Turk J Math
25 (2001) , 299 – 305.

A Class of Monoids Embeddable in a Group

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Abstract

In this paper, we develop a new method to show that a monoid, given by a certain kind of presentation, embeds in a group. A mathematical device called the *diamond condition* was used in [5] to prove that the singular braid monoid SB_n embeds. Motivated by this, we consider monoid presentations which have the basic properties of the presentation of the singular braid monoid. In the same way as in [5], we prove that the monoid embeds. The proof of the diamond condition is completely geometric in [5], but here we prove it by using elementary algebraic properties.

Key Words: Embedding, monoid presentation, group presentation, geometric braids.

§1. Introduction

Every monoid can be presented by a set of generators $\mathbf{X} = \{x_i \mid i \in \Omega\}$ and a set of relations $\mathbf{R} = \{u_i = v_i \mid i \in \Omega'\}$, where Ω, Ω' are some indexing sets and and u_i, v_i are words in the free monoid $F^+(\mathbf{X})$ on $\mathbf{X}$. The monoid presentation will be denoted by $[\mathbf{X} \mid \mathbf{R}]$. Two words $w, w' \in F^+(\mathbf{X})$ are the same element of M if there is a sequence of words, $w \equiv w_o, w_1, \dots, w_t \equiv w'$ where w_i is obtained from $w_{i-1},\ i = 1, \dots, t$, by a substitution of the form $au_ib \leftrightarrow av_ib$ for some defining relation $u_i = v_i \in \mathbf{R}$. We call this an *elementary substitution*. If a monoid M is given by the presentation $[\mathbf{X} \mid \mathbf{R}]$ then the group of M is defined to be the quotient group $F(\mathbf{X})/N(\mathbf{R})$, where $F(\mathbf{X})$ is the free group on $\mathbf{X}$ and $N(\mathbf{R})$ is the smallest normal subgroup of $F(\mathbf{X})$ containing the elements $g^{-1}u_iv_i^{-1}g$ where $g \in F(\mathbf{X})$ and $u_i = v_i \in \mathbf{R}$. We denote the group of M by $G(M)$ and we use $< \mathbf{X} \mid \mathbf{R} >$ to show that the presentation is a group presentation. The natural map from a monoid M to its group is not in general an injection even if left and right cancellations hold, (see for example [1] and [4]).

Proposition 1 *If a monoid M embeds in a group G then it embeds in its group $G(M)$. In other words, if there is an injection $M \hookrightarrow G$, then there is a commutative diagram*

$$\begin{array}{ccc} & & G(M) \\ & \nearrow & \downarrow \\ M & \hookrightarrow & G \end{array}$$

and $M \to G(M)$ is injective. $\square$

Lemma 2 *Let G be a group given by the presentation $< \mathbf{X} \mid \mathbf{R} >$ and let $u, v \in F(\mathbf{X})$ be such that $u =_G v$. Then v can be obtained from u by applying a finite sequence of elementary substitutions and the introduction or deletion of cancelling pairs of generators $axx^{-1}b \leftrightarrow ab \leftrightarrow ax^{-1}xb$.* $\square$

Theorem 3 *If a monoid $\mathcal{M}$ given by a presentation $\mathcal{P} = [\mathbf{A} \cup \mathbf{B} \mid \mathbf{R}]$ where*
$\mathbf{A} = \{a_1, \ldots, a_n\}$,
$\mathbf{B} = \{b_1, \ldots, b_m\}$ *and* $\mathbf{R} = \mathbf{R}_1 \cup \mathbf{R}_2 \cup \mathbf{R}_3 \cup \mathbf{R}_4$ *satisfies:*
$\mathbf{R}_1$ *consists of relations of the form $u = v$, $u, v \in F^+(\mathbf{A})$.*
$\mathbf{R}_2 = \{a_i u_i = u_i a_i = 1 \mid$ *for some $u_i \in F^+(\mathbf{A})$, for all $i = 1, \ldots, n\}$.*
$\mathbf{R}_3$ *consists of relations of the form $ub_j = b_k u$ for some $j, k = 1, \ldots, m$ and $u \in F^+(\mathbf{A})$.*
$\mathbf{R}_4$ *consists of relations of the form $b_j b_k = b_k b_j$ for some $j, k = 1, \ldots, m$;*
then $\mathcal{M}$ embeds in a group.

