Formula for the Highly Regularized Trace of the Sturm-Liouville Operator with Unbounded Operator Coefficients Having Singularity

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Abstract

In this work, a formula for the $n^{th}$ regularized trace of the Sturm-Liouville operator with unbounded operator coefficients having singularity is obtained.

1. Introduction

The regularized trace of the scalar differential operators has been calculated by I. M. Gelfand and B. M. Levitan [1], L. A. Dikiy [2], C. J. Halberg and V. A. Kramer [3] among other works. The list of published work on this subject can be found in B. M. Levitan and I. S. Sargsyan [4], I. C. Fulton and S. A. Pruess [5]. The regularized trace of differential operators with unbounded operator coefficients has been computed in [6, 8, 9, 10].

In this study, the $n$ ($n \in \mathbb{N}$) regularized trace of Sturm-Liouville operator with unbounded operator coefficients having singularity has been calculated in the interval $[0,1]$.

Let $H$ be a separable Hilbert Space. Denote by $H_1 = L_2(H : [0,1])$ the Hilbert space of all vector-valued measurable functions $y(t)$ ($t \in [0,1]$) with values in $H$ and such that

$$\int_0^1 \|y(t)\|^2 dt < \infty.$$ 

The scalar product vectors $y(t), z(t)$ in $H_1$ is defined by
\( (y, z) = \int_0^1 (y(t), z(t)) dt \).

Let us write

\[
    l_0(y) = -y''(x) + \frac{v^2 - 1}{x^2} y(x) + Ay(x)
\]

\[
    l(y) = -y''(x) + \frac{v^2 - 1}{x^2} y(x) + Ay(x) + Q(x)y(x), \quad v \geq \frac{1}{2}
\]

where \( A \) is an operator from \( D(A) \) to \( H \) which satisfies the following conditions such that \( D(A) = H \):

\[
    A = A^* \geq I, \quad A^{-1} \in \sigma_\infty(H).
\]

Expressions \( l_0(y) \) and \( l(y) \) form two operators \( L \) and \( L_0 \) in space \( H_1 \):

\[
    D(L_0) = \{ y(x) \in H_1 : y(x) \in D(A), y''(x), Ay(x) \text{ continuous in } [0,1] \text{ according to norm in } H \} \nonumber
\]

\[
    y(0) = y(1) = 0 \quad \text{and} \quad l_0(y) \in H_1
\]

\[
    L_0 y = l_0(y), \quad y \in D(L_0).
\]

Let us denote \( \overline{L_0} = L_0 \), where the overbar symbol shows closure of the operator. It can be shown that \( L_0 \) is a self-adjoint operator in \( H_1 \). Operator \( L = L_0 + Q(x) \) (\( Q(x) = Q^*(x) \) and is bounded in \( H_1 \); see condition 2) ) with domain \( D(L) = D(L_0) \) is a self-adjoint operator in \( H_1 \).

Let us accept that the operator function \( Q(x) \) in the expression \( l(y) \) satisfies the following conditions:

1) \( Q(x) \) has a weak derivative of the second order in \([0,1]\). For \( \forall x \in [0,1] \), \( Q^{(i)}(x) \) \((i=0,1,2)\) are self-adjoint operator from \( H \) to \( H \) and \( A^{n-1}Q^{(i)}(x) \in \sigma_1(H) \);

2) The functions \( ||A^{n-1}Q^{(i)}(x)||_{\sigma_1(H)} \) \((i=0,1,2)\) are bounded and measurable in \([0,1]\);
3) For $\forall f \in H$, $\int_0^1 (Q(x)f, f)dx = 0$;

4) For $\forall f \in H$ and $x \in [0, \delta]$, there is an operator $B \in \sigma_1(H)$ such that

$$|(Q(x)f, f)| \leq |(Bf, f)|.$$ 

Here, $\sigma_1(H)$ is the space of kernel operators from $H$ to $H$ as in [11], and $(., .)$ denotes the inner product in $H$. We denote the inner product by $(., .)_1$ and the norm of operator by $||.||_1$ in $H_1$.

