Remarks on the Paper “on the Commutant of the Ideal Centre”

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In memory of Yunus Aran (1976 - 2000)

Abstract

We continue with the work started in [4] and give a new sufficient condition on Riesz spaces having topologically full centres for $Z^-(E)_C = \text{Orth}(E^-)$ to hold.

If $E$ is a Riesz space $E^-$, the order dual of $E$ will be the Riesz space of all order bounded linear functionals on $E$. Riesz spaces considered in this note are assumed to have separating order duals. $Z(E)$ will denote the ideal centre, Orth $(E)$, will denote the orthomorphisms of $E$. If $E$ is a topological Riesz space $E'$ will denote continuous dual of $E$. When $T : E \to F$ is an order bounded operator between two Riesz spaces, the adjoint of $T$ carries $F^-$ into $E^-$ and it will be denoted by $T^\sim$. In all undefined terminology concerning Riesz spaces we will adhere to the definitions in [1], [5] and [8].

When the order dual $E^-$ separates the points of the Riesz space $E$, an order bounded operator $T : E \to E$ is an orthomorphism if and only if its adjoint $T^\sim : E^- \to E^-$ is an orthomorphism. Moreover, the operator $\psi : \text{Orth}(E) \to \text{Orth}(E^-); \psi(T) = T^\sim$ is a one to one Riesz homomorphism [1]. The image under $\psi$ of the centre $Z(E)$ will be denoted by $Z^-(E); Z^-(E)$ is a Riesz subspace of $Z(E^-)$.

Definition A Riesz space $E$, is said to have topologically full centre if, for each pair $x, y$ in $E$ with $0 \leq y \leq x$, there exists a net $(\pi_\alpha)$ in $Z(E)$ with $0 \leq \pi_\alpha \leq 1$ for each $\alpha$, such that $\pi_\alpha x \to y$ in $\sigma(E, E^-)$.

Banach lattices with topologically full centre were initiated in [7]. These spaces were also studied in [2],[3], [4] and [6]. The class of Riesz spaces and the class of Banach spaces have topologically full centres are quite large. $\sigma$-Dedekind complete Riesz spaces have topologically full centres. However, not all Riesz spaces have topologically full centres.

Order bounded maps on the Riesz space $E$ will be denoted by $L_b(E); Z(E)_C$ will denote the commutant of $Z(E)$ in $L_b(E)$. That is, $Z(E)_C = \{ T \in L_b(E) : T\pi = \pi T$ for each $\pi \in Z(E)\}$. The Riesz space Orth $(E)$ under composition is an Archimedean $f$-algebra and therefore it is commutative. Hence Orth $(E) \subset Z(E)_C$. 

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We have studied the commutant $Z(E)_C$ of the ideal centre $Z(E)$ in the order bounded operators $L_b(E)$ [4]. If $E$ is a Riesz space with topologically full centre, we have identified $Z(E)_C$ with $\text{Orth} (E)$.

If $E$ has topologically full centre, it was claimed that $Z^\sim (E)_C = \text{Orth} (E^\sim)$. However, as Arenson has pointed out, part of the proof of this claim contains an error. If $E = C(K)$, then we can embed $E'$ into $E''' = C(K)^{'''}$ in two different ways. One of these embeddings is the usual embedding of a Banach space into its bidual as: $\mu \in E' \rightarrow \hat{\mu} \in E'''$. Let $\hat{E}'$ denote the image of $E'$ in $E'''$. For $\psi \in E'''$, we consider $\mu = \psi |_{\hat{E}'}$. For each $\psi \in E'''$, $\psi - \hat{\mu} \in E'' \subset E'''$ with $\mu = \psi |_{\hat{E}'}$. Thus, $\psi = (\psi - \hat{\mu}) + \hat{\mu}$ implies that $E''' = \hat{E}' \oplus E''$. The correspondence $\psi \rightarrow \hat{\mu}$ is a positive operator which fails to be a lattice homomorphism.

