

The topology of symplectic manifolds

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1. Introduction

The purpose of this article is twofold: First, we provide an informal introduction to symplectic structures from a topological viewpoint. Second, we address the question of whether symplectic manifolds can ultimately be described as purely topological objects. We sketch work that will appear in [G2], pointing towards an affirmative answer to the question. The first section of the present article motivates and defines symplectic structures, and then discusses obstructions to their existence. In Section 2, we focus on a particular topological construction of symplectic structures, and in Section 3 we see that the construction is likely to be universal in the sense of realizing a dense subset of all symplectic structures on any given manifold. This would lead to a complete topological characterization of those manifolds that admit symplectic structures, and to a reinterpretation of a dense set of symplectic structures on a given manifold as a certain set that should ultimately be describable by purely topological means. Further details will appear in [G2]; see also [GS] for a discussion of the 4-dimensional case. For additional reading on symplectic topology, see e.g., [McS]. In this article, manifolds will always be assumed to be smooth, closed and oriented.

1.1. Why study symplectic manifolds?

While symplectic structures naturally arise in diverse contexts such as Hamiltonian mechanics and algebraic geometry, we focus on a topological application: the classification problem for simply connected 4-manifolds. The most direct approach to a classification problem is to begin by writing down examples. The main classical source of examples of simply connected 4-manifolds was complex surfaces (complex manifolds of complex dimension 2, hence real dimension 4). These can be constructed, for example, by writing down collections of homogeneous polynomials in $n + 1$ complex variables. The common zero locus then specifies a well-defined subset of projective space $\mathbb{C}\mathbb{P}^n = \mathbb{C}^{n+1} - \{0\}$ modulo multiplication by nonzero complex scalars. If this subset happens to be a manifold, it will automatically be complex. Complex surfaces arising in this manner are called *algebraic* surfaces. Many examples of simply connected algebraic surfaces are known — for example, any generic collection of $n - 2$ homogeneous polynomials will determine a

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simply connected algebraic surface in $\mathbb{C}\mathbb{P}^n$ (as will some nongeneric collections of more than $n - 2$ polynomials).

Once we have examples of simply connected 4-manifolds, we can construct many more by the *connected sum* operation: We remove the interior of a 4-ball from each of the 4-manifolds X_1 and X_2 , and glue along the resulting boundary 3-spheres so that the new manifold $X_1 \# X_2$ inherits the same orientation from each summand. This can be thought of as an unnecessary complication, however. We would like to restrict attention to those 4-manifolds that cannot be split as connected sums. Unfortunately, it is still unknown whether every 4-manifold homeomorphic to the 4-sphere is actually diffeomorphic to it, and it is even possible that *every* 4-manifold could split off arbitrarily many nontrivial summands homeomorphic to S^4 . Thus, we define a 4-manifold X to be *irreducible* if for every (smooth) decomposition $X \approx X_1 \# X_2$, one summand X_i must be homeomorphic (but not necessarily diffeomorphic) to S^4 .

We can now begin a brief history of the classification problem for simply connected, irreducible 4-manifolds. In the 1970's, virtually nothing was known. While there were many examples of simply connected complex surfaces, these could in general not be distinguished from each other (up to diffeomorphism) or shown to be irreducible. In fact, it was possible that a complete list of irreducible, simply connected 4-manifolds could be given by S^4 , $\pm\mathbb{C}\mathbb{P}^2$ (the complex projective plane with both orientations), $S^2 \times S^2$ ($= \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$) and $\pm K3$ (where $K3$ denotes the zero locus of a generic quartic polynomial in $\mathbb{C}\mathbb{P}^3$). Furthermore, it was unknown whether $K3$ could split as $X \# S^2 \times S^2$ for some unknown manifold X . In the 1980's, the situation began to change dramatically, due to techniques pioneered by Donaldson using gauge theory. While it now seems likely (in light of Freedman's breakthrough for topological 4-manifolds) that any simply connected (smooth) 4-manifold is *homeomorphic* to a connected sum of manifolds from the above list, the diffeomorphism classification is much more complicated. Our present knowledge about simply connected complex surfaces can be summed up as follows:

- There are many diffeomorphism types (sometimes infinitely many within a homeomorphism type).
- They are irreducible (when minimal).

Minimality is a technical condition that causes no essential difficulties here — any complex surface X can be “blown down” to a minimal complex surface Y , and then X is diffeomorphic to a connected sum of Y with copies of $-\mathbb{C}\mathbb{P}^2$. The $K3$ -surface, for example, is minimal and hence irreducible.

This breakthrough in understanding complex surfaces highlighted our ignorance regarding a related question: Are all simply connected, irreducible 4-manifolds ($\neq S^4$) complex? By the end of the 1980's, no counterexamples were known. An affirmative answer would have reduced the classification problem to that of understanding complex surfaces, much as the study of oriented surfaces can be reduced to that of complex curves (Riemann surfaces). In 1990, the question was answered in the negative: Infinitely many irreducible 4-manifolds homeomorphic to the $K3$ -surface were produced, and shown not

to admit complex structures with either orientation [GM]. Subsequently, many other families of counterexamples have been constructed. (See [GS] for a recent survey.) However, the underlying beauty of the question suggested looking for a generalization. It had long been known that every simply connected complex surface is algebraic (after deformation of the complex structure). But every algebraic manifold inherits a *Kähler* structure, i.e., a symplectic structure compatible with its complex structure. (See 1.2-3 for definitions.) Thus, we could generalize to the following question: Are all simply connected, irreducible 4-manifolds ($\neq S^4$) *symplectic*? Work in the early 1990's showed this to be a reasonable question. In fact:

- There are many diffeomorphism types of symplectic, noncomplex 4-manifolds [G1]. For example, the exotic $K3$ -surfaces of [GM] are symplectic. Dropping the simple connectivity hypothesis, we find that every finitely presented group is realized as the fundamental group of a symplectic 4-manifold, whereas fundamental groups of Kähler manifolds and complex surfaces are quite restricted.
- Minimal, simply connected, symplectic 4-manifolds are irreducible (Kotschick [K], after Taubes [T]), at least when $b_2^+ \neq 1$.

(The discussion of symplectic minimality is parallel to that of the holomorphic version discussed above. For b_2^+ , see 1.3.) At present, there are only a few known methods for constructing simply connected, irreducible, noncomplex 4-manifolds. These are highly restricted cut-and-paste constructions. (More general cut-and-paste constructions seem to invariably result in connected sums of simple manifolds.) These restricted operations can be shown to preserve symplectic structures under reasonable hypotheses [G1], [S]. Could it be that the only way to build an irreducible manifold is by equipping it with a symplectic structure? In fact, the answer is no: In 1996, Szabó [Sz] produced simply connected, irreducible 4-manifolds admitting no symplectic structures, by applying these operations under more general hypotheses. Subsequently, Fintushel and Stern [FS] generalized the method to produce an abundance of such examples. At present, there seems to be no reasonable question of this sort to ask to shed light on the structure of arbitrary simply connected, irreducible 4-manifolds.