In this study, we will use the following inequalities proved in [11]:

$$|trB_1| \leq ||B_1||_{\sigma_1(H)}$$

$$||B_1B_2||_{\sigma_1(H)} \leq ||B_1||_{\sigma_1(H)}.||B_2||$$

$$||B_2B_1||_{\sigma_1(H)} \leq ||B_1||_{\sigma_1(H)}.||B_2||,$$

where $B_1 \in \sigma_1(H)$ and $B_2 \in L(H, H)$.

Note that the operator $L_0$ has a pure discrete spectrum. Let $\mu_1 \leq \mu_2 \leq ...$ be the eigenvalues of this operator and $\psi_1(x), \psi_2(x), ...$ be the orthonormal eigenvectors corresponding to these eigenvalues. Here, every eigenvalue is repeated according to multiplicity number. Since $Q$ is a self-adjoint bounded operator from $H_1$ to $H_1$, $L = L_0 + Q$ is a self-adjoint operator in $H_1$ which satisfies $D(L) = D(L_0)$ and it has a pure discrete spectrum. Let $\lambda_1 \leq \lambda_2 \leq ...$ be the eigenvalues of this operator. Moreover, let $\gamma_1 \leq \gamma_2 \leq ...$ be the eigenvalues of the operator $A$ from $D(A)$ to $H$ and $\varphi_1, \varphi_2, ...$ be orthonormal eigenvectors corresponding to these eigenvalues.

If $\gamma_i \sim ai^\alpha$ while $i \to \infty$ $(a>0, \alpha>2)$, then it is proved in [7], [8] (and, respectively, for $\gamma=1/2; \gamma \geq 1/2$) that

$$\mu_m, \lambda_m \sim dm^{2/\alpha}$$

while $m \to \infty, (d > 0)$ (1)

is satisfied. By using this formula, we can show that the sequence $\{\mu_m\}_{m=1}^\infty$ has a subsequence $\{\mu_{mp}\}_{p=1}^\infty$ such that
\[ \mu_k - \mu_{m_p} \geq d_1 \left( k \frac{\mu_k}{m_p} - m_p \frac{\mu_k}{m_p} \right), \quad (k = m_p, m_p + 1, m_p + 2, \ldots), \]  

(2)

where \( d_1 \) is a positive constant. Let \( R_0^\lambda, R_\lambda \) be the resolvents of \( L_0 \) and \( L \), respectively. From (1), if \( \alpha > 2 \) and \( \lambda \neq \lambda_k, \mu_k \) \( (k = 1, 2, \ldots) \) then the series \( \sum_{k=1}^{\infty} |\mu_k - \lambda|^{-1} \) and \( \sum_{k=1}^{\infty} |\lambda_k - \lambda|^{-1} \) are convergent. Therefore \( R_0^\lambda \) and \( R_\lambda \) are the kernel operators. In this case, from [11] it is known that

\[ tr(R_\lambda - R_0^\lambda) = trR_\lambda - trR_0^\lambda = \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k - \lambda} - \frac{1}{\mu_k - \lambda} \right). \]

If we multiply this equation with \( \frac{\lambda^n}{2\pi i} \) and integrate along the circle

\[ |\lambda| = b_p = 2^{-1}(\mu_{m_p} + \mu_{m_p+1}) \]

then we obtain

\[ \frac{1}{2\pi i} \int_{|\lambda|=b_p} \lambda^n tr(R_\lambda - R_0^\lambda) d\lambda = \sum_{k=1}^{\infty} \left\{ \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{\lambda^n}{\lambda - \lambda_k} d\lambda - \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{\lambda^n}{\lambda - \mu_k} d\lambda \right\}. \]  

(3)

It can be shown that for large values of \( p, \)

\[ \{\lambda_k, \mu_k\}_{k=1}^{m_p} \subset K(0, b_p) = \{ \lambda : |\lambda| < b_p \}. \]  

(4)

Here, \( \lambda_k, \mu_k \notin K(0, b_p) = \{ \lambda : |\lambda| \leq b_p \}, (k \geq m_p + 1). \) From (3) and (4) we find

\[ \sum_{k=1}^{m_p} (\lambda_k^n - \mu_k^n) = \frac{1}{2\pi i} \int_{|\lambda|=b_p} \lambda^n tr(R_\lambda - R_0^\lambda) d\lambda. \]  

(5)

It is known that \( R_\lambda = R_\lambda^0 - R_\lambda Q R_0^\lambda. \) From here, for any natural number \( N \geq 2, \) we obtain