This is the main theorem of this paper. First let us notice that no b_j has a right or a left inverse. If $\mathbf{B} = \emptyset$ then $\mathcal{M}$ is given by the presentation $[\mathbf{A} \mid \mathbf{R}_1 \cup \mathbf{R}_2]$ and it is a group. In short, we consider a presentation where invertable generators can have any relation among each other but the non-invertable generators are subject to relations $\mathbf{R}_3$ and $\mathbf{R}_4$. The proof of the theorem will follow some definitions and observations.

In the set-up of the theorem, let $\mathbf{X} = \mathbf{A} \cup \mathbf{B}$ and consider the monoid $\bar{\mathcal{M}}$ which is presented by $[\mathbf{X} \cup \bar{\mathbf{B}} \mid \mathbf{R} \cup \mathbf{R}']$ where $\bar{\mathbf{B}} = \{\bar{b}_j \mid j = 1, \ldots, m\}$ and $\mathbf{R}'$ consists of the following relations:
$u\bar{b}_j = \bar{b}_k u$ if $ub_j = b_k u \in \mathbf{R}$
$\bar{b}_j b_k = b_k \bar{b}_j$ if $b_j b_k = b_k b_j \in \mathbf{R}$
$\bar{b}_j \bar{b}_k = \bar{b}_k \bar{b}_j$ if $b_j b_k = b_k b_j \in \mathbf{R}$.

Observe that $\mathcal{M}$ is contained in $\bar{\mathcal{M}}$ as a submonoid since, if two words x, y in the alphabet of $\mathbf{X}$ are equal in $\bar{\mathcal{M}}$, then there is a sequence of elementary substitutions from x to y. But for each of the substitution, we use a relation $u = v$ which is in $\mathbf{R}$ because any word in the extra relations of $\bar{\mathcal{M}}$ contains a letter which is not in $\mathbf{X}$. Therefore $x = y$ in $\mathcal{M}$.

Let $x \equiv x_1x_2 \in \bar{\mathcal{M}}$, where $\equiv$ shows letter by letter equality or, in other words, equality in the free monoid. Then we say that there is an *elementary expansion* from x to y in $\bar{\mathcal{M}}$ if $y \doteq x_1b_i\bar{b}_ix_2$ or $y \doteq x_1\bar{b}_ib_ix_2$ for some b_i, where $\doteq$ shows equality in $\bar{\mathcal{M}}$. We denote this by $x \nearrow\!\!\nearrow y$. Equivalently, we say that there is an *elementary collapse* from y to x and write $y \searrow\!\!\searrow x$.

We say that x *expands* to y and write $x \nearrow y$ if y is obtained from x by a finite sequence of elementary expansions. We also say that y *collapses* to x and write $y \searrow x$ if there is a finite sequence of elementary collapses from y to x. If $\sim$ denotes the equivalence relation generated by $x \nearrow\!\!\nearrow y$ then by lemma 2, $G(\mathcal{M}) \cong \bar{\mathcal{M}}/_\sim$, where $\bar{b}_i \in \bar{\mathcal{M}}$ is identified with $b_i^{-1} \in G(\mathcal{M})$ by the isomorphism. An element $y \in \bar{\mathcal{M}}$ is said to be *irreducible* if there is no x such that $y \searrow\!\!\searrow x$.

Theorem 4 (Diamond condition) *Let $x, y, z \in \bar{\mathcal{M}}$ such that $x \nearrow\!\!\nearrow y \searrow\!\!\searrow z$. Then either $x \doteq z$ or there is an element $w \in \bar{\mathcal{M}}$ such that $x \searrow\!\!\searrow w \nearrow\!\!\nearrow z$.*

Proof Let $y \equiv x_1b_i\bar{b}_ix_2$ and $x \equiv x_1x_2$. Suppose that $y \doteq y' \equiv z_1b_j\bar{b}_jz_2$ and $z \equiv z_1z_2$.

