}$ by $(U, f) \sim (V, g)$ if and only if $\exists W(\neq 0) \subset U \cap V$ such that $f = g$ on $W$. Since $M$ is a prime $\Gamma$-ring, it is possible to find a non-zero $W$ and so $\sim$ is an equivalence relation. This gives a chance for us to get a partition of $\mathcal{M}$. We then denote the equivalence class by $\text{Cl}(U, f) = \hat{f}$, where $\hat{f} := \{g : V \to M \mid (U, f) \sim (V, g)\}$, and denote by $Q$ the set of all equivalence classes. Now we define an addition “$+$” on $Q$ as follows:

$$\hat{f} + \hat{g} = \text{Cl}(U, f) + \text{Cl}(V, g) = \text{Cl}(U \cap V, f + g)$$

where $f + g : U \cap V \to M$ is a right $M$-module homomorphism. Then $Q$ is an additive abelian group (see [12]). Since $M\Gamma M \neq M$ and since $M$ is a prime $\Gamma$-ring, $M\Gamma M (\neq 0)$ is an ideal of $M$. We can take the homomorphism $1_{M\Gamma} : M\Gamma M \to M$ as a unit $M$-module homomorphism. Note that $M\beta M \neq 0$ for all $0 \neq \beta \in \Gamma$ so that $1_{M\beta} : M\beta M \to M$ is non-zero $M$-module homomorphism. Denote

$$\mathcal{N} := \{(M\beta M, 1_{M\beta}) \mid 0 \neq \beta \in \Gamma\},$$

and define a relation “$\approx$” on $\mathcal{N}$ by $(M\beta M, 1_{M\beta}) \approx (M\gamma M, 1_{M\gamma})$ if and only if $\exists W := M\alpha M(\neq 0) \subset M\beta M \cap M\gamma M$ such that $1_{M\beta} = 1_{M\gamma}$ on $W$. We can easily check that “$\approx$” is an equivalence relation on $\mathcal{N}$. Denote by $\text{Cl}(M\beta M, 1_{M\beta}) = \hat{\beta}$ the equivalence class containing $(M\beta M, 1_{M\beta})$ and by $\hat{\Gamma}$ the set of all equivalence classes of $\mathcal{N}$ with respect to $\approx$, that is,

$$\hat{\beta} := \{1_{M\gamma} : M\gamma M \to M \mid (M\beta M, 1_{M\beta}) \approx (M\gamma M, 1_{M\gamma})\}$$

and $\hat{\Gamma} := \{\hat{\beta} \mid 0 \neq \beta \in \Gamma\}$. Define an addition “$+$” on $\hat{\Gamma}$ as follows:

$$\hat{\beta} + \hat{\delta} = \text{Cl}(M\beta M, 1_{M\beta}) + \text{Cl}(M\delta M, 1_{M\delta}) = \text{Cl}(M\beta M \cap M\delta M, 1_{M\beta + 1_{M\delta}}).$$
for every $\beta(\neq 0), \delta(\neq 0) \in \Gamma$. Then $(\hat{\Gamma}, +)$ is an abelian group. Now we define a mapping $(-, -, -) : \hat{\Gamma} \times \hat{\Gamma} \to \hat{\Gamma}$, $(\hat{f}, \hat{\beta}, \hat{g}) \mapsto \hat{f}\hat{\beta}\hat{g}$, as follows:

$$\hat{f}\hat{\beta}\hat{g} = Cl(U, f)Cl(M\beta M, 1_{M\beta})Cl(V, g) = Cl(V\Gamma M\beta M, f1_{M\beta}g)$$

where

$$VT\Gamma M\beta M\Gamma U = \{ \sum v_i\gamma_i m_i\beta n_i\alpha_i u_i \mid v_i \in V, u_i \in U, m_i, n_i \in M \text{ and } \alpha_i, \gamma_i \in \Gamma \}$$

is an ideal of $M$ and $f1_{M\beta}g : VT\Gamma M\beta M\Gamma U \to M$ which is given by

$$f1_{M\beta}g(\sum v_i\gamma_i m_i\beta n_i\alpha_i u_i) = f(\sum g(v_i)\gamma_i m_i\beta n_i\alpha_i u_i)$$

is a right $M$-module homomorphism. Then $Q$ is a $\hat{\Gamma}$-ring with unity. Noticing that the mapping $\varphi : \Gamma \to \hat{\Gamma}$ defined by $\varphi(\beta) = \hat{\beta}$ for every $0 \neq \beta \in \Gamma$ is an isomorphism, we know that the $\hat{\Gamma}$-ring $Q$ is a $\Gamma$-ring (see [12]). For purposes of convenience, we use $q$ instead of $\hat{q} \in Q$.