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\[ R_\lambda - R_\lambda^0 = \sum_{j=1}^{N} (-1)^j R_\lambda^0 (QR_\lambda^0)^j + (-1)^{N+1} R_\lambda (QR_\lambda^0)^{N+1}. \]

If we substitute this expression into (5) then we find

\[
\sum_{k=1}^{m_p} (\lambda_k^n - \mu_k^n) = \frac{1}{2\pi i} \int_{|\lambda|=b_p} \lambda^n \text{tr} \left[ \sum_{j=1}^{N} (-1)^{j+1} R_\lambda^0 (QR_\lambda^0)^j + (-1)^{N} R_\lambda (QR_\lambda^0)^{N+1} \right] d\lambda
\]

\[ = \sum_{j=1}^{N} B_{pj} + B_p^{(N)}. \quad (6) \]

Here,

\[ B_{pj} = \frac{(-1)^{j+1}}{2\pi i} \int_{|\lambda|=b_p} \lambda^n \text{tr} \left[ R_\lambda^0 (QR_\lambda^0)^j \right] d\lambda, \quad (j = 1, 2, \ldots, N) \quad (7) \]

\[ B_p^{(N)} = \frac{(-1)^N}{2\pi i} \int_{|\lambda|=b_p} \lambda^n \text{tr} \left[ R_\lambda (QR_\lambda^0)^{N+1} \right] d\lambda. \quad (8) \]

For every natural number \( j \), it can be shown that the operator function \((QR_\lambda^0)^j\) is analytic according to the norm in \(\sigma_1(H_1)\) in the resolvent region \(\rho(L_0)\)of the operator \(L_0\). Moreover,

\[ \text{tr} \left[ R_\lambda^0 (QR_\lambda^0)^j \right] = \text{tr} \left[ (QR_\lambda^0)^{j-1} Q (R_\lambda^0)^2 \right] = \text{tr} \left[ (QR_\lambda^0)^{j-1} \frac{d}{d\lambda} (QR_\lambda^0) \right], \]

\[ \text{tr} \left\{ \frac{d}{d\lambda} [(QR_\lambda^0)^j] \right\} = j \text{tr} \left[ (QR_\lambda^0)^{j-1} \frac{d}{d\lambda} (QR_\lambda^0) \right]. \]
From the last two relations one obtains

$$\text{tr}[R^0_n(QR^0_n)^j] = \frac{1}{j}\text{tr}\left\{\frac{d}{d\lambda}[(QR^0_n)^j]\right\}.$$  

If this expression is substituted in (7), then

$$B_{pj} = \frac{(-1)^{j+1}}{2\pi ij} \int_{|\lambda|=b_p} \lambda^n \text{tr}\left\{\frac{d}{d\lambda}[(QR^0_n)^j]\right\} d\lambda$$

$$= \frac{(-1)^{j+1}}{2\pi ij} \int_{|\lambda|=b_p} \text{tr}\left\{\frac{d}{d\lambda}[\lambda^n(QR^0_n)^j] - n\lambda^{n-1}(QR^0_n)^j\right\} d\lambda$$

is found. So if we note that

$$\text{tr}\left\{\frac{d}{d\lambda}[\lambda^n(QR^0_n)^j]\right\} = \frac{d}{d\lambda}\{\text{tr}[\lambda^n(QR^0_n)^j]\}$$

then

$$B_{pj} = \frac{(-1)^j}{2\pi ij} \int_{|\lambda|=b_p} \lambda^{n-1} \text{tr}[(QR^0_n)^j] d\lambda + \frac{(-1)^{j+1}}{2\pi ij} \int_{|\lambda|=b_p} \frac{d}{d\lambda}\{\text{tr}[\lambda^n(QR^0_n)^j]\} d\lambda$$

is obtained. On the other hand, by using (4) we obtain

$$\int_{|\lambda|=b_p} \frac{d}{d\lambda}\{\text{tr}[\lambda^n(QR^0_n)^j]\} d\lambda = 0.$$  

Therefore,

$$B_{pj} = \frac{(-1)^j}{2\pi ij} \int_{|\lambda|=b_p} \lambda^{n-1} \text{tr}[(QR^0_n)^j] d\lambda$$  \hspace{1cm} (9)
From (6) and (10), we obtain
\[
B_{pj} = (-1)^j j^{-1} n \sum_{k=1}^{m_p} \text{Res} \left[ \lambda^{n-1} \text{tr} (QR_0^\lambda)^j \right].
\] (10)