We will observe how $x_1b_i\bar{b}_ix_2$ is transformed into $z_1b_j\bar{b}_jz_2$ through elementary substitutions and keep track of the inserted pair $b_i\bar{b}_i$ throughout the transformation. To make life easy we will write the inserted pair $b_i\bar{b}_i$ in bold $\mathbf{b}_i\bar{\mathbf{b}}_i$ and their images under the elementary substitutions will also be written in bold. So, for example, if we apply the substitution $ub_i \leftrightarrow b_ju$ we will write $u\mathbf{b}_i \leftrightarrow \mathbf{b}_ju$. Let $\phi = \phi_r \dots \phi_1$ be the chain of elementary substitutions which take $x_1\mathbf{b}_i\bar{\mathbf{b}}_ix_2$ to $z_1b_j\bar{b}_jz_2$. The pair $b_j\bar{b}_j$ may or may not be bold. We will study this in three cases below. Let $\phi_{\mathbf{b}}$ be the transformation ϕ, where ϕ_k is replaced by the identity transformation if ϕ_k is a substitution of a subword which contains $\mathbf{b}$ or $\bar{\mathbf{b}}$ (with some indices of course). This makes sense since all the defining relations of the monoid which involve elements of $\mathbf{B}$ are of the form $u\mathbf{b} = \mathbf{b}u$ or $u\bar{\mathbf{b}} = \bar{\mathbf{b}}u$, where $u \in F^+(A)$ or is a single b_j or $\bar{b}_j$. Replacing $\mathbf{b}$ by 1 will give the trivial substitution.

Case 1: $y' \equiv z_1\mathbf{b}_j\bar{\mathbf{b}}_jz_2$. Then clearly $\phi_{\mathbf{b}}(x) = z_1z_2$ and hence $x \doteq z$.

Case 2: $y' \equiv z_1\bar{\mathbf{b}}_j\mathrm{b}_jz_2$. The b_j (which is in roman typeset rather than bold to indicate that it is not the image of the bold $\mathbf{b}_i$ under ϕ) must appear in y as well possibly with a different index, say j_1, and hence $y \equiv x_{11}s_1x_{12}\mathbf{b}_i\bar{\mathbf{b}}_ix_{21}s_2x_{22}$, where $\{s_1, s_2\} = \{\mathrm{b}_{j_1}, 1\}$. Without loss of generality, let us assume that $s_2 = \mathrm{b}_{j_1}$. (Indeed, this has to be the case and can be observed by the fact that $b\bar{b} \neq \bar{b}b$ but this is not relevant to our argument.)

So $y \equiv x_1\mathbf{b}_i\bar{\mathbf{b}}_i x_{21}\mathrm{b}_{j_1}x_{22}$ and similarly $x \equiv x_1x_{21}\mathrm{b}_{j_1}x_{22}$, where $x_2 \equiv x_{21}\mathrm{b}_{j_1}x_{22}$. Also the bold $\mathbf{b}_i$ in y must appear in y', say $y' \equiv z_{11}\mathbf{b}_{i_1}z_{12}\bar{\mathbf{b}}_j\mathrm{b}_jz_2$ and hence $z \equiv z_{11}\mathbf{b}_{i_1}z_{12}z_2$ with $z_1 \equiv z_{11}\mathbf{b}_{i_1}z_{12}$. If $\phi(y) = y'$, then $\phi_{\mathbf{b}}(x) = z_{11}z_{12}\mathrm{b}_jz_2$.

Claim: $z_{11}z_{12}\mathrm{b}_jz_2 \doteq z_{11}\mathbf{b}_{i_1}z_{12}z_2$.

Let us duplicate the bold pair by expanding y to $w \equiv x_1\mathbf{b}_i\bar{\mathbf{b}}_i\mathbf{b}_i\bar{\mathbf{b}}_ix_{21}\mathrm{b}_{j_1}x_{22}$. We assume that the middle $\bar{\mathbf{b}}_i\mathbf{b}_i$ is the inserted pair. Define $\phi_1(w)$ as follows. Follow every elementary substitution which involves the old $\bar{\mathbf{b}}$ ($\mathbf{b}$) in ϕ by the same substitution applied to the new $\mathbf{b}$ ($\bar{\mathbf{b}}$). To see that this is possible, it is enough to look at the relations of the presentation and observe that in w all the bold letters are adjacent. So $\phi_1(w) = z_{11}\mathbf{b}_{i_1}\bar{\mathbf{b}}_{i_1}z_{12}\mathbf{b}_j\bar{\mathbf{b}}_j\mathrm{b}_jz_2$. On the other hand, let $\phi_2(w)$ be the transformation where the inserted pair $\bar{\mathbf{b}}_i\mathbf{b}_i$ follows the old $\bar{\mathbf{b}}$. Then we get $\phi_2(w) = z_{11}\mathbf{b}_{i_1}z_{12}\bar{\mathbf{b}}_j\mathbf{b}_j\bar{\mathbf{b}}_j\mathrm{b}_jz_2$.