**Definition 3.1.** Let $M$ be a $\Gamma$-ring with unity. An element $u$ in $M$ is called a *unit* of $M$ if it has a multiplicative inverse in $M$. If every nonzero element of $M$ is a unit, we say that $M$ is a $\Gamma$-*division ring*. A $\Gamma$-ring $M$ is called a $\Gamma$-*field* if it is a commutative $\Gamma$-division ring.

**Definition 3.2.** The set

$$C_\Gamma := \{ g \in Q \mid g\gamma f = f\gamma g \text{ for all } f \in Q \text{ and } \gamma \in \Gamma \}$$

is called the *extended centroid* of a $\Gamma$-ring $M$.

**Lemma 3.3.** Let $M$ be a prime $\Gamma$-ring. Then the extended centroid $C_\Gamma$ of $M$ is a $\Gamma$-field.

**Proof.** Noticing that $C_\Gamma$ is a commutative ring with unity, it is sufficient to show that every nonzero element of $C_\Gamma$ is invertible. If $c(\neq 0) \in C_\Gamma$, then $c = Cl(U, \mu)$. Thus, by Lemma 2.7., there is a nonzero ideal $U$ of $M$ such that $\mu(U) \subset M$. Clearly, $0 \neq V = \mu(U)$ is an ideal of $M$. Since $U\Gamma M \subset U$, therefore $\mu(U)\Gamma M \subset \mu(U)$. Hence we can define a mapping $f : \mu(U) \to M$ by $f(\mu(u)) = u$ for all $u \in U$, and this is a right $M$-module.
homomorphism. In fact, let \( v_1, v_2 \in V = \mu(U) \) and so there exists \( u_1, u_2 \in U \) such that 
\( v_1 = \mu(u_1) \) and \( v_2 = \mu(u_2) \). It follows that 
\[
 f(v_1 + v_2) = f(\mu(u_1) + \mu(u_2)) 
= f(\mu(u_1 + u_2)) = u_1 + u_2 
= f(\mu(u_1)) + f(\mu(u_2)) 
= f(v_1) + f(v_2). 
\]

Now, for any \( v \in V, \, m \in M \) and \( \gamma \in \Gamma \), we have 
\[
 f(v \gamma m) = f(\mu(u) \gamma m) = f(\mu(u \gamma m)) = u \gamma m = f(\mu(u)) \gamma m = f(v) \gamma m. 
\]

Finally, considering \( d = \text{Cl}(V, f) \), we get 
\[
 d \gamma c = \text{Cl}(V, f) \text{Cl}(M \gamma M, 1_M \gamma M) \text{Cl}(U, \mu) 
= \text{Cl}(U \Gamma M \gamma M \Gamma, f 1_M \gamma M) 
= \text{Cl}(U \Gamma M \gamma M \Gamma \mu(U), 1) = I. 
\]

This completes the proof. \( \square \)

**Definition 3.4.** For the extended centroid \( C_\Gamma \) of a prime \( \Gamma \)-ring \( M \), we say that \( S := M \Gamma C_\Gamma \) is the central closure of \( M \).

**Remark 3.5.** For \( a, b \in S \), if \( a \Gamma S b = 0 \) then \( a \Gamma M \Gamma C_\Gamma b = 0 \) and so \( a \Gamma M \Gamma b \Gamma M a \Gamma C_\Gamma b = 0 \). Since \( M \) is a prime \( \Gamma \)-ring, it follows that \( a \Gamma M \Gamma b = 0 \) or \( a \Gamma C_\Gamma b = 0 \) so \( a = 0 \) or \( b = 0 \).

Thus \( S \) is a prime \( \Gamma \)-ring.