Let \( \beta_1 < \beta_2 < \ldots \) be the non-negative roots of the Bessel function \( J_\nu(x) \). The eigenvalues of the operator \( L_0 \) and its eigenvectors corresponding to these eigenvalues are of the respective forms
\[
\beta_q^2 + \gamma_i, \quad (q, i = 1, 2, \ldots)
\]

\[
\sqrt{2 \pi} \frac{J_\nu(\beta_q x)}{J_{\nu+1}(\beta_q)} \varphi_i, \quad (q, i = 1, 2, \ldots).
\]

Therefore, the eigenvalues \( \mu_1 \leq \mu_2 \leq \ldots \) of the operator \( L_0 \)
\[
\mu_k = \beta_q^2 + \gamma_{i_k} \quad (k = 1, 2, \ldots)
\]

and the orthonormal eigenvectors \( \psi_k(x) \) corresponding to these eigenvalues can be in the form
\[
\psi_k(x) = \sqrt{2 \pi} \frac{J_\nu(\beta_q x)}{J_{\nu+1}(\beta_q)} \varphi_{i_k} \quad (k = 1, 2, \ldots).
\]

From formula (9) we have
\[
B_{p1} = -\frac{n}{2\pi i} \int_{|\lambda|=b_p} \lambda^{n-1} \text{tr} \left[ (QR_0^\lambda)^j \right] d\lambda.
\]
Moreover, \( tr(QR^0_k) = \sum_{k=1}^{\infty} (QR^0_k \psi_k, \psi_k)_1 = \sum_{k=1}^{\infty} (\mu_k - \lambda)^{-1}(Q \psi_k, \psi_k)_1 \). Hence,

\[
B_{p1} = n \sum_{k=1}^{\infty} (Q \psi_k, \psi_k)_1 \frac{1}{2\pi i} \int_{|\lambda| = b_p} \frac{\lambda^{n-1}}{\lambda - \mu_k} d\lambda
\]

\[
= n \sum_{k=1}^{m_p} \mu_k^{n-1}(Q \psi_k, \psi_k)_1
\]

\[
= n \sum_{k=1}^{m_p} (\beta^2_{q_k} + \gamma_{i_k})^{n-1}(Q \psi_k, \psi_k)_1
\]

\[
= n \sum_{k=1}^{m_p} \sum_{l=0}^{n-1} C_{n-1}^l \beta^{2(n-1-l)}_{q_k} \gamma^{l}_{i_k} (Q \psi_k, \psi_k)_1
\]

\[
= n \sum_{k=1}^{m_p} \sum_{l=0}^{n-2} C_{n-1}^l \beta^{2(n-1-l)}_{q_k} \gamma^{l}_{i_k} (Q \psi_k, \psi_k)_1 + n \sum_{k=1}^{m_p} \gamma^{n-1}_{i_k} (Q \psi_k, \psi_k)_1
\]

(12)

From (11) and (12) we find

\[
\sum_{k=1}^{m_p} \left\{ \lambda^n_k - \mu^n_k - n \sum_{j=2}^{N} (-1)^j j^{-1} \text{Res}_{\lambda=\mu_k} [\lambda^{n-1} tr(QR^0_k)^j] - n \sum_{l=0}^{n-2} C_{n-1}^l \beta^{2(n-1-l)}_{q_k} \gamma^{l}_{i_k} (Q \psi_k, \psi_k)_1 \right\}
\]

\[
= n \sum_{k=1}^{m_p} \gamma^{n-1}_{i_k} (Q \psi_k, \psi_k)_1 + \beta^N_p,
\]

(13)

where the indices \( q_k \) and \( i_k \) are the natural numbers such that \( \mu_k = \beta^2_{q_k} + \gamma_{i_k} (\mu_1 \leq \mu_2 \leq \ldots \) and \( k = 1, 2, \ldots \)). In the following we will compute the limit

\[
\lim_{p \to \infty} \sum_{k=1}^{m_p} \left\{ \lambda^n_k - \mu^n_k - n \sum_{j=2}^{N} (-1)^j j^{-1} \text{Res}_{\lambda=\mu_k} [\lambda^{n-1} tr(QR^0_k)^j] - n \sum_{l=0}^{n-2} C_{n-1}^l \beta^{2(n-1-l)}_{q_k} \gamma^{l}_{i_k} (Q \psi_k, \psi_k)_1 \right\}
\]