On the other hand, $\hat{E}'$ can be identified with the space of order continuous linear functionals on $E'' = C(K)''$. Consequently, $\hat{E}'$ is a band in $E'''$ and there exists an order projection $P : E''' \rightarrow \hat{E}'$. $P$ is an orthomorphism and $E''' = \hat{E}' \oplus (I - P)E'''$. However, $E'' \not\cong (I - P)E'''$ and $P \psi \neq \psi |_{E''}$ as it was erroneously claimed in [4].

The next example of Arenson’s (private communication) explains the situation even better.

**Example:** (Arenson) Let $K$ be a compact Hausdorff space with no isolated points and $E$ be $C(K)$. Then $Z(E) = E$ and $E^\sim = Z(E)'$ is the space of measures on $K$. If $Q$ is the Stone compact space of the Banach lattice $Z(E)'$, we identify $Z(E^\sim)$ with $C(Q)$. Since $Z(E)$ and $Z^\sim (E)$ are isometrically isomorphic, we are able to identify $Z^\sim (E)$ with $C(Q)$.

Let us note that $Z(E^\sim)' = C(Q)$ and $Z^\sim (E)' = C(K)'$. Therefore we have $Z(E^\sim)' = C(Q)' = C(K)^{'''} = Z^\sim (E)^{'''}$. Let $j$ be the natural embedding of $Z^\sim (E)' = C(K)'$ into $C(K)^{'''} = Z(E^\sim)'$ and let $H_1 = j(Z^\sim (E)'_1), H_2 = H_1^d.$ $H_1$ is a band of $Z(E^\sim)'$ as $Z^\sim (E)'$ is an AL-space. Therefore $Z(E^\sim)' = C(Q)' = H_1 \oplus H_2$. It is well known that $H_1$ is the class of order continuous functionals on $C(Q)$ and therefore:

1. If $\mu \in H_1$ then the support of $\mu$ is a closed and open subset of $Q$;

2. If the support of $\mu \in C(Q)'$ is nowhere dense then $\mu \in H_2$.

Under this circumstances $\{Z^\sim (E)^0\}^d = \{0\}$ and $P = 0$. To see this, let $S(\mu) = j(\mu \mid_{Z^\sim(E)}) S : H_2 \rightarrow H_1$ be the restriction map. If $\vartheta$ is a nonzero measure in $H_2$ then the measure $\mu = \vartheta - S(\vartheta)$ is in $Z^\sim(E)^0$ and $|\mu| \land |\vartheta| = |\vartheta| \neq 0$. Therefore $P(\vartheta) = 0$. If $\mu$ is a non-zero measure in $H_1$, then by the following lemma, there exists a measure $\vartheta \in H_2$ with $S(\vartheta) = \mu$. The measure $\eta = \vartheta - \mu$ is an element of $Z^\sim(E)^0$ and $|\eta| \land |\mu| = |\mu| \neq 0$. 

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Therefore $P(\mu) = 0$.

Let us note that if $Q_1$ is a nowhere dense closed subset of $Q$ then $C(Q_1)'$ (considered as the space of measures on $Q$ whose supports are contained in $Q_1$) is contained in $H_2$. To complete the proof of $(Z^\sim(E))' = 0$ we only need to prove the following lemma.

**Lemma 1.** There exists a nowhere dense closed subset $Q_1$ of $Q$ such that $S(C(Q_1)') = H_1$.

**Proof.** Let $\varphi : Q \to K$ be the continuous surjection which gives rise the natural embedding $\pi \to \pi \cdot \varphi$ of $C(K)$ into $C(Q)$.

For each $t \in K$, let $\delta_t$ be the point evaluation at $t$, i.e., $\delta_t = \pi \mapsto \pi(t)$ on $C(K)$. Similarly, for each $q \in Q$, let $\Delta_q$ be the functional $\pi \mapsto \pi(q)$ on $C(Q)$. If $t = \varphi(q)$, then $\Delta_q |_{C(K)} = \delta_t$.