In summary, we are left with the following classes of simply connected, irreducible 4-manifolds (up to diffeomorphism):

$$\emptyset \subset \{\text{complex}\} \subset \{\text{symplectic}\} \subset \{\text{arbitrary}\} .$$

We have seen that each class contains many elements not in the previous one — in fact, there seems to be a sense in which “most” elements of a given class lie outside the previous one. At present, there seems to be little hope of classifying arbitrary simply connected, irreducible 4-manifolds, so we might hope to simplify the problem by restricting to one of the other classes. Complex surfaces, on the other hand, are difficult for a topologist to study. There is little hope of cutting and pasting, due to the rigid nature of holomorphic functions, so one must resort to methods of algebraic geometry. Symplectic manifolds, however, are accessible by topological methods. The main constructions of symplectic, noncomplex manifolds are of a cut-and-paste nature (e.g., [G1], [S]). In Sections 2 and 3

we will discuss a different topological construction, motivated by fiber bundles, that (at least in dimension 4) provides a complete topological characterization of those manifolds admitting symplectic structures. Thus, one can consider the diffeomorphism classification of simply connected, irreducible, symplectic 4-manifolds as a purely topological problem that may be more accessible than the original classification problem for smooth 4-manifolds.

1.2. Symplectic structures

Definition 1.1. A *symplectic manifold* is a $2n$ -manifold X together with a *symplectic form* ω on X , i.e., a differential 2-form that is closed ($d\omega = 0$) and nondegenerate.

Here, nondegeneracy has its usual meaning in the context of bilinear forms: For any nonzero $v \in T_x X$ there is a vector $w \in T_x X$ such that $\omega(v, w) \neq 0$. An equivalent condition is that the top exterior power ω^n of ω should be nowhere zero, i.e., a volume form on X . (This indicates why X must have even dimension.) The volume form ω^n determines an orientation on X ; we will always use this orientation when considering X as an oriented manifold. For example, we will see that $\mathbb{C}\mathbb{P}^2$ admits a symplectic structure while $-\mathbb{C}\mathbb{P}^2$ does not.

It is instructive to compare the above definition with Riemannian geometry. If ω were symmetric rather than skew-symmetric, nondegeneracy would imply that ω was a Riemannian or Lorentzian metric. The condition that $d\omega = 0$ can be compared with requiring a Riemannian metric to have constant curvature. In each case, the relevant partial differential equation guarantees the absence of local structure — two Riemannian n -manifolds with the same constant curvature are locally identical (i.e., any two points have isometric neighborhoods), and the same holds for symplectic $2n$ -manifolds (any two points have symplectomorphic neighborhoods). Thus, symplectic structures can be thought of as skew-symmetric analogs of constant curvature metrics. In the Riemannian case, constant curvature allows a classification theory, which reduces to a study of discrete groups of isometries of Euclidean, hyperbolic or spherical space. One might hope to similarly reduce the study of symplectic manifolds to a topological or combinatorial problem. One cannot hope to generalize the Riemannian theory directly, since there is no symplectic analog of geodesics, and since the classification problem is already difficult for simply connected symplectic manifolds. We will use a different approach to this problem in Section 3.

1.3. Obstructions to constructing symplectic structures

The study of which manifolds admit symplectic structures has two directions: existence and nonexistence. We now discuss the three known sources of obstructions to existence, and defer the discussion of constructing symplectic manifolds to the next section.

To obtain the first obstruction, note that since a symplectic form is closed, it determines a cohomology class $[\omega] \in H_{dR}^2(X) \cong H^2(X; \mathbb{R})$. Nondegeneracy implies that the top exterior power $[\omega]^n = [\omega^n] \in H_{dR}^{2n}(X) \cong \mathbb{R}$ (for X connected) is positive relative to the

given orientation on X . Thus, a symplectic structure cannot exist unless there is a class $\alpha \in H^2(X; \mathbb{R})$ with $\alpha^n > 0$.

Examples. S^{2n} admits no symplectic structure for $n > 1$, since $H^2(S^{2n}; \mathbb{R}) = 0$. $S^2 \times S^{2n-2}$ admits no symplectic structure for $n > 2$, for although $H^2(S^2 \times S^{2n-2}; \mathbb{R}) \cong \mathbb{R}$, the generator α has $\alpha \wedge \alpha = 0$. Similarly, $-\mathbb{C}\mathbb{P}^2$ admits no symplectic structure since the generator of $H^2(-\mathbb{C}\mathbb{P}^2; \mathbb{R}) \cong \mathbb{R}$ has negative square.

For the second source of obstructions, we forget the closure condition on ω , and consider arbitrary nondegenerate 2-forms on X . Such a 2-form reduces the structure group of the tangent bundle TX from $GL(2n, \mathbb{R})$ to the subgroup $Sp(2n)$ consisting of isomorphisms of \mathbb{R}^{2n} preserving the standard symplectic form $dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$. (This corresponds to the reduction to $O(n) \subset GL(n; \mathbb{R})$ in the Riemannian case.) The group $Sp(2n)$ is noncompact, but it deformation retracts onto its maximal compact subgroup $U(n) \subset GL(n; \mathbb{C})$ (where we identify \mathbb{R}^{2n} with \mathbb{C}^n in the obvious way). Since the latter inclusion is also a homotopy equivalence, the homotopy classification of nondegenerate 2-forms on X is equivalent to the homotopy classification of *almost-complex structures*, i.e., complex vector bundle structures on TX . This is a classical problem in obstruction theory. For example, a homotopy class of nondegenerate 2-forms inherits Chern classes from the corresponding homotopy class of almost-complex structures.

Examples. $\mathbb{C}\mathbb{P}^2 \# \mathbb{C}\mathbb{P}^2$ admits no symplectic structure, even though it has classes $\alpha \in H^2(\mathbb{C}\mathbb{P}^2 \# \mathbb{C}\mathbb{P}^2; \mathbb{R})$ with $\alpha \wedge \alpha > 0$, because it admits no almost-complex structure. In fact, standard characteristic class theory shows that such a structure would have a Chern class with $c_1^2 = 2\chi + 3\sigma = 14$ (where χ is the Euler characteristic and σ is the signature of the wedge product pairing on H^2), but a routine computation shows that no integral cohomology class has square 14. More generally, a 4-manifold X cannot admit an almost-complex structure unless the invariant $\frac{1}{2}(\chi + \sigma) = 1 - b_1(X) + b_2^+(X)$ is even, where b_2^+ is the dimension of a maximal positive definite subspace of H^2 under the wedge product. In contrast, $S^3 \times S^1$ admits a complex structure as $\mathbb{C}^2 - \{0\}$ modulo multiplication by 2, and this automatically determines an almost-complex structure on $S^3 \times S^1$. Thus, this manifold admits nondegenerate 2-forms. Such forms cannot be closed, however, since $H^2(S^3 \times S^1; \mathbb{R}) = 0$.