This limit will be called the \( n^{th} \) regularized trace of the operator \( L \).
Theorem 1. If the conditions 1)-4) are satisfied then

\[
\sum_{q=1}^{\infty} \sum_{i=1}^{\infty} \left| \gamma_i \right|^{n-1} \int_0^1 2x \frac{J_q^2(\beta_q x)}{J_{q+1}^2(\beta_q)} (Q(x)\varphi_i, \varphi_i) dx < \infty.
\]

Proof. Let \( f_i(x) = (Q(x)\varphi_i, \varphi_i) \). Then

\[
\int_0^1 2x \frac{J_q^2(\beta_q x)}{J_{q+1}^2(\beta_q)} (Q(x)\varphi_i, \varphi_i) dx = \int_0^1 2x \frac{J_q^2(\beta_q x)}{J_{q+1}^2(\beta_q)} f_i(x) dx
\]

\[
+ \int_0^{\beta_q^{-1+\varepsilon}} 2x \frac{J_q^2(\beta_q x)}{J_{q+1}^2(\beta_q)} [f_i(x) - f_i(0)] dx + f_i(0)
\]

(14)

where \( \varepsilon \in (0, \frac{1}{2}) \). It has been shown in [8] that the asymptotic formula

\[
2x \frac{J_q^2(\beta_q x)}{J_{q+1}^2(\beta_q)} = 1 + \sin(2\beta_q x - \nu \pi) - \frac{\cos(2\beta_q x - \nu \pi)}{8\beta_q x} (4\nu^2 - 1) + O((\beta_q x)^{-2})
\]

(15)

is satisfied for the large value of \( q \), where \( x \in [\beta_q^{-1+\varepsilon}, 1) \).

From (14) and (15) we write

\[
\sum_{q=1}^{\infty} \sum_{i=1}^{\infty} \gamma_i^{n-1} \left| \int_0^1 2x \frac{J_q^2(\beta_q x)}{J_{q+1}^2(\beta_q)} f_i(x) dx \right| \leq \text{const} \sum_{q=1}^{\infty} \sum_{i=1}^{\infty} \left| \beta_q^{-1+\varepsilon} \int_0^1 [f_i(x) - f_i(0)][2x \frac{J_q^2(\beta_q x)}{J_{q+1}^2(\beta_q)} dx] +
\]

\[
\int_0^{\beta_q^{-1+\varepsilon}} (f_i(x) - f_i(0))(1 + \sin(2\beta_q x - \nu \pi)) dx + f_i(0)
\]

\[
+ \int_0^{\beta_q^{-1+\varepsilon}} (f_i(x) - f_i(0)) \frac{\cos(2\beta_q x - \nu \pi)}{8\beta_q x} dx + f_i(0)
\]

(16)

By using the inequality \( |\sqrt{x}J_{\nu}(x) - C \) and conditions 1), 4) for the first integral on the right hand side of (16), we get
\[
\gamma_i^{-1}\left| \int_0^1 (f_i(x) - f_i(0))(1 + \sin(2\beta_q x - \nu \pi))dx + f_i(0) \right| = \gamma_i^{-1}\left| \int_0^1 (f_i(x) - f_i(0))dx \right|
\]

\[
\leq \gamma_i^{-1}\int_0^1 |f_i(x)|dx + \gamma_i^{-1}\left| \frac{\cos(2\beta_q x - \nu \pi)(f_i(1) - f_i(0))}{2\beta_q} \right|
\]

\[
- \frac{\cos(2\beta_q x - \nu \pi)(f_i(\beta_q^{-1+\varepsilon}) - f_i(0))}{2\beta_q} - \frac{1}{2\beta_q} \int f_i'(x) \cos(2\beta_q x - \nu \pi)dx
\]

\[
\leq \text{Const} \left[ \left| (B\varphi, \varphi) \right| \beta_q^{-2+2\varepsilon} + \gamma_i^{-1} \beta_q^{-2} (|f_i(0)| + |f_i(1)|) + |(B\varphi, \varphi)| \beta_q^{-2+\varepsilon} \right]
\]