For each $t \in K$, there is a unique point in $Q$, say $\psi(t)$, such that $j(\delta_t) = \Delta_{\varphi(t)} \cdot \psi(t)$ is an isolated point of $Q$ and $\psi : K \to Q$ is discontinuous and maps $K$ onto an open subset $V = \psi(K)$ of $Q$. Let $Q_1 = V \setminus V$. $Q_1$ is nowhere dense and closed. To prove the lemma, it suffices to show that $\varphi(Q_1) = K$. Let $t \in K$. As there are no isolated points in $K$, there exists a net $\{t_\alpha\}, t_\alpha \not\to t$ for each $\alpha$ in $K$ with $t = \lim_{\alpha} t_\alpha$. Let $q$ be a cluster point of the net $\{\psi(t_\alpha)\}$. Then $q \in Q_1$ and $\varphi(q) = t$ as $t_\alpha = \varphi(\psi(t_\alpha))$ for each $\alpha$. \hfill $\square$

Let us note that the conclusion $Z^\sim(E)_C = Orth(E^\sim)$ remains valid for Arenson’s example. The details are below.

We now give a sufficient condition for $Z^\sim(E)_C = Orth(E^\sim)$. We first give a lemma that will be needed.

**Lemma 2.** Let $E$ be a Riesz space with topologically full centre and satisfying $(E^\sim)^\sim = (E^\sim)_\sim$. Then the bilinear map

$$(f,F) \mapsto \psi_{f,F} \text{ of } E^\sim \times (E^\sim)^\sim \to Z^\sim(E)^\sim \text{ defined by } \psi_{f,F}(\tilde{\pi}) = F(\tilde{\pi}f)$$

is a bi-lattice homomorphism.

**Proof.** For each $f \in E^\sim_+$, the map $\psi_f : (E^\sim)^\sim \to Z^\sim(E)^\sim$ defined by $F \mapsto \psi_{f,F}$ is positive. Hence we have $\psi_f(F)^+ \leq \psi_f(F^+)$ for each $F \in (E^\sim)^\sim$. Let $\tilde{\pi} \in Z^\sim(E)_+$ be arbitrary, then

$$\psi_f(F^+)(\tilde{\pi}) = \psi_{f,F^+}(\tilde{\pi}) = F^+(\tilde{\pi}f) = \sup\{F(g) : 0 \leq g \leq \tilde{\pi}f\}$$

If $0 \leq g \leq \tilde{\pi}f$, we claim there exists $\{\pi_\alpha\}$ in $Z(E)$ satistying $0 \leq \pi_\alpha \leq I$ for each $\alpha$ and $\pi_\alpha(\tilde{\pi}f) \to g$ in $\sigma(E^\sim,(E^\sim)^\sim)$. As $E^\sim$ is Dedekind complete, we can find $S \in Z(E^\sim)$
with \( 0 \leq S \leq I \) and \( S(\pi f) = g \). The Arens homomorphism \( m : Z(E)^{\prime\prime} \rightarrow Z(E^{\sim}) \) is surjective and continuous when the domain is equipped with \( \sigma(Z(E)^{\prime\prime}, Z(E)^{\prime}) \) and the range has the \( \sigma(E^{\sim}, (E^{\sim})^{\sim}_{0}) \) operator topology [6]. Therefore there exists \( F \) in \( Z(E)^{\prime\prime} \) with \( 0 \leq F \leq I \) satisfying \( m(F) = S \). Using the fact that \( Z(E) \) is \( \sigma(Z(E)^{\prime\prime}, Z(E)^{\prime}) \) dense in \( Z(E)^{\prime\prime} \), we can find a net \( \{\pi_{\alpha}\} \) in \( Z(E) \) satisfying \( 0 \leq \pi_{\alpha} \leq I \) for each \( \alpha \) and \( \pi_{\alpha} \rightarrow F \) in \( \sigma(Z(E)^{\prime\prime}, Z(E)^{\prime}) \). Continuity of the map \( m : Z(E)^{\prime\prime} \rightarrow Z(E^{\sim}) \) imply that \( m(\pi_{\alpha}) = \tilde{\pi}_{\alpha} \rightarrow m(F) = S \) in \( \sigma(E^{\sim}, (E^{\sim})^{\sim}_{0}) \) operator topology. This is to say \( G(\tilde{\pi}_{\alpha}h) \rightarrow G(Sh) \) for each \( h \in E^{\sim} \) and \( G \in (E^{\sim})^{\sim}_{0} \). Thus \( \tilde{\pi}_{\alpha}(\pi f) \rightarrow g \) in \( \sigma(E^{\sim}, (E^{\sim})^{\sim}_{0}) \).