We will find it useful to link symplectic structures more explicitly with almost-complex structures. First note that the latter structures can be specified by choosing the effect of multiplication by i on each tangent space. Thus, an almost-complex structure can be thought of as a linear bundle isomorphism $J : TX \rightarrow TX$ (covering id_X) such that $J^2 = -\text{id}_{TX}$. It is routine to verify that such an isomorphism actually does specify a complex structure; we require the induced orientation to agree with the given one on TX .

Definition 1.2. A 2-form ω *tames* an almost-complex structure J if for any nonzero tangent vector v we have $\omega(v, Jv) > 0$. If, in addition, $\omega(Jv, Jw) = \omega(v, w)$ for any two tangent vectors v, w lying in the same tangent space, we say that ω and J are *compatible*.

Thus, ω tames J if it is a positive area form on each complex line (in the complex orientation). The compatibility condition, that J preserves ω (i.e., $J \in \text{Sp}(TX)$), corresponds to orthogonality of J in the Riemannian case. A compatible pair (ω, J) determines a Hermitian structure on TX via the metric $g(v, w) = \omega(v, Jw)$. For a fixed nondegenerate form on X , the spaces of tamed and compatible almost-complex structures are nonempty and contractible (e.g., [McS]), so either condition exhibits the above correspondence of homotopy classes. In the remaining sections, we will make extensive use of the following

Observations. (1) If a 2-form tames some almost-complex structure, it is obviously nondegenerate. Hence, a closed, taming form is automatically symplectic.

(2) If $\omega_1, \dots, \omega_k$ tame a fixed J , then any convex combination $\sum_{i=1}^k t_i \omega_i$ (all $t_i \geq 0$, $\sum t_i = 1$) will also tame J .

(3) The taming condition is *open*, i.e., preserved under sufficiently small perturbations of ω and J .

To verify the last observation, note that the taming condition $\omega(v, Jv) > 0$ is satisfied provided that it holds for vectors v in the unit sphere bundle $\Sigma \subset TX$ (given by any preassigned metric). Since Σ is compact, taming implies that $\omega(v, Jv)$ is bounded below by a positive constant on Σ , so it will remain positive under small perturbations of ω and J . Note that compatibility is not an open condition. For this reason, we will mainly use the taming condition in subsequent sections, although compatibility appears more commonly in the literature.

Examples. While every symplectic manifold (X, ω) has a compatible almost-complex structure J , this latter structure may not come from a complex structure on X . (For J to be a complex structure on X , it must be locally identical to \mathbb{C}^n , which is equivalent to requiring J to satisfy a certain partial differential equation.) If J actually is a complex structure on X , the triple (X, J, ω) is called a *Kähler manifold*. A standard example of this is $\mathbb{C}\mathbb{P}^n$, which inherits both J and ω in simple ways from \mathbb{C}^{n+1} . (Restrict the standard $\omega = \sum_{i=0}^n dx_i \wedge dy_i$ from \mathbb{C}^{n+1} to S^{2n+1} , then note that its projection to $\mathbb{C}\mathbb{P}^n$ is well-defined by $U(1)$ -invariance and the fact that all tangent vectors projecting to 0 pair trivially with TS^{2n+1} .) Since a complex submanifold of a Kähler manifold is Kähler, it follows immediately that any algebraic manifold is Kähler. (More generally, if ω tames J and $Y \subset X$ is J -holomorphic, i.e., each $T_y Y \subset T_y X$ is a J -complex subspace, then $\omega|_Y$ tames $J|_Y$.)

The third and final known source of obstructions to the existence of symplectic structures is the Seiberg-Witten invariants (from gauge theory) on 4-manifolds [T] (cf. also [GS], [K]). An important example is that minimal, simply connected symplectic 4-manifolds with $b_2^+ \neq 1$ must be irreducible. Similarly, if an arbitrary symplectic 4-manifold has a connected sum splitting, the wedge product pairing on H^2 must be negative definite for all but one summand. The Seiberg-Witten invariants are much more subtle than the previously discussed invariants, and will not be needed in the subsequent sections.

Example. $\mathbb{C}\mathbb{P}^2 \# \mathbb{C}\mathbb{P}^2 \# \mathbb{C}\mathbb{P}^2$ has no symplectic structure, since the pairing on $H^2(\mathbb{C}\mathbb{P}^2)$ is not negative definite. However, it clearly has a cohomology class α with $\alpha \wedge \alpha > 0$, and can be shown to admit an almost-complex structure. Similarly, a connected sum of 3 copies of $\mathbb{C}\mathbb{P}^2$ with 19 copies of $-\mathbb{C}\mathbb{P}^2$ does not admit a symplectic structure, but it is actually homeomorphic (although clearly not diffeomorphic) to a Kähler manifold. The homeomorphism is covered by an isomorphism of tangent bundles. This shows that the obstructions from Seiberg-Witten theory are more subtle than the homotopy-theoretic ones discussed previously.

2. Constructing symplectic structures

We turn to the construction of symplectic manifolds. Historically, the first examples of (compact) symplectic manifolds were the Kähler manifolds, obtained largely by algebrogeometric methods. We will consider in detail the first construction of symplectic manifolds admitting no Kähler structure. (For other topological constructions, see e.g., [G1], [Mc], [S].) We will then generalize the construction into a form suitable for the applications in Section 3.

2.1. Symplectic forms on bundles

The original construction of symplectic, nonKähler manifolds, due to Thurston [Th] (see also [McS]), consists of finding a symplectic form on the total space of a fiber bundle. The basic method is quite simple, and reminiscent of techniques previously introduced into complex analysis by Grauert. We state the simplest version of Thurston's theorem, in which the fibers are 2-dimensional.

Theorem 2.1. [Th] *Let $f : X^{2n} \rightarrow Y^{2n-2}$ be a bundle map, with X connected, Y symplectic and $[f^{-1}(y)] \neq 0 \in H_2(X; \mathbb{R})$. Then X admits a symplectic structure.*

Recall that all manifolds are assumed to be compact and oriented. Thus, the fibers $f^{-1}(y)$ are all closed, oriented surfaces and homologous, so the homological condition makes sense and is independent of y . To see that this condition is necessary, consider the bundle map $f : S^3 \times S^1 \rightarrow S^2$ obtained by projecting to S^3 and applying the Hopf fibration. The theorem generalizes to bundles with higher dimensional fibers. In that case one also needs the fibers to be symplectic and the transition functions to be symplectomorphisms. These additional conditions are automatically satisfied when the fibers have dimension 2, since a symplectic form on a surface is the same as an area form.