If \( M \) has a unit element, then \( C_\Gamma = Z(S) \), the centre of \( S \). If \( M \) is a simple \( \Gamma \)-ring with unity, then \( Q = S = M \). Because the only non-zero ideal of \( M \) is \( M \) itself. In this case; \( M \) is its own central closure.

Throughout, we shall use \( M \) as a prime \( \Gamma \)-ring such that \( M \Gamma M \neq M \).

**Theorem 3.6.** Let \( C_\Gamma \) be the extended centroid of a prime \( \Gamma \)-ring \( M \). If \( a \) is a nonzero element of \( M \) such that \( a \gamma_1 x \gamma_2 a \beta_1 y \beta_2 a = a \beta_1 y \beta_2 a \gamma_1 x \gamma_2 a \) for all \( x, y \in M, \gamma_1, \gamma_2, \beta_1, \beta_2 \in \Gamma \) then \( S = M \Gamma C_\Gamma \) is a primitive \( \Gamma \)-ring with minimal right (left) ideal and the commuting ring of \( S \) on this right (left) ideal is merely \( C_\Gamma \) itself.

**Proof.** Let fixed \( a \gamma_1 x \gamma_2 a \) element in the relation \( (a \gamma_1 x \gamma_2 a) \beta_1 y \beta_2 a = a \beta_1 y \beta_2 (a \gamma_1 x \gamma_2 a) = 0 \) then, from Lemma 2.8 we get \( a \gamma_1 x \gamma_2 a = \lambda(x) a a \), where \( \lambda(x) \in C_\Gamma \) and \( a \in \Gamma \) and for
all $x \in M$. Similarly we also get $a\beta_1 y\beta_2 a = \lambda(y)\alpha a$, where $\lambda(y) \in C_\Gamma$ and $\alpha \in \Gamma$ and for all $y \in M$. Thus, since $a\beta_1 y\beta_2 a = \lambda(y)\alpha a \in C_\Gamma \Gamma a$ we get $a\Gamma S\Gamma a \subseteq C_\Gamma \Gamma a$. Since $a \neq 0$ and $S$ is prime $\Gamma$-ring, there is some $y_o \in S$ such that $a\beta_1 y_o\beta_2 a \neq 0$ for some $\beta_1, \beta_2 \in \Gamma$. Thus, $a\beta_1 y_o\beta_2 a = \lambda(y_o)\alpha a$, where $0 \neq \lambda(y_o) \in C_\Gamma$. Similarly we get $a\gamma_1 x_o\gamma_2 a = \lambda(x_o)\alpha a$, where $0 \neq \lambda(x_o) \in C_\Gamma$. If $x_o = \lambda^{-1}(y_o)\alpha y_o$, then $a\gamma_1 x_o\gamma_2 a = a\gamma_1 \lambda^{-1}(y_o)\alpha y_o \gamma_2 a = \lambda^{-1}(y_o)\alpha a\gamma_1 y_o \gamma_2 a = \lambda^{-1}(y_o)\alpha \lambda(y_o)\alpha a = a$. Thus, let $e = a\gamma_1 x_o$. $e^2 e = (a\gamma_1 x_o)\gamma_2 (a\gamma_1 x_o) = (a\gamma_1 x_o\gamma_2 a)\gamma_1 x_o = a\gamma_1 x_o = e$. From this we will have $e$ idempotent. In this case; $e\Gamma S\Gamma e = (a\gamma_1 x_o)\Gamma S\Gamma (a\gamma_1 x_o) \subseteq C_\Gamma (a\gamma_1 x_o) = C_\Gamma e$. Thus $e\Gamma S$ is a minimal right ideal of $S$ and $C_\Gamma e$ is the commuting ring of $S$ on $e\Gamma S$ by Lemma 3.3. Since $S$ is prime $\Gamma$-ring and has a minimal right ideal. $S$ is primitive $\Gamma$-ring by Theorem 2.2.