$$\begin{array}{ccccc}
w & \doteq & z_{11}\mathbf{b}_{i_1}\bar{\mathbf{b}}_{i_1}z_{12}\mathbf{b}_j\bar{\mathbf{b}}_j\mathrm{b}_jz_2 & \doteq & z_{11}\mathbf{b}_{i_1}z_{12}\bar{\mathbf{b}}_j\mathbf{b}_j\bar{\mathbf{b}}_j\mathrm{b}_jz_2 \\
\Downarrow & & \Downarrow & & \Downarrow \\
y & \doteq & z_{11}z_{12}\mathbf{b}_j\bar{\mathbf{b}}_j\mathrm{b}_jz_2 & \doteq & z_{11}\mathbf{b}_{i_1}z_{12}\bar{\mathbf{b}}_j\mathbf{b}_jz_2 \\
\Downarrow & & \Downarrow & & \Downarrow \\
x & \doteq & z_{11}z_{12}\mathrm{b}_jz_2 & \doteq & z_{11}\mathbf{b}_{i_1}z_{12}z_2.
\end{array}$$

The collapses give the same word by the first case and this proves case 2.

Case 3: Let $y' \equiv z_1\mathrm{b}_j\bar{\mathrm{b}}_jz_2$. Then we have $y \equiv x_{11}s_1x_{12}s_2x_{13}\mathbf{b}_i\bar{\mathbf{b}}_ix_{21}s_3x_{22}s_4x_{23}$ and $y' \equiv z_{11}t_1z_{12}t_2z_{13}\mathrm{b}_j\bar{\mathrm{b}}_jz_{21}t_3z_{22}t_4z_{23}$, where $\{s_1, s_2, s_3, s_4\} = \{\mathrm{b}, \bar{\mathrm{b}}, 1, 1\}$ and $\{t_1, t_2, t_3, t_4\} = \{\mathbf{b}, \bar{\mathbf{b}}, 1, 1\}$. Then $x \equiv x_1x_2 \equiv x_{11}s_1x_{12}s_2x_{13}x_{21}s_3x_{22}s_4x_{23}$ and $z \equiv z_1z_2 \equiv z_{11}t_1z_{12}t_2z_{13}z_{21}t_3z_{22}t_4z_{23}$:

$$\begin{array}{ccccccc}
y & \equiv & x_{11}s_1x_{12}s_2x_{13}\mathbf{b}_i\bar{\mathbf{b}}_ix_{21}s_3x_{22}s_4x_{23} & \doteq & z_{11}t_1z_{12}t_2z_{13}\mathrm{b}_j\bar{\mathrm{b}}_jz_{21}t_3z_{22}t_4z_{23} & \equiv & y' \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
x & \equiv & x_{11}s_1x_{12}s_2x_{13}x_{21}s_3x_{22}s_4x_{23} & & z_{11}t_1z_{12}t_2z_{13}z_{21}t_3z_{22}t_4z_{23} & \equiv & z \\
 & & \downarrow \phi_{\mathbf{b}} & & \downarrow (\phi^{-1})_{\mathrm{b}} & & \\
 & & z_{11}z_{12}z_{13}\mathrm{b}_j\bar{\mathrm{b}}_jz_{21}z_{22}z_{23} & & x_{11}x_{12}x_{13}\mathbf{b}_i\bar{\mathbf{b}}_ix_{21}x_{22}x_{23} & & \\
 & & \Downarrow & & \Downarrow & & \\
 & & z_{11}z_{12}z_{13}z_{21}z_{22}z_{23} & \overset{(\phi_{\mathbf{b}})_{\mathrm{b}}^{-1}}{\longrightarrow} & x_{11}x_{12}x_{13}x_{21}x_{22}x_{23} & &
\end{array}$$