Example. It is easy to construct a torus bundle over the torus whose total space X has $b_1(X) = 3$. For example, begin with a torus bundle over S^1 whose monodromy is a Dehn twist, then cross with S^1 . This example clearly has a section, so $[f^{-1}(y)] \neq 0$, and Theorem 2.1 provides a symplectic structure on X . However, it is a basic fact that the odd-degree Betti numbers of a Kähler manifold must be even, so X is not even homotopy equivalent to a Kähler manifold. This example from Thurston's paper was also known to Kodaira. It can also be seen as the quotient of $(\mathbb{R}^4, dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$ by the discrete

group of symplectomorphisms generated by unit translations along the x_1, y_1 and x_2 axes and the map $(x_1, y_1, x_2, y_2) \mapsto (x_1 + y_1, y_1, x_2, y_2 + 1)$.

2.2. Symplectic forms induced by J -holomorphic maps

Although Theorem 2.1 has a simple direct proof, we will proceed by the alternate method of generalizing the theorem and supplying a proof of the generalization that is not significantly harder than the original proof. To motivate the generalization, first recall that the symplectic manifold Y in Theorem 2.1 automatically has a compatible almost-complex structure J_Y . We can easily construct an almost-complex structure J on X for which f is J -holomorphic, i.e., $df \circ J = J_Y \circ df$. (That is, each $df_x : T_x X \rightarrow T_{f(x)} Y$ is complex linear.) For example, choose a metric on X and let $H \subset TX$ be the subbundle of orthogonal complements to the fibers of f . Clearly, $df|_H : H \rightarrow TY$ is an isomorphism on each fiber; let $J|_H$ be the pullback of J_Y . Define J on the tangent spaces to the fibers of f to be $\frac{\pi}{2}$ counterclockwise rotation (using the metric and preimage orientation and the fact that these spaces are 2-dimensional). J is now uniquely determined on TX by linearity, and f is J -holomorphic by construction. Thus, the hypotheses of Theorem 2.1 are hiding almost-complex structures on X and Y making f J -holomorphic. Once we observe this, we find that the bundle structure is completely unnecessary! We obtain the following theorem:

Theorem 2.2. *Let $f : X \rightarrow Y$ be a J -holomorphic map of almost-complex manifolds. Let ω_Y be a symplectic form on Y taming J_Y . Fix a class $c \in H_{dR}^2(X)$. Suppose that for each $y \in Y$, $f^{-1}(y)$ has a neighborhood W_y with a closed 2-form η_y such that $[\eta_y] = c|_{W_y} \in H_{dR}^2(W_y)$, and such that η_y tames $J|_{\ker df_x}$ for each $x \in W_y$. Then X admits a symplectic structure.*

Note that $\ker df_x$ is a J -complex subspace of $T_x X$ (since f is J -holomorphic); the taming condition means $\eta_y(v, Jv) > 0$ for each nonzero $v \in \ker df_x$.

To motivate the remaining hypotheses of Theorem 2.2, we show that it implies Theorem 2.1. This is essentially the first part of Thurston's proof. We leave it as an exercise to state and deduce the analog of Theorem 2.1 for bundles with higher dimensional fibers.

Proof of Theorem 2.1. We assume the hypotheses of Theorem 2.1 and deduce those of Theorem 2.2; the conclusion follows. We have already obtained the first two sentences of Theorem 2.2. We may assume the fibers $f^{-1}(y)$ are connected, by passing to a finite cover of Y if necessary. Let c be any class for which $\langle c, f^{-1}(y) \rangle = 1$; such classes exist since $[f^{-1}(y)] \neq 0$ in $H_2(X; \mathbb{R})$ and $H_{dR}^2(X)$ is dual to this space. For each $y \in Y$, let D_y be an open disk containing y and let $W_y = f^{-1}(D_y) \approx D_y \times f^{-1}(y)$. Choose an area form on $f^{-1}(y)$ with area 1 (and inducing the preimage orientation on $f^{-1}(y)$), and let η_y be the pullback of this form to W_y via the projection $W_y \rightarrow f^{-1}(y)$. Since $H_2(W_y)$ is generated by $[f^{-1}(y)]$, the equalities $\langle \eta_y, f^{-1}(y) \rangle = 1 = \langle c, f^{-1}(y) \rangle$ show that $[\eta_y] = c|_{W_y} \in H_{dR}^2(W_y)$. Since f is a bundle map, $\ker df_x$ is the tangent plane to the fiber at x , so the required taming condition is just the obvious statement that η_y tames J when restricted to each fiber of f in W_y . \square

Proof of Theorem 2.2. The proof follows Thurston's paper, except for two deviations where we exploit the almost-complex structures. The first step is to splice together the locally defined forms η_y into a global closed form η on X satisfying the corresponding hypotheses that $[\eta] = c$ and η tames $J| \ker df_x$ for all $x \in X$. Unfortunately, splicing the forms in the obvious way by a partition of unity destroys the closure condition, so we use a trick: Fix a representative ζ of the deRham class $c = [\zeta]$. Now for each $y \in Y$ we have $\eta_y = \zeta + d\alpha_y$ on W_y , for some 1-form α_y on W_y (since $[\eta_y] = c|_{W_y}$). We splice the forms η_y by splicing the 1-forms α_y as follows. Let $\{\rho_i\}$ be a partition of unity on Y , subordinate to a sufficiently fine cover. Pull back by f to obtain the corresponding partition of unity $\{\rho_i \circ f\}$ on X , and let $\eta = \zeta + d\sum_i(\rho_i \circ f)\alpha_{y_i}$, where the sum splices the forms α_y in the usual way via $\{\rho_i \circ f\}$. Clearly, η is a closed 2-form on X with $[\eta] = c$. To verify the taming condition, we carry out the differentiation to obtain $\eta = \zeta + \sum_i(\rho_i \circ f)d\alpha_{y_i} + \sum_i(d\rho_i \circ df)\alpha_{y_i}$. The last term clearly vanishes when applied to a pair of vectors in $\ker df_x$, so on $\ker df_x$ we have $\eta = \zeta + \sum_i(\rho_i \circ f)d\alpha_{y_i} = \sum_i(\rho_i \circ f)\eta_{y_i}$. This is a convex combination of forms taming $J| \ker df_x$, so it tames as required (by Observation 2 of 1.3). Note how the almost-complex structures guide the construction here — an arbitrary convex combination of symplectic forms need not be symplectic, e.g., any symplectic form ω satisfies $-\omega + \omega = 0$, but $\pm\omega$ are both symplectic for the same oriented manifold if the dimension is divisible by 4.