**Theorem 3.7.** Let $M$ be a simple $\Gamma$-ring with unity. Suppose that for some $a \neq 0$ in $M$ we have $a\gamma_1 x\gamma_2 a\beta_1 y\beta_2 a = a\beta_1 y\beta_2 a\gamma_1 x\gamma_2 a$ for all $x, y \in M$ and $\gamma_1, \gamma_2, \beta_1, \beta_2 \in \Gamma$. Then $M$ is isomorphic onto the $\Gamma$-ring $D_{n,m}$, where $D_{n,m}$ is the additive abelian group of all rectangular matrices of type $n \times m$ over a division ring $D$ and $\Gamma$ is a nonzero subgroup of the additive abelian group of all rectangular matrices of type $m \times n$ over a division ring $D$. Furthermore $M$ is the $\Gamma$-ring of all $n \times n$ matrices over the field $C_\Gamma$.

**Proof.** Since $M$ is simple $\Gamma$-ring we have $M = S$ and from Theorem 3.6 we get $M$ has a minimal right ( left ) ideal of $M$. In this case, $M$ is the sum of minimal right ( left ) ideals by Theorem 2.4, that is, $M$ is the sum of minimal right ideals $N_i$, where $N_i = x_i \Gamma N$ ( $N$ is a non-zero minimal right ideal of $M$ ) for some $x_i \in M$. Also, since $M$ has unit ( $1 \in M$ ), $1 \in N_1 + \ldots + N_n$ for some $n$, we get $M = N_1 + \ldots + N_n$ and so $M$ is the sum of a finite number of minimal right ideals, each of which is an irreducible right $M$-module. Thus $M$, as a $M$-module, has a composition series. Thus $M$ has min-r condition and so $M$ is primitive $\Gamma$-ring by Theorem 2.3. In this case, by Theorem 3.6, the commuting ring of $M$ on an irreducible module is $C_\Gamma = Z(M)$, the center of $M$. Thus, this finishes the proof of the theorem by Theorem 2.6.

**Theorem 3.8.** Let $M$ be prime $\Gamma$-ring and $C_\Gamma$ the extended centroid of $M$. If $a$ and $b$ are non-zero elements in $S = M \Gamma C_\Gamma$ such that $a\gamma x\beta b = b\beta x\gamma a$ for all $x \in M$ and $\gamma, \beta \in \Gamma$, then $a$ and $b$ are $C_\Gamma$-dependent.

**Proof.** Firstly, we assume that $a \neq 0$ and $b \neq 0$. Let $U$ be a non-zero ideal of
M such that \(a\Gamma U \subseteq M\) and \(a\Gamma U \subseteq M\), and set \(V = U\Gamma a\Gamma U = \{ \sum x_i\gamma_i a\beta_i y_i \mid x_i, y_i \in U, \gamma_i, \beta_i \in \Gamma \}\). We define a mapping \(f : V \rightarrow M\) defined by \(v \mapsto f(v) = f(\sum x_i\gamma_i a\beta_i y_i) = \sum x_i\gamma_i b\beta_i y_i\), for all \(x_i, y_i \in U\) and \(\gamma_i, \beta_i \in \Gamma\). We suppose that \(\sum x_i\gamma_i a\beta_i y_i = 0\). Then,

\[
0 = ba_i m\sigma_i \sum x_i\gamma_i a\beta_i y_i = \sum ba_i (m\sigma_i x_i)\gamma_i a\beta_i y_i = \sum \alpha_i (m\sigma_i x_i)\gamma_i b\beta_i y_i = a\alpha_i m\sigma_i \sum x_i\gamma_i b\beta_i y_i
\]

Thus, we get, for all \(x_i, y_i \in U\) and \(\gamma_i, \beta_i \in \Gamma\)

\[
a\Gamma^M \Gamma(\sum x_i\gamma_i b\beta_i y_i) = 0
\]

and so since \(a \neq 0\) and \(M\) is prime \(\Gamma\)-ring we get \(\sum x_i\gamma_i b\beta_i y_i = 0\). Therefore, \(f\) is well defined. Also, specially \(f((x\gamma\alpha\beta y)am) = x\gamma b\beta yam = f(x\gamma\alpha\beta y)am\) for all \(x, y \in U\) and \(m \in M\) and \(\gamma, \beta, \alpha \in \Gamma\) and so \(f\) is a \(M\)-module homomorphism. Let \(q\) denote the element of \(Q\) determined by \(f\), that is, \(q = Cl(V, f)\). Let \(p\) be any element of \(Q\) with \(p(W) \subseteq M\) for some non-zero ideal \(W\) of \(M\) by Lemma 2.7. In this case,