\[
0 \leq \tilde{\pi}_{\alpha}(\pi f) \leq \tilde{\pi}(f) \quad \text{for each } \alpha, \text{ so that } F(\tilde{\pi}_{\alpha}(\pi f)) \leq \psi_{f}(F)^{+}(\tilde{\pi})
\]

which yields

\[
F(g) \leq \psi_{f}(F)^{+} \text{ for each } g \text{ with } 0 \leq g \leq \tilde{\pi}f. \text{ Hence } \psi_{f}(F^{+}) \leq \psi_{f}(F)^{+}.
\]

We now show that \( \psi_{F} : E^{\sim} \rightarrow Z^{\sim}(E)^{\sim} \) is a lattice homomorphism for an arbitrary \( F \in (E^{\sim})^{\sim}_{0} \). Let \( f \land g = 0 \) in \( E^{\sim} \). As \( I \) is a strong order unit in \( Z^{\sim}(E) \), it suffices to show \([\psi_{f}(f) \land \psi_{F}(g)](I) = 0\).

\[
[\psi_{f}(f) \land \psi_{F}(g)](I) = (\psi_{f,F} \land \psi_{g,F})(I)
\]

\[
= \inf\{\psi_{f,F}(\pi_{1}) + \psi_{g,F}(\pi_{2}) : \pi_{1}, \pi_{2} \in Z^{\sim}(E)_{+}; \pi_{1} + \pi_{2} = I\}
\]

\[
= \inf\{F(\pi_{1}f) + F(\pi_{2}g) : \pi_{1}, \pi_{2} \in Z^{\sim}(E)_{+}; \pi_{1} + \pi_{2} = I\}
\]

As \( E^{\sim} \) is Dedekind complete, the principal band generated by \( f, B_{f} \) is a projection band and let \( P_{f} : E^{\sim} \rightarrow B_{f} \) be this projection. \( P_{f} \in Z(E^{\sim}), P_{f}(g) = 0, (I - P_{f})(f) = 0 \) and \((I - P_{f}) + P_{f} = I\). Arguing as above, we can find a net \( (\pi_{\alpha}) \) in \( Z(E), 0 \leq \pi_{\alpha} \leq I \) and \( \tilde{\pi}_{\alpha} \rightarrow P_{f} \) in \( \sigma(E^{\sim}, (E^{\sim})^{\sim}_{0}) \) operator topology.

Thus,

\[
[\psi_{f}(f) \land \psi_{F}(g)](I) \leq F(I - \tilde{\pi}_{\alpha})f + F(\tilde{\pi}_{\alpha}g) \quad \text{for each } \alpha
\]

\[
\leq F(I - P_{f})f + F(P_{f}g) = 0.
\]

**Proposition.** Let \( E \) be a Riesz space with \( (E^{\sim})^{\sim} = (E^{\sim})^{\sim}_{0} \) and having topologically full centre. Then \( Z^{\sim}(E)_{C} = \text{Orth}(E^{\sim}) \).

**Proof.** Let \( T \in Z^{\sim}(E)_{C} \) be arbitrary; let \( f, g \in E^{\sim} \) satisfying \( f \perp g \). For each \( F, G \) in \( (E^{\sim})^{\sim} \), we have \( \psi_{F,F} \perp \psi_{g,G} [3] \).

Thus for \( f \in E^{\sim} \) and \( F \in (E^{\sim})^{\sim} \),

\[
\psi_{T,f,F}(\tilde{\pi}) = F(\tilde{T}(Tf)) = F(T(\tilde{\pi}f)) = \tilde{T}(F)(\tilde{\pi}f) = \psi_{f,T}(\tilde{\pi}f)
\]

which yields \([\psi_{T,F}(f) \land \psi_{g,F}] = \psi_{T,f,g,F} \land \psi_{g,F} = 0 \). Therefore \( F([Tf] \land [g]) = 0 \) for each \( F \in (E^{\sim})^{\sim} \) which gives \( Tf \perp g \) and \( T \) is an orthomorphism.
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