As in Thurston's proof, we now wish to show that the closed form $\omega_t = t\eta + f^*\omega_Y$ on X is symplectic for sufficiently small $t > 0$. By Observation 1 of 1.3, it suffices to show that ω_t tames J for small $t > 0$, so we only need to verify that $\omega_t(v, Jv) > 0$ on the unit tangent bundle $\Sigma \subset TX$. But

$$\begin{aligned} \omega_t(v, Jv) &= t\eta(v, Jv) + \omega_Y(df(v), df(Jv)) \\ &= t\eta(v, Jv) + \omega_Y(df(v), J_Y df(v)) , \end{aligned}$$

where the last line uses J -holomorphicity of df ($df \circ J = J_Y \circ df$). Since ω_Y tames J_Y , the last term is ≥ 0 , with equality if and only if $v \in \ker df$. On the other hand, η tames J on $\ker df$, so by openness of the taming condition (Observation 3 of 1.3), $\eta(v, Jv) > 0$ for v in some neighborhood U of the subset $\Sigma \cap \ker df$ in Σ . Thus, $\omega_t(v, Jv) > 0$ for all $t > 0$ when $v \in U$. But $\Sigma - U$ is compact, so on $\Sigma - U$, $\eta(v, Jv)$ is bounded and the last displayed term is bounded below by a *positive* constant (since it is positive away from $\ker df$). It is now clear that for sufficiently small $t > 0$, $\omega_t(v, Jv) > 0$ for all $v \in \Sigma$, as required. \square

3. Characterizing symplectic manifolds

Fiber bundles form an interesting but relatively small class of manifolds. We wish to find more general structures to which Theorem 2.2 can be applied. We ultimately define structures with sufficient generality that they can probably be found in any symplectic manifold, providing our desired topological characterization of manifolds admitting symplectic forms. Such a structure determines an essentially unique symplectic form, and one should be able to realize a dense subset of all symplectic forms in this manner. This

could lead to a purely topological way of understanding the set of symplectic structures on any given manifold.

3.1. Lefschetz pencils

We begin by considering what topological structure can be found on an algebraic surface X . By definition, X is a holomorphic submanifold of $\mathbb{C}\mathbb{P}^N$ for some N . Let $A \subset \mathbb{C}\mathbb{P}^N$ be a generic linear subspace of complex codimension 2 (so it is a copy of $\mathbb{C}\mathbb{P}^{N-2}$ cut out by two homogeneous linear equations $p_0(z) = p_1(z) = 0$). Then A intersects X transversely in a finite set B called the *base locus*. The set of all hyperplanes through A is parametrized by $\mathbb{C}\mathbb{P}^1$. (They are given by the equations $y_0 p_0(z) + y_1 p_1(z) = 0$, for $(y_0, y_1) \in \mathbb{C}^2 - \{0\}$ up to scale.) These hyperplanes intersect X in a family of (possibly singular) complex curves $\{F_y \mid y \in \mathbb{C}\mathbb{P}^1\}$. Since the hyperplanes fill $\mathbb{C}\mathbb{P}^N$ and any two intersect precisely in A , we have $\bigcup_{y \in \mathbb{C}\mathbb{P}^1} F_y = X$ and $F_y \cap F_{y'} = B$ for $y \neq y'$. The canonical map $\mathbb{C}\mathbb{P}^N - A \rightarrow \mathbb{C}\mathbb{P}^1$ induced by the hyperplanes restricts to a holomorphic map $f : X - B \rightarrow \mathbb{C}\mathbb{P}^1$ determined by the condition $f^{-1}(y) = F_y - B$. Since A intersects X transversely, each F_y is smooth near B , and f can be locally identified with projectivization $\mathbb{C}^2 - \{0\} \rightarrow \mathbb{C}\mathbb{P}^1$ there. (In fact, the hyperplanes restrict to the complex lines through 0 on the tangent plane to X at each $b \in B$.) Since A is generic, so is the function f . This means f is the complex analog of a Morse function, i.e., its critical points are complex quadratic. The structure we have defined here is called a *Lefschetz pencil* on X , and can be generalized from holomorphic to smooth manifolds.

Definition 3.1. A *Lefschetz pencil* on a 4-manifold X is a finite *base locus* $B \subset X$ and a map $f : X - B \rightarrow \mathbb{C}\mathbb{P}^1$ such that

- (1) each $b \in B$ has an orientation-preserving local coordinate map to $(\mathbb{C}^2, 0)$ under which f corresponds to projectivization $\mathbb{C}^2 - \{0\} \rightarrow \mathbb{C}\mathbb{P}^1$, and
- (2) each critical point of f has an orientation-preserving local coordinate chart in which $f(z_1, z_2) = z_1^2 + z_2^2$ for some holomorphic local chart in $\mathbb{C}\mathbb{P}^1$.

Note that there is no analog of the Morse index, since $-z^2 = +(iz)^2$. In the literature, additional conditions are sometimes imposed. For example, after perturbing f we can assume that f is injective on the (finite) set of critical points. In addition, our algebraic prototype has the property that each component of $F_y - \{\text{critical points}\}$ intersects B (since its closure is a complex curve and hence homologically essential in the corresponding hyperplane); some version of this condition is needed for constructing symplectic structures (e.g., to rule out the torus bundle $f : S^3 \times S^1 \rightarrow S^2$).

Like Morse functions in the real-valued setting, Lefschetz pencils determine the topology of the underlying 4-manifolds. A useful way to exploit this is to “blow up” the base locus B , compactifying $X - B$ by one-point compactifying each fiber separately at each $b \in B$. This changes X by connected summing with a copy of $-\mathbb{C}\mathbb{P}^2$ for each $b \in B$. We obtain a singular fibration $X \# k(-\mathbb{C}\mathbb{P}^2) \rightarrow \mathbb{C}\mathbb{P}^1$ called a *Lefschetz fibration*, characterized by having only complex quadratic critical points as above. Explicit

handle diagrams can be drawn for Lefschetz fibrations, using the fact that each critical point corresponds to a 2-handle. Alternatively, one can remove the critical values from $\mathbb{C}\mathbb{P}^1 = S^2$, and delete the corresponding singular fibers from X , obtaining an honest fiber bundle over $S^* = S^2 - (\text{finite set})$. The monodromy around each critical value will be a right-handed Dehn twist of the fiber (assuming $f|_{\{\text{critical points}\}}$ is injective), and the monodromy representation $\pi_1(S^*) \rightarrow \text{Map}(F)$ (into the group of orientation-preserving diffeomorphisms of the fiber up to isotopy) will determine the Lefschetz fibration if the fiber has genus ≥ 2 . Thus, the study of Lefschetz fibrations reduces to a purely combinatorial problem about the mapping class group $\text{Map}(F)$. A similar reduction can be made for Lefschetz pencils, using diffeomorphisms of the fiber that fix a point and its tangent plane for each $b \in B$. (In this case, one must remove an extra point from S^* , around which the monodromy is nontrivial due to the twisted normal bundles of the exceptional spheres.) Lefschetz fibrations on 4-manifolds have recently become a particularly active area of research. For example, many Lefschetz fibrations have been directly constructed for which the underlying manifold X admits no complex structure. In particular, one can use monodromy representations to construct Lefschetz fibrations whose fundamental groups include all finitely presented groups [ABKP]. (Recall that most finitely presented groups cannot be realized by complex surfaces.) For a recent (but rapidly becoming outdated) survey of Lefschetz pencils and fibrations, see [GS].