\[
+ \frac{\gamma_i^{-1}}{2\beta_q} \left| f_i'(1) \sin(2\beta_q - \nu \pi) - f_i'(\beta_q^{-1+\varepsilon}) \sin(2\beta_q - \nu \pi) \right|
\]

\[
- \frac{1}{\beta_q^{-1+\varepsilon}} \left| f_i''(x) \sin(2\beta_q x - \nu \pi) \right| \leq \text{Const} \left[ \left| (B\varphi, \varphi) \right| \beta_q^{-2+2\varepsilon} \right]
\]

\[
+ \gamma_i^{-1} \beta_q^{-2} (|f_i(0)| + |f_i(1)| + |f_i'(1)|) + \gamma_i^{-1} \beta_q^{-2} \frac{1}{0} \left| f_i''(x) \right|dx.
\]

From the conditions 1) to 4) and by using the asymptotic formula \( \beta_q = (q + \frac{1}{2} - \frac{1}{4}) \pi + \text{O}(q^{-1}) \) we obtain

\[
\gamma_i^{-1}\left| \int_0^1 (f_i(x) - f_i(0))(1 + \sin(2\beta_q x - \nu \pi))dx + f_i(0) \right| = \gamma_i^{-1}\left| \int_0^1 (f_i(x) - f_i(0))dx \right|
\]

\[
\leq \gamma_i^{-1}\int_0^1 |f_i(x)|dx + \gamma_i^{-1}\left| \frac{\cos(2\beta_q x - \nu \pi)(f_i(1) - f_i(0))}{2\beta_q} \right|
\]

Now, let us give a bound the third integral on the right hand side of (16). First we write

\[
\left| \int_0^1 (f_i(x) - f_i(0)) \frac{\cos(2\beta_q x - \nu \pi)}{\beta_q x} dx \right| = \left| \int_0^\delta (f_i(x) - f_i(0)) \frac{\cos(2\beta_q x - \nu \pi)}{\beta_q x} dx \right|
\]

\[
+ \left| \int_\delta^1 (f_i(x) - f_i(0)) \frac{\cos(2\beta_q x - \nu \pi)}{\beta_q x} dx \right|.
\]
Finally let us give a bound for the forth integral on the right hand side of (16). 

\[
\gamma_i^{n-1} \left| \int_{\beta_q^{-1+\varepsilon}}^1 \frac{f_i(x) - f_i(0)}{\beta_q x} \cdot \cos(2\beta_q x - \nu x)dx \right| \leq \gamma_i^{n-1} \int_{\beta_q^{-1+\varepsilon}}^1 \frac{f_i(x) - f_i(0)}{\beta_q x} \cdot \cos(2\beta_q x - \nu x)dx \\
- \frac{\gamma_i^{n-1}}{\beta_q} \int_{\beta_q^{-1+\varepsilon}}^1 \frac{f_i'(x) - f_i'(0)}{x^2} \cdot \sin(2\beta_q x - \nu x)dx \\
+ \frac{\gamma_i^{n-1}}{\beta_q^2} \int_{\beta_q^{-1+\varepsilon}}^1 \frac{f_i(x) - f_i(0)}{\beta_q x} \cdot \sin(2\beta_q x - \nu x)dx \\
\leq \text{Const.} |(B\phi_i, \phi_i)| \beta_q^{-2} + \text{Const.} \gamma_i^{n-1} \beta_q^{-2} \left[ |f_i(0)| + |f_i(1)| + \int_0^1 (|f_i(x)| + |f_i'(x)|)dx \right].
\]

(19)