So we have $x \Searrow z_{11}z_{12}z_{13}z_{21}z_{22}z_{23} \doteq x_{11}x_{12}x_{13}x_{21}x_{22}x_{23} \Nearrow z$. This proves the final case. Note that the only other possibilities are $y' \equiv z_1\bar{\mathrm{b}}_j\mathbf{b}_jz_2$ or $y' \equiv z_1\bar{\mathrm{b}}_j\mathrm{b}_jz_2$ which are equivalent to second and third cases respectively. Also $y \equiv x_1\bar{\mathbf{b}}_i\mathbf{b}_ix_2$ is just the symmetric of the cases above and does not make any difference in the proof. □

Proposition 5 *Let $x, y, z \in \bar{\mathcal{M}}$, where x, z are words in the alphabet of* **X** *only. If there is a sequence $x \nearrow\!\!\!\nearrow y \searrow\!\!\!\searrow z$ then $x =_{M} z$.*

Proof Every element of $\mathcal{M}$ is irreducible so there is no y' such that $x \searrow\!\!\!\searrow y'$. Therefore by the diamond condition $x \doteq z$ in $\bar{\mathcal{M}}$. Since $\mathcal{M}$ injects into $\bar{\mathcal{M}}$ the equality is true in $\mathcal{M}$. □

Proof of Theorem 3 Let G be the group of $\mathcal{M}$. If $u, v \in \mathcal{M}$ are equal in G then by lemma 2 there exists a sequence $u = u_o \nearrow u_1 \searrow u_2 \nearrow \ldots \searrow u_k = v$. Note that the first step is always an expansion and the last always a collapse since u, v are irreducible. Every expansion/collapse is a sequence of elementary expansions/collapses. Consider $u_o \nearrow u_1 \searrow u_2$ where $u_o \nearrow u_1$ is given by $u_o = x_o \nearrow\!\!\!\nearrow x_1 \nearrow\!\!\!\nearrow \ldots \nearrow\!\!\!\nearrow x_{k_o} = u_1$ and $u_1 \searrow u_2$ is given by $u_1 = y_o \searrow\!\!\!\searrow y_1 \searrow\!\!\!\searrow \ldots \searrow\!\!\!\searrow y_{k_1}$. So we have an intermediate stage $x_{k_o-1} \nearrow\!\!\!\nearrow u_1 \searrow\!\!\!\searrow y_1$. Since $\bar{\mathcal{M}}$ is diamond, we can either replace x_{k_o-1} by y_1 or we can replace $x_{k_o-1} \nearrow\!\!\!\nearrow u_1 \searrow\!\!\!\searrow y_1$ by $x_{k_o-1} \searrow\!\!\!\searrow u_1' \nearrow\!\!\!\nearrow y_1$. In both cases the length of the expansion and the collapse reduces by one. Applying this procedure finitely many times we will obtain a new sequence $u = v_o \nearrow\!\!\!\nearrow v_1 \searrow\!\!\!\searrow v_2 \nearrow\!\!\!\nearrow \ldots \searrow\!\!\!\searrow v_l = v$. Now since $u \in \mathcal{M}$ by the above proposition, $u =_{\mathcal{M}} v_1 =_{\mathcal{M}} \ldots =_{\mathcal{M}} v$. This is exactly what we want to prove, i.e. $u =_{\mathcal{M}} v$. □

Application

Baez [2] and Birman [3] introduced the singular braid monoid SB_n. In [6], we study some properties of a larger class of monoids, which we call the geometric singular braids. Geometric singular braids form a monoid which admits a presentation satisfying the conditions of theorem 3 and therefore they embed in a group. This result helps us to imagine the "inverses" of singular points as geometric objects. Consider the braid diagrams on n-strings including singular points of any order $k \leq n$. We can perform an operation on these diagrams by putting two diagrams α and β side by side, joining the initial points of β to the final points of α to obtain the diagram $\alpha\beta$. There is a natural equivalence on the diagrams corresponding to the isotopies of $\mathbb{R}^3$ fixing the end points and leaving the diagram unchanged in a small planar neighbourhood of a singular point. We call the equivalence class of diagrams *geometric braids*. With the above operation geometric braids form a monoid which is denoted by MB_n. Note that the monoid SB_n is a submonoid of MB_n.