Our construction of Lefschetz pencils on algebraic surfaces can be generalized to algebraic manifolds of any dimension. If we continue to require $\text{codim}_{\mathbb{C}} A = 2$, we obtain a map $f : X - B \rightarrow \mathbb{C}\mathbb{P}^1$, where B is a submanifold of (complex) codimension 2. The map f will look like projectivization in the directions normal to B , and the critical points of f will be locally modeled by $f(z_1, \dots, z_n) = \sum_{i=1}^n z_i^2$. (These correspond to n -handles.) Such structures are still called Lefschetz pencils. They were first used by Lefschetz to study the topology of algebraic manifolds. (See [L].) One can analyze them using the monodromy representation as in the 4-dimensional case, although at present, little work has been done on this. For a further generalization, we can allow A to have complex codimension $k + 1 \geq 2$, and consider linear subspaces with codimension k containing A . We then obtain a map $f : X - B \rightarrow \mathbb{C}\mathbb{P}^k$ with $\text{codim}_{\mathbb{C}} B = k + 1$. (For example, we can make B finite by setting $k = \dim_{\mathbb{C}} X - 1$. For larger k , B vanishes entirely.) The map f will still be projectivization on normal slices to the manifold B , but critical points will no longer be isolated and they may require higher degree terms in their local models. (For example, the $k = 2$ case can be thought of as a Lefschetz pencil of pencils. A single Lefschetz pencil has isolated critical points, but these will sweep out sheets as we vary through a pencil of pencils, and for some values of the parameter, quadratic critical points will coalesce to form those of higher-degree.) Algebraic geometers call structures of this more general form *linear systems*. In principle, one could try to analyze their topology via monodromies and induction on dimension. Note that if X has a linear system for a given k , then it has them for all smaller values of k : Simply compose f with the canonical

projection map $\mathbb{C}\mathbb{P}^k - \{\text{pt.}\} \rightarrow \mathbb{C}\mathbb{P}^{k-1}$. (This corresponds to choosing a new A containing the old one with codimension 1.) Thus, the information content of a linear system increases with k .

3.2. Hyperpencils

Linear systems $f : X - B \rightarrow \mathbb{C}\mathbb{P}^k$ provide the sort of “fibrationlike” structure on a $2n$ -manifold X that allows us to construct symplectic forms by the method of Section 2. Our present goal is to carefully define such a structure in topological terms, in such a way as to guarantee the existence of symplectic forms. A plausible starting place would be the $k = n$ case, where B is empty and f is a sort of singular branched covering. However, it seems best to start with the weakest possible definition guaranteeing a symplectic structure, meaning we should use the smallest possible value of k . But if $k \leq n - 2$, the fibers will have (real) dimension > 2 , and theorems producing symplectic structures will require hypotheses guaranteeing that the fibers and transition functions will be symplectic. Thus, for a theorem without symplectic hypotheses, the optimal case seems to be $k = n - 1$, where generic fibers are surfaces and hence are automatically symplectic. We will call such a structure a *hyperpencil*, with the prefix indicating that k should be changed from 1 (for a pencil) to complex codimension 1. The definition is analogous to that of Lefschetz pencils. However, the critical points are necessarily more complicated, so we allow them to be modeled by *any* holomorphic function (provided that within each fiber they are isolated). In fact, the situation is not significantly complicated by taking the function to be just locally J -holomorphic with respect to *almost*-complex structures (subject to a certain technical condition that is automatically satisfied in the holomorphic case or when $n \leq 3$). We allow these almost-complex structures to be C^0 rather than smooth, both for convenience and to emphasize that their primary function is homotopy-theoretic in nature, controlling monodromies. (For example, if we allow orientation-reversing charts at critical points in our definition of Lefschetz pencils, so that some monodromies are given by *left*-handed Dehn twists, then we can construct such structures on manifolds admitting no symplectic forms; see e.g. [GS].) As a final generalization, we allow the almost-complex structure on $\mathbb{C}\mathbb{P}^{n-1}$ to be different for different points on a given fiber, by using locally defined almost-complex structures on the bundle $f^*T\mathbb{C}\mathbb{P}^{n-1}$ rather than on $T\mathbb{C}\mathbb{P}^{n-1}$ itself. (The reader can simplify the setup by pretending these are given by the standard holomorphic structure on $\mathbb{C}\mathbb{P}^{n-1}$.) We require these structures to be compatible with the standard symplectic form $\omega_{\mathbb{C}\mathbb{P}^{n-1}}$ on $\mathbb{C}\mathbb{P}^{n-1}$ (pulled back to a skew-symmetric pairing on $f^*T\mathbb{C}\mathbb{P}^{n-1}$). We then obtain the following definition (which can, and probably should, be generalized even further).

Definition 3.2. A *hyperpencil* on a $2n$ -manifold X is a finite set $B \subset X$ and a map $f : X - B \rightarrow \mathbb{C}\mathbb{P}^{n-1}$ such that

- (1) each $b \in B$ has an orientation-preserving local coordinate map to $(\mathbb{C}^n, 0)$ under which f corresponds to projectivization $\mathbb{C}^n - \{0\} \rightarrow \mathbb{C}\mathbb{P}^{n-1}$,

- (2) each fiber $F_y = \text{cl } f^{-1}(y) \subset X$ contains only finitely many critical points of f , each locally modeled by a holomorphic map if $n \geq 4$, and each critical point has a neighborhood U with C^0 almost-complex structures on U and $f^*T\mathbb{C}\mathbb{P}^{n-1}|_U$ for which the latter is compatible with $\omega_{\mathbb{C}\mathbb{P}^{n-1}}$ and f is J -holomorphic, and
- (3) each component of $F_y - \{\text{critical points}\}$ intersects B .

The construction of Section 2 can be used to produce a symplectic structure on any manifold with a hyperpencil. In particular, any 4-manifold with a Lefschetz pencil (with $B \neq \emptyset$) admits a symplectic structure. (While a Lefschetz pencil need not satisfy condition (3) above, the condition is actually unnecessary in this case, provided $B \neq \emptyset$; see [GS]. However, without the condition, we lose both the control of $[\omega]$ and the uniqueness statement given below.) The construction allows us to control the cohomology class $[\omega]$. Recall that $H_{dR}^2(\mathbb{C}\mathbb{P}^{n-1}) \cong \mathbb{R}$ has a canonical generator h , the *hyperplane class*, Poincaré dual to $[\mathbb{C}\mathbb{P}^{n-2}]$. The class f^*h is defined on $X - B$, but $H_{dR}^2(X - B) \cong H_{dR}^2(X)$ for $n > 1$, so we can think of f^*h as a class in $H_{dR}^2(X)$ determined by the hyperpencil. (For $n = 1$, it is natural to identify f^*h with the Poincaré dual of $[B] \in H_0(X; \mathbb{R})$.) The construction allows us to arrange $[\omega] = f^*h$. The form ω is then completely determined by the construction, up to isotopy. (Two symplectic forms ω_0, ω_1 on X are *isotopic* if there is a diffeomorphism φ isotopic to the identity with $\varphi^*\omega_1 = \omega_0$. Thus, isotopic symplectic forms only differ by a deformation of X .) Furthermore, the isotopy class is unchanged if we deform the hyperpencil. (A deformation of hyperpencils should be roughly thought of as a bundle Z over a path connected parameter space S , whose fibers are hyperpencils. More precisely, we naturally generalize the definition of hyperpencil to this parametrized setting. For example, the base locus becomes a finite covering $B \rightarrow S$, and the local almost-complex structures at a critical point of X become continuous families of fiber-wise almost-complex structures defined near a point in Z .) More specifically, we obtain our main theorem:

Theorem 3.1. *A deformation class of hyperpencils uniquely determines an isotopy class of symplectic forms. This isotopy class is characterized as being the unique class containing representatives ω for which $[\omega] = f^*h \in H_{dR}^2(X)$ and ω tames a given hyperpencil in the deformation class.*

We say that ω *tames* a hyperpencil $f : X - B \rightarrow \mathbb{C}\mathbb{P}^{n-1}$ if there is a C^0 almost-complex structure $J_{\mathbb{C}\mathbb{P}^{n-1}}$ compatible with $\omega_{\mathbb{C}\mathbb{P}^{n-1}}$ on $f^*T\mathbb{C}\mathbb{P}^{n-1}$, such that each $x \in X$ has a neighborhood with a C^0 almost-complex structure tamed by ω and making f J -holomorphic. It can be shown that if ω tames f then there is a *global* almost-complex structure J on X with ω taming J and f J -holomorphic. Similarly, the local almost-complex structures in (2) of the definition of hyperpencils can be made global. In each case, the global structures (including $J_{\mathbb{C}\mathbb{P}^{n-1}}$) can be arranged to be standard near B . Similar statements apply in the setting of deformations. See [G2] for details.

3.3. Proof of Theorem 3.1

We sketch the proof; for further details, see [G2]. To prove existence, we fix a hyperpencil $f : X - B \rightarrow \mathbb{C}\mathbb{P}^{n-1}$ in the given deformation class, and establish the hypotheses of Theorem 2.2. As remarked in the previous paragraph, there are global structures J on X and $J_{\mathbb{C}\mathbb{P}^{n-1}}$ on $f^*T\mathbb{C}\mathbb{P}^{n-1}$, standard near B , making f J -holomorphic. Let $c = f^*h \in H_{dR}^2(X)$. For each $y \in \mathbb{C}\mathbb{P}^{n-1}$, we construct the required W_y and η_y : At each critical point $x \in F_y$, $(T_x X, J)$ can be identified with \mathbb{C}^n . Thus, the standard linear symplectic form on \mathbb{C}^n tames J at x . Extend this to a closed 2-form η_y near x ; we can assume η_y tames J on this neighborhood by openness of the taming condition. Since F_y is a J -complex curve, $\eta_y|_{F_y}$ is an area form defined near the singular points of F_y . Extend this to an area form on all of F_y whose total area on each component F_i of $F_y - \{\text{critical points}\}$ is $\langle f^*h, c|_{F_i} \rangle$ (which is positive since it equals the number of points in $F_i \cap B$; cf. (3) of Definition 3.2). W_y and η_y can now be constructed in a manner analogous to the proof of Theorem 2.1, by pulling back $\eta_y|_{F_y}$ by a suitable map. By construction, $[\eta_y] = c|_{W_y}$. Also, η_y tames J on each $T_x X$ near critical points, and on $\ker df_x = T_x F_{f(x)}$ elsewhere (for W_y sufficiently small). (We have ignored minor technical difficulties arising, e.g., if F_y has nonconelike singularities.) A slight generalization of Theorem 2.2 now gives a symplectic form ω on $X - B$ taming J . (Note that $f : X - B \rightarrow \mathbb{C}\mathbb{P}^{n-1}$ fails the hypotheses of Theorem 2.2 in that the domain is noncompact; this can be fixed by working relative to a standard symplectic form defined near B .) Unfortunately, ω is singular at B — it has the form $t\eta + f^*\omega_{\mathbb{C}\mathbb{P}^{n-1}}$, and the second term is singular. Fortunately, we have an explicit description of ω and J near each $b \in B$ (on a neighborhood identified with a neighborhood of 0 in \mathbb{C}^n , with f given by projectivization). This local model shows that we can dilate X at b and glue in a symplectic ball to make ω smooth everywhere, without losing the taming condition. (This construction is essentially equivalent to blowing up B , applying Theorem 2.2 to the resulting singular fibration on a compact manifold, and blowing back down. However it bypasses some technical difficulties involving working with the blown-up points.) We now have the desired symplectic form. It obviously tames f via J , and $[\omega] = [t\eta + f^*\omega_{\mathbb{C}\mathbb{P}^{n-1}}] = tc + f^*[\omega_{\mathbb{C}\mathbb{P}^{n-1}}] = (t+1)f^*h$ (since $[\omega_{\mathbb{C}\mathbb{P}^{n-1}}] = h$), so ω satisfies the required conditions after rescaling.

To prove uniqueness, we start with symplectic forms ω_0 and ω_1 on X , satisfying the two conditions in Theorem 3.1 with respect to deformation equivalent pencils f_0 and f_1 , respectively. We show that ω_0 and ω_1 are isotopic, completing the proof of the theorem. We can assume the deformation is parametrized by the interval $I = [0, 1]$. Each form ω_i is given to tame f_i , so as indicated at the end of the previous subsection, we can find a global C^0 almost-complex structure J_i on X making f_i J_i -holomorphic, and with ω_i taming J_i . Using the same paragraph (in the parametrized version rel $\{0, 1\}$ without a taming ω), we can extend J_0, J_1 to a continuous family J_t of almost-complex structures, $0 \leq t \leq 1$, with f_t J_t -holomorphic for some family $J_{t, \mathbb{C}\mathbb{P}^{n-1}}$ on $f_t^*T\mathbb{C}\mathbb{P}^{n-1}$. (This is the one place where Definition 3.2 requires the condition for $n \geq 4$, and compatibility with $\omega_{\mathbb{C}\mathbb{P}^{n-1}}$ rather than taming.) For each hyperpencil f_t in the deformation and each

structure J_t , construct ω_t taming J_t as in the previous paragraph. While the resulting forms ω_t , $0 \leq t \leq 1$, will be *a priori* unrelated to each other, we can make the family smooth by a trick: By openness of the taming condition, each ω_t will tame each J_s in some neighborhood of $t \in I$. Thus, we can cover I by neighborhoods U_α on which a single ω_t tames each J_s , $s \in U_\alpha$. Using a partition of unity $\{\rho_\alpha\}$ on I subordinate to $\{U_\alpha\}$, splice together these forms ω_t . The resulting smooth family (still called ω_t) will consist of closed forms (since each ρ_α is constant on each fiber of the deformation), and each ω_t will tame the corresponding J_t (since it is a convex combination of taming forms). Thus, we have a smooth family of symplectic forms on X . Furthermore, $[\omega_t] = f_t^*h$ is independent of t (since we can assume B is fixed and invoke homotopy invariance of induced maps). The theorem now follows from:

Theorem 3.2 (Moser [M]). *Let ω_t , $0 \leq t \leq 1$, be a smooth family of symplectic forms on X , with $[\omega_t] \in H_{dR}^2(X)$ independent of t . Then there is an isotopy $\varphi_t : X \rightarrow X$ with $\varphi_0 = \text{id}_X$ and $\varphi_t^*\omega_t = \omega_0$.*

(The proof of Moser's Theorem is actually quite short. One simply writes down a suitable formula for a time-dependent vector field, then integrates to obtain φ_t .) \square

3.4. Characterization

We now turn to the question of how general the hyperpencil construction of symplectic structures is, addressing the topological characterization of symplectic manifolds. The answer lies in work of Donaldson [D] followed by Auroux [A], the roots of which go back to Kodaira in the holomorphic setting (the Kodaira Embedding Theorem, e.g., [GH]). If $\sigma_0, \dots, \sigma_k$ are sections of a complex line bundle $L \rightarrow X$, then because each fiber L_x is canonically \mathbb{C} up to (complex) scale, the vector $(\sigma_0(x), \dots, \sigma_k(x))$ in $(L_x)^{k+1}$ determines an element of \mathbb{C}^{k+1} up to scale. Thus, projectivizing gives a well-defined map $f = [\sigma_0 : \dots : \sigma_k] : X - B \rightarrow \mathbb{C}\mathbb{P}^k$, where B is the common zero locus of $\sigma_0, \dots, \sigma_k$. Kodaira used this idea to characterize which complex manifolds are algebraic (i.e., embed holomorphically in $\mathbb{C}\mathbb{P}^N$) in terms of line bundles: Given the existence of a suitable holomorphic line bundle L over a complex manifold X , one can obtain arbitrarily many holomorphic sections by taking sufficiently large tensor powers $L^{\otimes m}$ of the line bundle, eventually yielding an embedding $f : X \hookrightarrow \mathbb{C}\mathbb{P}^N$. It automatically follows that X is Kähler with symplectic form $\omega = f^*\omega_{\mathbb{C}\mathbb{P}^N}$ satisfying $[\omega] = c_1(L^{\otimes m}) = mc_1(L)$. Donaldson's contribution was to extend this idea to the symplectic setting. Starting with a symplectic manifold (X, ω) with ω *integral* (i.e., $[\omega] \in \text{Im}(H^2(X; \mathbb{Z}) \rightarrow H_{dR}^2(X))$), he chose a compatible almost-complex structure J , and arranged a line bundle $L \rightarrow X$ with Chern class $c_1(L) = [\omega]$ to be J -holomorphic in a suitable sense. Unfortunately, in this setting holomorphic sections rarely exist. However, by defining a suitable notion of "approximately holomorphic" sections and applying hard analysis on the line bundles $L^{\otimes m}$, Donaldson was able to construct a Lefschetz pencil $X - B \rightarrow \mathbb{C}\mathbb{P}^1$ on any integral symplectic manifold. Subsequently, Auroux [A] generalized the method to construct a linear system $X - B \rightarrow \mathbb{C}\mathbb{P}^k$ for $k = 2$, and he is currently extending his work to arbitrary k .

In particular, the case $k = n - 1$ should yield a hyperpencil (with particularly nice local properties) tamed by the original symplectic form ω , and with $f^*h = c_1(L^{\otimes m}) = m[\omega]$ for any given sufficiently large integer m . (It may be useful to slightly weaken the definition of a hyperpencil here.) This would complete the proof of the following conjecture, which currently seems to be established in dimensions ≤ 6 (by the above cases $k = 1, 2$.)

Conjecture 3.3. *For any integral symplectic manifold (X, ω) and sufficiently large $m \in \mathbb{Z}$, there is a hyperpencil on X for which the canonical isotopy class of symplectic forms contains $m\omega$.*

Now note that the nondegeneracy condition for symplectic forms is open, and $H^2(X; \mathbb{Q})$ is dense in $H_{dR}^2(X)$. Thus, rational symplectic forms on a manifold X are dense in the space of all symplectic forms. Furthermore, any rational cohomology class can be rescaled to an integral one. Thus, up to scale, the hyperpencil construction should give a dense subset of *all* symplectic forms on any given manifold. That is:

Proposition 3.4. *Let $\mathcal{P}(X)$ be the set of deformation classes of hyperpencils on a manifold X , $\mathcal{S}(X)$ be the set of isotopy classes of rational symplectic structures, and $\Omega : \mathcal{P}(X) \rightarrow \mathcal{S}(X)$ be the map given by Theorem 3.1. Suppose that all integral symplectic structures on X satisfy Conjecture 3.3. Then the induced map $\overline{\Omega} : \mathcal{P}(X) \rightarrow \mathcal{S}(X)/\mathbb{Q}_+$ is surjective. Equivalently, there is a surjection $\tilde{\Omega} : \mathcal{P}(X) \times \mathbb{Q}_+ \rightarrow \mathcal{S}(X)$, where $\tilde{\Omega}(f, q)$ is obtained from $\Omega(f)$ by rescaling so that $[\tilde{\Omega}(f, q)]$ is q times a primitive integral class. \square*

Corollary 3.5. *In dimensions where Conjecture 3.3 holds (e.g. dimensions ≤ 6), a manifold admits a symplectic structure if and only if it admits a hyperpencil. A 4-manifold admits a symplectic structure if and only if it admits a Lefschetz pencil with $B \neq \emptyset$. \square*

(For the 4-dimensional version, see also [GS].) Thus, Conjecture 3.3 topologically characterizes those manifolds admitting symplectic structures. From there, to completely determine, in topological terms, the dense subset $\mathcal{S}(X)$ of the space of symplectic forms on X , we only need to identify the point preimages of $\tilde{\Omega}$ (which are the same as for $\overline{\Omega}$).

Conjecture 3.6. *The point preimages of $\tilde{\Omega}$ (or equivalently, $\overline{\Omega}$) can be given by a topologically defined equivalence relation on $\mathcal{P}(X)$.*

To do this, it may be useful to strengthen the definition of hyperpencils. The main evidence for this conjecture is that the theorems of Donaldson and Auroux come with uniqueness statements, up to “stabilization,” or multiplying m by large integers (taking tensor powers of the relevant line bundle). This suggests that one should be able to topologically define stabilization maps $\sigma_k : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$, $k \in \mathbb{Z}_+$, with $\sigma_1 = \text{id}_{\mathcal{P}(X)}$, $\sigma_k \circ \sigma_\ell = \sigma_{k\ell}$ and $\Omega \circ \sigma_k = k\Omega$, and that the required equivalence relation should be given by $f \sim g$ if and only if $\sigma_k(f) = \sigma_\ell(g)$ for some $k, \ell \in \mathbb{Z}_+$. However, these stabilization maps seem complicated, even in dimension 4.

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