\[
(f1_{M\alpha} p)(\sum w_i\gamma_i m_i \alpha_i \beta_i x_i\gamma_i a\beta_i y_i)
\]

\[
= f(\sum p(w_i)\gamma_i m_i \alpha_i \beta_i x_i\gamma_i a\beta_i y_i)
\]

\[
= \sum p(w_i)\gamma_i m_i \alpha_i \beta_i x_i\gamma_i b\beta_i y_i.
\]

\[
= p(\sum w_i\gamma_i m_i \alpha_i \beta_i x_i\gamma_i b\beta_i y_i)
\]

\[
= p(1_{M\alpha} f(\sum w_i\gamma_i m_i \alpha_i \beta_i x_i\gamma_i a\beta_i y_i))
\]

\[
= (p1_{M\alpha} f)(\sum w_i\gamma_i m_i \alpha_i \beta_i x_i\gamma_i a\beta_i y_i)
\]

and so \(q \circ p = Cl(WT M\alpha MTV, f1_{M\alpha} P) = Cl(WT M\alpha MTV, P1_{M\alpha} f) = p \circ q\). Thus, we get \(q \in C_\Gamma\). For \(\gamma, \beta, \alpha \in \Gamma\),

\[
q\gamma(x\alpha a\beta y) = Cl(V, f)Cl(M\gamma M, 1_{M\gamma})Cl(\hat{V}, x\alpha a\beta y)
\]

\[
= Cl(\hat{V}T M\gamma MTV, f1_{M\gamma}(x\alpha a\beta y))
\]

\[
= Cl(\hat{V}T M\gamma MTV, x\alpha a\beta y)
\]

\[
= xab\beta y,
\]
Hence we have \((x\gamma qa - xab)\beta y = 0\) for all \(x, y \in U\) and \(\gamma, \alpha \in \Gamma\). Therefore, since \(M\) is prime \(\Gamma\)-ring we get \(x\gamma qa - xab = 0\) for all \(x, y \in U\) and \(\gamma, \alpha \in \Gamma\). Now writing \(\alpha + \gamma\) for in the previous equation we get, \(x\gamma (q\alpha - b) = 0\) for all \(\gamma \in \Gamma\) and \(x \in U\). Thus, since \(M\) is prime \(\Gamma\)-ring, we get, \(q\gamma a = b\) for all \(\gamma \in \Gamma\) and so this completes the proof. \(\square\)

**Theorem 3.9.** Let \(M\) be prime \(\Gamma\)-ring, \(Q\) quotient \(\Gamma\)-ring of \(M\) and \(C\) the extended centroid of \(M\). If \(q\) is non-zero element in \(Q\) such that \(q\gamma_1 x\gamma_2 y_1 y_2 = q\beta_1 x\gamma_2 q\gamma_1 x\gamma_2 y\) for all \(x, y \in M, \gamma_1, \gamma_2, \beta_1, \beta_2 \in \Gamma\) then \(S\) is a primitive \(\Gamma\)-ring with minimal right (left) ideal such that \(e\Gamma S\), where \(e\) is idempotent and \(C\Gamma e\) is the commuting ring of \(S\) on \(e\Gamma S\).

**Proof.** If \(q \in M\), then the proof finishes from Theorem 3.6. If \(q \in Q\) then one can pick \(a \in M\) such that \(q = qa\) is a non-zero element of \(M\) by Lemma 2.7. Also, \(\hat{q}\) satisfies \(\hat{q}\gamma_1 x\gamma_2 \hat{q}\beta_1 y_2 \hat{q} = \hat{q}\beta_1 x\gamma_2 \hat{q}\gamma_1 x\gamma_2 y\) for all \(x, y \in M, \gamma_1, \gamma_2, \beta_1, \beta_2 \in \Gamma\) and so this completes the proof. \(\square\)

**References**


M. Ali ÖZTÜRK
Department of Mathematics,
Faculty of Arts and Science,
Cumhuriyet University,
58140, Sivas-TURKEY
e-mail: maozturk@cumhuriyet.edu.tr

Young Bae JUN
Department of Mathematics Education,
Gyeongsang National University,
660-701, Chinju-KOREA
e-mail: ybjun@nongae.gsnu.ac.kr

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