We can present MB_n with generators $\mathbf{A} = \{\sigma_i,\ \sigma_i^{-1},\ i = 1, \cdots, n-1\}$ $\mathbf{B} = \{\mu_{i,k}(\pi),\ i = 1, \cdots, n-1,\ i+k \leq n+1,\ \pi \in S_k\}$. Here, $\sigma_i^{\pm 1}$ correspond to positive and negative crossings of the $i, i+1$ strings as in classical braids and $\mu_{i,k}(\pi)$ is the singular point where $i, i+1, \cdots, i+k-1$ strings intersect with permutation π. For example the braid in the figure is $\mu_{1,3}(\pi)\sigma_2\mu_{1,2}(\rho)$, where $\pi = (13)$ and $\rho = (12)$. The relations are

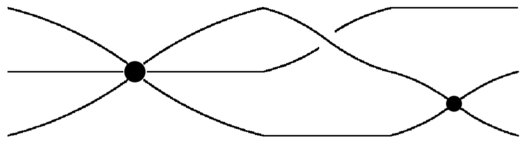

Figure 1. A geometric braid diagram on 3 strings

$\mathbf{R}_1 \cup \mathbf{R}_2 \cup \mathbf{R}_3 \cup \mathbf{R}_4$ where:

$$\mathbf{R}_1 = \left\{ \begin{array}{ccll} \sigma_i\sigma_j & = & \sigma_j\sigma_i & \text{if } \mid i-j \mid > 1 \\ \sigma_i\sigma_j\sigma_i & = & \sigma_j\sigma_i\sigma_j & \text{if } \mid i-j \mid = 1 \end{array} \right\}.$$

These are the classical braid relations and correspond to the isotopies away from the singular points.

$$\mathbf{R}_2 = \left\{\sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1,\ i = 1, \cdots, n-1\right\}.$$

$\mathbf{R}_3$ consist of relations of the form

$\mu_{i,k}(\pi)\sigma_j = \sigma_j\mu_{i,k}(\pi)$ if $i+k < j+1$ or $j+1 < i$,

$\mu_{i,k}(\pi)\delta_{i,k} = \delta_{i,k}\mu_{i-1,k}(\pi)$ where $\delta_{i,k} = \sigma_{i-1}\sigma_i \cdots \sigma_{i+k-2}$ and $1 < i \leq n-k+1$,

$\mu_{i,k}(\pi)\delta'_{i,k} = \delta'_{i,k}\mu_{i+1,k}(\pi)$ where $\delta'_{i,k} = \sigma_{i+k-1} \cdots \sigma_i$ and $1 \leq i \leq n-k$,

$\mu_{i,k}(\pi)\Delta_{i,k} = \Delta_{i,k}\mu_{i,k}(\pi')$ where $\Delta_{i,k} = \delta_{i+1,k-1} \cdots \delta_{i+1,1}$ and

$$\pi' = \begin{pmatrix} 1 & \cdots & k-1 & k \\ k+1-a_k & \cdots & k+1-a_2 & k+1-a_1 \end{pmatrix} \text{if } \pi = \begin{pmatrix} 1 & 2 & \cdots & k \\ a_1 & a_2 & \cdots & a_k \end{pmatrix}.$$

The first of these relations is the commutation of a crossing with a singular point if they are on disjoint strings. The next two are moving an arc over or under a singular point and the last one is a twist around a singular point. Note that the permutation of the singular point changes only in the last relation, because in this case we are rotating the point 180 degrees so we read the permutation from upside down. $\mathbf{R}_4$ indicates that the singular points on disjoint strings commute.

$$\mathbf{R}_4 = \{\mu_{i,k}(\pi)\mu_{j,l}(\rho) = \mu_{j,l}(\rho)\mu_{i,k}(\pi) \text{ if } j+1 > i+k \text{ or } i+1 > j+l\}.$$

This monoid clearly satisfies the condition of Theorem 3 and hence embeds in a group. Let $GB_n =< \mathbf{A} \cup \mathbf{B} \mid \mathbf{R}_i,\ i = 1,2,3,4\}$ be the group of geometric braids. This group does not have a real geometric meaning since the cancellation of a singular point is not an isotopy. But because of the embedding we can visualize an element of the group by a diagram denoting the inverse of a singular point with say a white blob while denoting the singular point by a black blob. The inverses have the same geometry as the singular points. Moreover, by putting restrictions on the types of crossings we can obtain submonoids and subgroups of the geometric braid monoid and group, respectively.

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Received 24.08.2000