From (1),(2) and (4) we have

\[
\gamma_i^{n-1} \left| \int_{\beta_q^{-1+\varepsilon}}^1 \frac{f_i(x) - f_i(0)}{\beta_q x} \cdot \cos(2\beta_q x - \nu x)dx \right| \leq \gamma_i^{n-1} \int_{\beta_q^{-1+\varepsilon}}^1 \frac{f_i(x) - f_i(0)}{\beta_q x} \cdot \cos(2\beta_q x - \nu x)dx \\
- \frac{\gamma_i^{n-1}}{\beta_q} \int_{\beta_q^{-1+\varepsilon}}^1 \frac{f_i'(x) - f_i'(0)}{x^2} \cdot \sin(2\beta_q x - \nu x)dx \\
+ \frac{\gamma_i^{n-1}}{\beta_q^2} \int_{\beta_q^{-1+\varepsilon}}^1 \frac{f_i(x) - f_i(0)}{\beta_q x} \cdot \sin(2\beta_q x - \nu x)dx \\
\leq \text{Const.} |(B\phi_i, \phi_i)| \beta_q^{-2} + \text{Const.} \gamma_i^{n-1} \beta_q^{-2} \left[ |f_i(0)| + |f_i(1)| + \int_0^1 (|f_i(x)| + |f_i'(x)|)dx \right].
\]

(20)

From the relations (16),(17),(18),(19) and (20) we obtain

\[
\sum_{q=1}^{\infty} \sum_{i=1}^{\infty} \gamma_i^{n-1} \left| 2x \beta_q^2 \frac{\gamma_i}{\frac{1}{\beta_q^{-1+\varepsilon}}(\beta_q x)} f_i(x)dx \right| \leq \text{Const.} \left( \sum_{q=1}^{\infty} \beta_q^{-2+2\varepsilon} \right) \sum_{i=1}^{\infty} |(B\phi_i, \phi_i)| \\
+ \text{Const.} \sum_{q=1}^{\infty} \beta_q^{-2} \sum_{i=1}^{\infty} \gamma_i^{n-1} \left[ |f_i(0)| + |f_i(1)| + |f_i'(1)| + \int_0^1 (|f_i(x)| + |f_i'(x)| + |f_i''(x)|)dx \right].
\]

(21)

By assumption, since $B \in \sigma_1(H)$ and $\|A^{n-j}Q^{(j)}(x)\|_{\sigma_1(H)} \leq \text{Const.} (j = 0, 1, 2)$, we have

\[
\sum_{i=1}^{\infty} |(B\phi_i, \phi_i)| \leq \|B\|_{\sigma_1(H)}.
\]

(22)
and

\[
\sum_{i=1}^{\infty} \gamma_i^{n-1} \int_0^1 \left( |f_i(x)| + |f'_i(x)| + |f''_i(x)| \right) \, dx \\
= \lim_{N \to \infty} \int_0^1 \sum_{i=1}^{N} \gamma_i^{n-1} \left( |f_i(x)| + |f'_i(x)| + |f''_i(x)| \right) \, dx \\
\leq \int_0^1 \left[ \sum_{i=1}^{\infty} \gamma_i^{n-1} \left( \sum_{j=0}^{2} |f_i^{(j)}(x)| \right) \right] \, dx \\
\leq \int_0^1 \left[ \sum_{j=0}^{2} \| A^{n-1} Q^{(j)}(x) \|_{\sigma_1(H)} \right] \, dx < \text{Const.} \tag{23}
\]

If we take into account \( \lim_{q \to \infty} \frac{2}{q} = 1 \), then from (21),(22) and (23) we obtain

\[
\sum_{q=1}^{\infty} \sum_{i=1}^{\infty} \left| \gamma_i^{n-1} \int_0^1 2x J_2^2(\beta_q x) (Q(x) \varphi_i, \varphi_i) \, dx \right| < \text{Const.}
\]

Thus the theorem is proved. \( \square \)

The main result of this work is given in the following theorem.

**Theorem 2.** If the conditions 1)-4) are satisfied and \( \lim_{i \to \infty} \frac{\gamma_i}{q} = 1 \) \((a > 0, \alpha > 2)\), then for the \( n \)th regularized trace of the operator \( L \) the formula

\[
\lim_{\nu \to \infty} \sum_{k=1}^{N} \left\{ \lambda_k^0 - \mu_k^0 - n \sum_{j=2}^{N} (-1)^j \text{Res} \left[ \lambda^{n-1} \text{tr}(Q R_0^0) \right] \right. \\
- n \sum_{l=0}^{n-2} C_{\nu-1}^l \beta_k^2 (n-1-l) \gamma_{i_k} \nu_k (Q \psi_k, \psi_k) \nu_k \left. \right\} = -\frac{\pi}{4} 2n \text{tr} (A^{n-1} Q(0)) + \text{tr} (A^{n-1} Q(1))
\]

is satisfied, where \( N = \left[ \frac{2(2a+2)}{\alpha-2} \right] + 1 \).

**Proof.** From theorem 1, we observe that the sequence \( \sum_{k=1}^{\infty} \gamma_{i_k}^{n-1} (Q \psi_k, \psi_k) \nu_k \) is convergent and

\[
\sum_{k=1}^{\infty} \gamma_{i_k}^{n-1} (Q \psi_k, \psi_k) \nu_k = \sum_{i=1}^{\infty} \sum_{q=1}^{\infty} \gamma_i^{n-1} \int_0^1 2x J_2^2(\beta_q x) (Q(x) \varphi_i, \varphi_i) \, dx.
\]

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From these and the equality

$$\sum_{q=1}^{\infty} \int_0^1 2x J_q^2(\beta_q x) (Q(x) \phi_1, \phi_1) dx = \frac{1}{4} [2\nu (Q(0) \phi_1, \phi_1) + (Q(1) \phi_1, \phi_1)]$$

proved in [8], we obtain

$$\sum_{k=1}^{\infty} \gamma_k^{-1}(Q\psi_k, \psi_k)_1 = -\frac{1}{4} \left[ 2\nu \sum_{i=1}^{\infty} \gamma_i^{-1}(Q(0) \phi_1, \phi_1) + \sum_{i=1}^{\infty} \gamma_i^{-1}(Q(1) \phi_1, \phi_1) \right]$$

$$= -\frac{1}{4} [2\nu tr(A^{-1} Q(0)) + tr(A^{-1} Q(1))]$$  \hspace{1cm} (24)

By making use of inequality (2), it can be proved that the inequalities

$$\|R^{(0)}_\lambda\|_{H^1_0} \leq Const. m_p^{1-\delta}$$

$$\|R^{(0)}_\lambda\|_1 \leq Const. m_p^{-\delta}$$

$$\|R_\lambda\|_1 \leq Const. m_p^{-\delta}$$

are satisfied on the circle $|\lambda| = b_p$ where $\delta = \frac{\omega - 2}{\sigma + 2}$. Now, let us give a bound for $B^{(N)}_p$ by using these inequalities. From the (8), we find

$$B^{(N)}_p \leq \frac{1}{2\pi} \int_{|\lambda| = b_p} |\lambda|^n |tr[R_\lambda (QR^{(0)}_\lambda)^{N+1}]| d\lambda$$

$$\leq b_p^n \int_{|\lambda| = b_p} \|R_\lambda (QR^{(0)}_\lambda)^{N+1}\|_{H^1_0} d\lambda$$

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\begin{align*}
&\leq b_p^n \int_{|\lambda|=b_p} \|R_{\lambda,1}\|(QR_0^N)^{N+1}\|\sigma_1(H_1)\|d\lambda \\
&\leq b_p^n \int_{|\lambda|=b_p} \|R_{\lambda,1}\| \|QR_0^N\|\|Q\|\|R_0^N\|\|\sigma_1(H_1)\|d\lambda \\
&\leq b_p^n \int_{|\lambda|=b_p} \|R_{\lambda,1}\| \|QR_0^N\|\|Q\|\|R_0^N\|\|\sigma_1(H_1)\|d\lambda \\
&\leq \text{Const.} b_p^{n+1} m_p^{-(N+1)\delta} m_p^{1-\delta}. 
\end{align*}

Taking into account \( b_p \leq \text{Const.} m_p^{1+\delta} \), one then obtains

\[ |B_p^{(N)}| \leq \text{Const.} m_p^{(n+1)(1+\delta)-(N+1)\delta+1-\delta}. \]

Hence, if

\[ N = \left\lceil \delta^{-1}(n + 2 + n\delta - \delta) \right\rceil + 1 = \left\lceil \frac{2\alpha n + \alpha + 6}{\alpha - 2} \right\rceil + 1 \]

then

\[ \lim_{p \to \infty} B_p^{(N)} = 0. \]

From (13), (24) and this last equality, we obtain

\[ \lim_{p \to \infty} \sum_{k=1}^{m_p} \left\{ \lambda_k^n - \mu_k^n - n \sum_{j=2}^{N} (-1)^j j^{-1} \text{Res} \left[ \lambda^{n-1} \text{tr}(QR_0^N) \right] \right\} \]

Thus, the proof is done.
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References


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